Note on a Casimir Energy Calculation for a Purely Dielectric Cylinder by Mode Summation

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Abstract. We comment on a recent calculation of the zero-point energy for a dilute and infinitely long cylinder of purely-dielectric material. The vanishing result predicted by integration of van der Waals potentials is obtained.

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The Casimir effect is a change in the electromagnetic vacuum fluctuations brought about by the presence of boundaries. Particularly, cylindrical surfaces limiting dielectric media were considered in [1]. One of the first versions of that paper inspired an unpublished calculation, by Romeo, of the van der Waals energy for a purely dielectric cylinder in the dilute-dielectric approximation, which yielded a null result. That calculation found a tribune in appendix B of the final version of [1] and, eventually, unpublished work by Milonni and ref.[2] by Barton provided independent confirmations.

This finding aroused curiosity about the corresponding Casimir energy, which would have to show the predicted equality between both quantities [3] and, therefore, was expected to vanish similarly. The divergences of this problem were studied through its heat kernel coefficents in [4], and the expected vanishing was first verified in [5], where the Casimir pressure was obtained from the expectation value of the stress-energy tensor using Green's functions. Next, a calculation of the Casimir energy based on the mode summation method [6] was completed. The present note offers a comment on that work.

Let J_m, H_m denote the Bessel and Hankel functions (for $y > 0$, $H_m(y) \equiv H_m^{(1)}(y)$). Given an infinitely long cylinder of radius a, oriented along the z-axis, with permittivity and permeability (ε_1, μ_1), surrounded by a medium with permittivity and permeability (ε_2, μ_2), the eigenfrequencies ω of the Maxwell equations with the adequate boundary

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conditions are the solutions of:

$$f_m(k_z,\omega)=0,\quad m\in\mathbb{Z},\quad k_z\in\mathbb{R},$$
$$f_m(k_z,\omega)\equiv\frac{1}{\Delta^2}\left[\Delta_m^{\rm TE}(x,y)\,\Delta_m^{\rm TM}(x,y)-m^2\frac{a^4\omega^2k_z^2}{x^2y^2}(\varepsilon_1\mu_1-\varepsilon_2\mu_2)^2J_m^2(x)\,H_m^2(y)\right] \tag{1}$$

(see [7, 1]), where

$$\begin{aligned}\Delta&=-\frac{2i}{\pi},\\ \Delta_m^{\rm TE}(x,y)&=\mu_1 yJ'_m(x)\,H_m(y)-\mu_2 xJ_m(x)\,H'_m(y),\\ \Delta_m^{\rm TM}(x,y)&=\varepsilon_1 yJ'_m(x)\,H_m(y)-\varepsilon_2 xJ_m(x)\,H'_m(y),\\ x=\lambda_1 a,\quad y&=\lambda_2 a,\quad \lambda_i^2=\varepsilon_i\mu_i\omega^2-k_z,\quad i=1,2.\end{aligned} \tag{2}$$

The m index is the azimuthal quantum number, k_z is the momentum along the cylinder axis, and p labels the zeroes of $f_m(k_z,\omega)$. In fact $f_m=-\Delta^{-2}\Xi$, being Ξ the same object as in [5] and Δ^{-2} a factor introduced for convenience. The velocities of light in each media are $c_i=(\varepsilon_i\mu_i)^{-1/2}$, $i=1,2$.

If medium 1 is purely dielectric and medium 2 is vacuum, $\varepsilon_1=\varepsilon$, $\mu_1=1$, $\varepsilon_2=\mu_2=1$ (obviously, $c_2=1$). Further,

$$\omega=a^{-1}(y^2+\widehat{k}^2)^{1/2},\qquad x^2=y^2+(\varepsilon-1)(y^2+\widehat{k}^2),\qquad \widehat{k}\equiv k_z a. \tag{3}$$

