How does Casimir energy fall? II. Gravitational acceleration of quantum vacuum energy

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It has been demonstrated that quantum vacuum energy gravitates according to the equivalence principle, at least for the finite Casimir energies associated with perfectly conducting parallel plates. We here add further support to this conclusion by considering parallel semitransparent plates, that is, δ-function potentials, acting on a massless scalar field, in a spacetime defined by Rindler coordinates (τ, x, y, ξ). Fixed ξ in such a spacetime represents uniform acceleration. We calculate the force on systems consisting of one or two such plates at fixed values of ξ. In the limit of large Rindler coordinate ξ (small acceleration), we recover (via the equivalence principle) the situation of weak gravity, and find that the gravitational force on the system is just $M\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration and M is the total mass of the system, consisting of the mass of the plates renormalized by the Casimir energy of each plate separately, plus the energy of the Casimir interaction between the plates. This reproduces the previous result in the limit as the coupling to the δ-function potential approaches infinity.

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I. INTRODUCTION

The subject of quantum vacuum energy, or of Casimir energy, has engendered a certain controversy from the beginning because of the presence of divergences, which make it difficult to extract self-energies for single bodies [1–4]. Although it appears that many of these divergences can be consistently isolated when calculating the Casimir forces between distinct bodies, the issue of how divergent and finite Casimir energies couple to gravity remains unclear.

In the last few years there have been several calculations of the gravitational acceleration imparted to the Casimir energy associated with a pair of perfectly conducting plates [5–9]. The results were inconsistent, and there was no consensus that the gravitational force agreed with the equivalence principle. Recently, we have shown that indeed the gravitational force on a Casimir apparatus is exactly that required by the equivalence principle, that is, that the gravitational mass of the Casimir energy is just the Casimir energy itself [10]. Other authors now agree with our conclusion [11]. However, that calculation included only the finite Casimir energy of the two plates, so the question of what happens to the divergent contributions remains unanswered.

Here we answer that question. We describe a uniformly accelerated system by Rindler coordinates [12], which naturally represent frames undergoing hyperbolic motion. We consider both a single plate, and two plates, both represented by δ-function potentials, what are sometimes called semitransparent plates. In Minkowski space, the Casimir energies for such systems have been considered by many authors [3, 13–15]. Saharian et al. [16] considered Dirichlet, Neumann, and perfectly conducting plates in Rindler coordinates, and showed for rigid acceleration of those plates, in the limit of large Rindler coordinate, which corresponds to the weak gravitational field limit, that the finite Casimir energy undergoes the normal acceleration. We carry out that calculation here for semitransparent plates (which reduce to Dirichlet plates in the strong-coupling limit) and find that both for a single plate, and for two parallel plates (both orthogonal to the Rindler spatial coordinate) both the divergent and finite parts of the Casimir energy gravitate according to the equivalence principle, and that the divergent energies serve to renormalize the inertial and gravitational masses of each separate plate.[1]

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[1] Acceleration of semitransparent plates has been considered by many authors in connection with quantum radiation [17–21].

II. GREEN'S FUNCTIONS IN RINDLER COORDINATES

Relativistically, uniform acceleration is described by hyperbolic motion,

$$z = \xi \cosh \tau \quad \text{and} \quad t = \xi \sinh \tau. \tag{2.1}$$

Here the proper acceleration of the particle described by these equations is ξ^{-1}, and we have chosen coordinates so that at time $t = 0$, $z(0) = \xi$. Here we are going to consider the corresponding metric

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 = -\xi^2 d\tau^2 + d\xi^2 + dx^2 + dy^2. \tag{2.2}$$

In these coordinates, the d'Alembertian operator takes on cylindrical form

$$-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 + \boldsymbol{\nabla}_\perp^2 = -\frac{1}{\xi^2}\left(\frac{\partial}{\partial \tau}\right)^2 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\right) + \boldsymbol{\nabla}_\perp^2, \tag{2.3}$$

where $\perp$ refers to the x-y plane.

A. Green's function for one plate

For a scalar field in these coordinates, subject to a potential $V(x)$, the action is

$$W = \int d^4x \sqrt{-g(x)} \mathcal{L}(\phi(x)), \tag{2.4}$$

where $x \equiv (\tau, x, y, \xi)$ represents the coordinates, $d^4x = d\tau\, d\xi\, dx\, dy$ is the coordinate volume element, $g_{\mu\nu}(x) = \text{diag}(-\xi^2, +1, +1, +1)$ defines the metric, $g(x) = \det g_{\mu\nu}(x) = -\xi^2$ is the determinant of the metric, and the Lagrangian density is

$$\mathcal{L}(\phi(x)) = -\frac{1}{2} g_{\mu\nu}(x) \partial^\mu \phi(x) \partial^\nu \phi(x) - \frac{1}{2} V(x) \phi(x)^2, \tag{2.5}$$

where for a single semitransparent plate located at ξ_1

$$V(x) = \lambda \delta(\xi - \xi_1), \tag{2.6}$$

and $\lambda > 0$ is the coupling constant having dimensions of mass. More explicitly we have

$$W = \int d^4x \, \frac{\xi}{2} \left[\frac{1}{\xi^2} \left(\frac{\partial \phi}{\partial \tau}\right)^2 - \left(\frac{\partial \phi}{\partial \xi}\right)^2 - (\boldsymbol{\nabla}_\perp \phi)^2 - V(x)\phi^2 \right]. \tag{2.7}$$

Stationarity of the action under an arbitrary variation in the field leads to the equation of motion

$$\left[-\frac{1}{\xi^2} \frac{\partial^2}{\partial \tau^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} + \boldsymbol{\nabla}_\perp^2 - V(x) \right] \phi(x) = 0. \tag{2.8}$$

The corresponding Green's function satisfies the differential equation

$$-\left[-\frac{1}{\xi^2} \frac{\partial^2}{\partial \tau^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} + \boldsymbol{\nabla}_\perp^2 - V(x) \right] G(x, x') = \frac{\delta(\xi - \xi')}{\xi} \delta(\tau - \tau') \delta(\mathbf{x}_\perp - \mathbf{x}'_\perp). \tag{2.9}$$

Since in our case $V(x)$ has only ξ dependence we can write this in terms of the reduced Green's function $g(\xi, \xi')$,

$$G(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} e^{-i\omega(\tau - \tau')} e^{i\mathbf{k}_\perp \cdot (\mathbf{x} - \mathbf{x}')_\perp} g(\xi, \xi'), \tag{2.10}$$

where $g(\xi, \xi')$ satisfies

$$-\left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} + \frac{\omega^2}{\xi^2} - k_\perp^