The Casimir energy per unit length stems from the mode sum

$$\mathcal{E}_C=\frac{1}{2}\hbar\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_m\sum_p\omega_{m,p,k_z}, \tag{4}$$

which is divergent, and will be regularized appropriately (see below). Reference [4] tells us that, up through the order of $(\varepsilon-1)^2$, there are no ambiguities, because the heat kernel coefficient which would multiply them is of $\mathcal{O}((\varepsilon-1)^3)$. Thus, we may just set

$$\mathcal{E}_C(s)=\frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_m\sum_p\omega_{m,p,k_z}^{-s}=\frac{\hbar}{2}\,a^{s-1}\int_{-\infty}^{\infty}\frac{d\widehat{k}}{2\pi}\sum_m\sum_p(y_{m,p}^2+\widehat{k}^2)^{-s/2}, \tag{5}$$

without any additional mass scale. $\mathcal{E}_C(s)$ is a function of the complex variable s, and our idea is to redefine (4) by analytic continuation of this function to $s=-1$, i.e.,

$$\mathcal{E}_C=\lim_{s\to-1}\mathcal{E}_C(s). \tag{6}$$

Once that $\widehat{k}$, m have specific values, the sum over p is expressed as a contour integral in complex y plane:

$$\mathcal{E}_C(s)=\frac{\hbar}{2}\,a^{s-1}\int_{-\infty}^{\infty}\frac{d\widehat{k}}{2\pi}\sum_{m=-\infty}^{\infty}\frac{s}{2\pi i}\int_C dy\,y\,(y^2+\widehat{k}^2)^{-s/2-1}\,\ln f_m, \tag{7}$$

where C is a circuit enclosing all the y values corresponding to the positive zeroes of f_m (the argument principle [8] derived from the residue theorem). When applying this method, one sometimes finds an asymptotic form $f_{m,\rm as}$ of f_m and then subtracts $\ln f_{m,\rm as}$ from $\ln f_m$ in the integrand. In fact, the factors introduced in (1) relative to the original

f_m of [1] have the same effect as having divided that function by the leading part of $f_{m,\mathrm{as}}$.

At this point, the logarithm function of (7) is expanded in powers of $(\varepsilon - 1)$, taking y as an independent variable and x as a function of y, $\widehat{k}$, ε (see (3)). Then,

$$\begin{aligned}\ln f_m = \; & \left[L^0_{m1}(y) + L^1_{m1}(y)(y^2+\widehat{k}^2)\right](\varepsilon-1) \\ & + \Big[L^{00}_{m2}(y) + L^{10}_{m2}(y)(y^2+\widehat{k}^2) + L^{20}_{m2}(y)(y^2+\widehat{k}^2)^2 \\ & \quad + L^{11}_{m2}(y)(y^2+\widehat{k}^2)\,\widehat{k}^2\Big](\varepsilon-1)^2 \\ & + \mathcal{O}((\varepsilon-1)^3),\end{aligned} \tag{8}$$

where

$$\begin{aligned} L^0_{m1}(y) &= \frac{1}{\Delta} y\, J'_m(y) H_m(y), \\ L^1_{m1}(y) &= \frac{1}{\Delta y} \Delta^{(1,0)}_m(y), \\ L^{00}_{m2}(y) &= -\frac{1}{2\Delta^2} y^2\, {J'_m}^2(y) H^2_m(y), \\ L^{10}_{m2}(y) &= -\frac{1}{2\Delta^2}\Big[\Delta^{(1,0)}_m(y) J'_m(y) H_m(y) \\ &\qquad + \frac{\Delta}{y}\left(J'_m(y) + y\left(1-\frac{m^2}{y^2}\right)J_m(y)\right)H_m(y)\Big], \\ L^{20}_{m2}(y) &= L^{20A}_{m2}(y) + L^{20B}_{m2}(y), \quad \begin{cases} L^{20A}_{m2}(y) = \dfrac{1}{4\Delta y^2}\left(\Delta^{(2,0)}_m(y) - \dfrac{1}{y}\Delta^{(1,0)}_m(y)\right), \\ L^{20B}_{m2}(y) = -\dfrac{1}{4\Delta^2 y^2}\left(\Delta^{(1,0)}_m(y)\right)^2, \end{cases} \\ L^{11}_{m2}(y) &= -\frac{m^2}{\Delta^2 y^4} J^2_m(y) H^2_m(y), \end{aligned} \tag{9}$$

with

$$\begin{aligned} \Delta^{(1,0)}_m(y) &= -\frac{1}{y}\left[y^2 J'_m(y) H'_m(y) + (y^2-m^2) J_m(y) H_m(y)\right] - (J_m(y) H_m(y))', \\ \Delta^{(2,0)}_m(y) &= \left(\Delta^{(1,0)}_m(y)\right)' - \left(1 - \frac{m^2+1}{y^2}\right)\Delta, \qquad (\Delta^{(1,0)}_m(y))' \equiv \frac{d}{dy}\Delta^{(1,0)}_m(y). \end{aligned} \tag{10}$$

Now, (8) is inserted into (7). The obtained expression involves integrals of the form

$$I \equiv \int_{-\infty}^{\infty} d\widehat{k} \int_C dy\, y\, F(y)\,(y^2+\widehat{k}^2)^{-\alpha}\,\widehat{k}^{2\beta}, \tag{11}$$

where C is the contour of (7) and F satisfies $F(-iv) = F(iv)$ for $v \in \mathbb{R}$, as well as having good asymptotic properties (the role of F is played by the L_m's of (9),(10)). Examining the $(y^2 + \widehat{k}^2)$ powers in (7), (8), one sees that, in the required cases, $\alpha = s/2+1, s/2, s/2-1$, and $\beta = 0$ except for one integral with $\beta = 1$. Analytic continuation in s obviously amounts to analytic continuation in α. Following [6], the value of I is given by

$$I = -2i\,\mathrm{B}\left(\beta + \frac{1}{2}, 1-\alpha\right)\sin(\pi\alpha)\int_0^{\infty} dv\, v^{2-2\alpha+2\beta}\, F(iv), \tag{12}$$

where B denotes the Euler beta function (about the mathematical basis, see also [9, 10]). Note that for $s=-1$, i.e., $\alpha = 1/2, -1/2, -3/2$, and for $\beta = 0, 1$, the beta and sine functions are finite. Application of formula (12) to Eqs. (7), (8) gives:

$$\mathcal{E}_C(s) = \mathcal{E}_{C1}(s)(\varepsilon-1) + \mathcal{E}_{C2}(s)(\varepsilon-1)^2 + \mathcal{O}((\varepsilon-1)^3), \tag{13}$$

where

$$\mathcal{E}_{C1}(s) = \mathcal{E}_{C1}^0(s) + \mathcal{E}_{C1}^1(s),$$
$$\begin{cases} \mathcal{E}_{C1}^0(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2}, -\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{-s} L_{m1}^0(iv), \\ \mathcal{E}_{C1}^1(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2}, 1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{2-s} L_{m1}^1(iv), \end{cases} \tag{14}$$

and

$$\mathcal{E}_{C2}(s) = \mathcal{E}_{C2}^{00}(s) + \mathcal{E}_{C2}^{10}(s) + \mathcal{E}_{C2}^{20A}(s) + \mathcal{E}_{C2}^{20B}(s) + \mathcal{E}_{C2}^{11}(s),$$
$$\begin{cases} \mathcal{E}_{C2}^{00}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2}, -\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{-s} L_{m2}^{00}(iv), \\ \mathcal{E}_{C2}^{10}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2}, 1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{2-s} L_{m2}^{10}(iv), \\ \mathcal{E}_{C2}^{20A,B}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2}, 2-\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{4-s} L_{m2}^{20A,B}(iv), \\ \mathcal{E}_{C2}^{11}(s) = -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{3}{2}, 1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{4-s} L_{m2}^{11}(iv). \end{cases} \tag{15}$$

With $\mathcal{E}_{C1}^0(s)$ taken from (14), and $L_{m1}^0(iv)$ from (9), we arrive at

$$\mathcal{E}_{C1}^0(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\frac{1}{2}, -\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{1-s}\, I_m'(v)K_m(v). \tag{16}$$

The beta and sine functions are already finite at $s=-1$, and the integral will be reexpressed by introducing the factor $1 = -vW[I_m(v), K_m(v)] = -v[I_m(v)K_m'(v) - I_m'(v)K_m(v)]$ for every m:

$$\begin{gathered}\int_0^\infty dv\, v^{1-s}\sum_{m=-\infty}^{\infty} I_m'(v)K_m(v) = \\ -\int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty} I_m(v)I_m'(v)K_m(v)K_m'(v) + \int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty} {I_m'}^2(v)K_m^2(v).\end{gathered} \tag{17}$$

The summations over m will be performed by taking advantage of the addition theorem for the modified Bessel functions:

$$\begin{aligned} \sum_{m=-\infty}^{\infty} I_m(kr)K_m(k\rho)\, e^{im\phi} &= K_0(kR(r,\rho,\phi)) \\ R(r,\rho,\phi) &= \sqrt{r^2+\rho^2-2r\rho\cos\phi}, \quad \rho > r. \end{aligned} \tag{18}$$

Suitable manipulations of this identity ([11, 12, 5, 6]) yield:

$$
\begin{aligned}
\int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} I_m'^{\,2}(v) K_m^2(v) &= \\
\int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} K_m'^{\,2}(v) I_m^2(v) &= \\
\int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} I_m(v) I_m'(v) K_m(v) K_m'(v) &= \frac{1}{8\pi^{1/2}} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \\
\int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} m^2 I_m(v) I_m'(v) K_m(v) K_m'(v) &= \frac{1}{16\pi^{1/2}} \frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \\
\int_0^\infty dv\, v^{4-s} \sum_{m=-\infty}^{\infty} I_m'^{\,2}(v) K_m'^{\,2}(v) &= \frac{1}{8\pi^{1/2}} \left[\frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \right. \\
&\quad + \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} \\
&\quad \left. + \frac{1}{4} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)} \right] \\
\int_0^\infty dv\, v^{4-s} \sum_{m=-\infty}^{\infty} I_m^2(v) K_m^2(v) &= \frac{1}{8\pi^{1/2}} \frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)} \\
\int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} m^2 I_m^2(v) K_m^2(v) &= \frac{1}{16\pi^{1/2}} \frac{\Gamma\left(\frac{7-s}{2}\right)\Gamma^2\left(\frac{5-s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} \\
\int_0^\infty dv\, v^{-s} \sum_{m=-\infty}^{\infty} m^4 I_m^2(v) K_m^2(v) &= \frac{1}{8\pi^{1/2}} \left[\frac{3}{4} \frac{\Gamma^4\left(\frac{5-s}{2}\right)}{\Gamma(5-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \right. \qquad (19) \\
&\quad + \frac{1}{2} \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} \\
&\quad \left. + \frac{1}{4} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s-4}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)} \right] \\
\int_0^\infty dv\, v^{3-s} \sum_{m=-\infty}^{\infty} I_m'^{\,2}(v) K_m(v) K_m'(v) &= \\
\int_0^\infty dv\, v^{3-s} \sum_{m=-\infty}^{\infty} K_m'^{\,2}(v) I_m(v) I_m'(v) &= -\frac{1}{8\pi^{1/2}} \left[\frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \right. \\
&\quad \left. + \frac{1}{2} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} \right] \\
\int_0^\infty dv\, v^{3-s} \sum_{m=-\infty}^{\infty} I_m^2(v) K_m(v) K_m'(v) &= \\
\int_0^\infty dv\, v^{3-s} \sum_{m=-\infty}^{\infty} K_m^2(v) I_m(v) I_m'(v) &= -\frac{1}{8\pi^{1/2}} \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} \\
\int_0^\infty dv\, v^{1-s} \sum_{m=-\infty}^{\infty} m^2 I_m^2(v) K_m(v) K_m'(v) &= \\
\int_0^\infty dv\, v^{1-s} \sum_{m=-\infty}^{\infty} m^2 K_m^2(v) I_m(v) I_m'(v) &= -\frac{1}{16\pi^{1/2}} \frac{\Gamma^2\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)}{\Gamma(4-s)} \frac{\Gamma\left(\frac{s-2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}.
\end{aligned}
$$

Although the left hand side of each integral is not initially defined for $s = -1$, the right hand side together with the remaining s dependent factors in $\mathcal{E}_C(s)$ will eventually provide the desired extension to negative s through the existing analytic continuations of the involved functions. Then, the poles at $s = -1, -3, -5, \ldots$ in the last dividing gamma functions, will give rise to zeros at these points.

Going back to $\mathcal{E}^0_{C1}(s)$, since (19) show that the two integrals in the second line of (17) have the same value,

$$\mathcal{E}^0_{C1}(s) = 0, \tag{20}$$

even before setting $s = -1$.

Formulas (14) tell us that $\mathcal{E}^1_{C1}(s)$ involves the integration of the $L^1_{m1}(iv)$ function, defined by (9), (10). Therefore,

$$\begin{aligned} \mathcal{E}^1_{C1}(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right) \\ \times \sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{2-s}\left[I'_m(v)K'_m(v) - \left(1+\frac{m^2}{v^2}\right)I_m(v)K_m(v) + \frac{1}{v}(I_m(v)K_m(v))'\right]. \end{aligned} \tag{21}$$

We multiply, again, each term in the m summation of (21) by $1 = -vW[I_m(v), K_m(v)]$, and turn the initial expression into a linear combination of integrals with summations of products of four Bessel functions. That linear combination yields an identically null result — one that is zero for any s value — by virtue of the symmetries observed in (19) under interchange of different Bessel function types (see also comment after Eqs. (80) in [5]). As a result,

$$\mathcal{E}^1_{C1}(s) = 0. \tag{22}$$

Equation (21) admits the following reinterpretation. Taking into account the fact that I_m, K_m satisfy the modified Bessel equation, we apply partial integration to (21) omitting a 'boundary term' which vanishes for a given s range that does not include $s = -1$ yet. Doing so, we find

$$\begin{aligned} \mathcal{E}^1_{C1}(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right) \\ \times\left[\int_0^\infty dv\, v^{1-s}\sum_{m=-\infty}^{\infty}(I_m(v)K_m(v))' + \frac{2}{1-s}\int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty}I_m(v)K_m(v)\right]. \end{aligned} \tag{23}$$

These integrals cannot be straightforwardly taken at $s = -1$ but, if this is ignored, we may formally put $s = -1$ and get

$$\mathcal{E}^1_{C1}(-1) \to -\frac{\hbar}{8\pi a^2}\int_0^\infty dv\, v^2\sum_{m=-\infty}^{\infty}(I_m(v)K_m(v))' - \frac{\hbar}{8\pi a^2}\int_0^\infty dv\, v^3\sum_{m=-\infty}^{\infty}I_m(v)K_m(v). \tag{24}$$

The first part could arguably be dismissed as a mere contact term because, from (18), it may be shown that it is local in v (In fact it is possible to obtain $\lim_{\phi\to 0}\sum_{m=-\infty}^{\infty}(I_m(v)K_m(v))'e^{im\phi} = -\frac{1}{v}$). The second part of (24) cancels the bulk contribution found in [5] (See formulas (72), (78) there and recall that the Casimir radial pressure is $P_C = \frac{1}{\pi a^2}\mathcal{E}_C$.)

Viewed in a different way, by the arguments in [13] (and references therein) all linear terms in $(\varepsilon_2 - \varepsilon_1)$ have to be removed because they are the self-energy of the electromagnetic field due to polarizable particles. By that rule, one simply must take out the linear part, regardless of its particular form. This is actually a re-statement of the physical reason for the removal of the bulk contribution.

When going on to second order in $(\varepsilon-1)$, we take first the piece called $\mathcal{E}^{20A}_{C2}(s)$, as its calculation is most similar to that of $\mathcal{E}^{0}_{C1}(s)$, $\mathcal{E}^{1}_{C1}(s)$. From the $\mathcal{E}^{20A}_{C2}(s)$ given in (15), the $L^{20A}_{m2}(y)$ in (9), expressions (10) with $y = iv$, introducing, once more, $1 = -vW[I_m(v), K_m(v)]$, and using the same reasoning that led to (22), one gets

$$\mathcal{E}^{20A}_{C2}(s) = 0. \tag{25}$$

Now, selecting the lines in (15), which determine $\mathcal{E}^{00}_{C2}(s)$, $\mathcal{E}^{10}_{C2}(s)$, $\mathcal{E}^{20B}_{C2}(s)$, $\mathcal{E}^{11}_{C2}(s)$, the parts of (9) which define $L^{00}_{m2}(y)$, $L^{10}_{m2}(y)$, $L^{20B}_{m2}(y)$, $L^{11}_{m2}(y)$, the form of $\Delta^{(1,0)}_m(y)$ dictated by (10) (its square for the case of $L^{20B}_{m2}(y)$), and setting $y = iv$, we come to

$$\begin{aligned}
\mathcal{E}^{00}_{C2}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{4\pi^2}\,\mathrm{B}\left(\frac{1}{2},-\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty} {I'_m}^2(v)K_m^2(v),\\
\mathcal{E}^{10}_{C2}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{4\pi^2}\,\mathrm{B}\left(\frac{1}{2},1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right)\int_0^\infty dv\, v^{2-s}\\
&\quad\times\sum_{m=-\infty}^{\infty}\Big[2I_m(v)I'_m(v)K_m(v)K'_m(v) + v\,{I'_m}^2(v)K_m(v)K'_m(v)\\
&\qquad -\left(v+\frac{m^2}{v}\right)I_m^2(v)K_m(v)K'_m(v)\Big],\\
\mathcal{E}^{20B}_{C2}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{8\pi^2}\,\mathrm{B}\left(\frac{1}{2},2-\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\int_0^\infty dv\, v^{2-s}\\
&\quad\times\sum_{m=-\infty}^{\infty}\Big[{I'_m}^2(v)K_m^2(v) + I_m^2(v){K'_m}^2(v)\\
&\qquad +2(1-v^2-m^2)I_m(v)I'_m(v)K_m(v)K'_m(v)\\
&\qquad +v^2{I'_m}^2(v){K'_m}^2(v) + \left(v^2+2m^2+\frac{m^4}{v^2}\right)I_m^2(v)K_m^2(v)\\
&\qquad +2v\left({I'_m}^2(v)K_m(v)K'_m(v) + I_m(v)I'_m(v){K'_m}^2(v)\right)\\
&\qquad -2\left(v+\frac{m^2}{v}\right)\left(I_m(v)I'_m(v)K_m^2(v) + I_m^2(v)K_m(v)K'_m(v)\right)\Big],\\
\mathcal{E}^{11}_{C2}(s) &= \frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\frac{3}{2},1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right)\int_0^\infty dv\, v^{-s}\sum_{m=-\infty}^{\infty} m^2 I_m^2(v)K_m^2(v).
\end{aligned} \tag{26}$$

The outcome of replacing the results (19) into (26) and expanding in $(s+1)$ is:

$$\mathcal{E}^{00}_{C2}(s) + \mathcal{E}^{10}_{C2}(s) + \mathcal{E}^{20B}_{C2}(s) + \mathcal{E}^{11}_{C2}(s) = \frac{\hbar}{a^2}\widehat{\mathcal{E}}(s+1) + \mathcal{O}((s+1)^2), \tag{27}$$

with $\widehat{\mathcal{E}} = \frac{23}{5760\pi}$. Formulas (25) and (27) make evident that

$$\lim_{s\to-1}\mathcal{E}_{C2}(s) = 0, \tag{28}$$

i.e., the $(\varepsilon-1)^2$ contribution to the Casimir energy per unit length in the dilute-dielectric approximation is zero, as we wished to prove.

Employing a regularization which analytically continues the vacuum energy as a function of the eigenmode power, we have found a pure Casimir term (in the sense of [2]) that is seen to vanish to the order of $(\varepsilon - 1)^2$. Remarkably, for the analogous problem with light velocity conservation condition [1, 12] the result is null to the order of $\xi^2 \equiv \left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)^2$. In fact, we have applied a form of zeta function regularization, whose links to other techniques have been studied in e.g. [14]. The sight of (19) makes us evoke the words of [15] and proclaim that a forest of gamma functions has grown out of an analytic continuation.

A divergence at third order in $(\varepsilon - 1)$ introduces an unavoidable ambiguity [4] (for further discussions on divergences see [16].) No universal agreement exists on the physical interpretation of the technique used, as commented in [15]. The nature of a third order divergence, viewed as a weak-coupling limit, has been considered in [17].

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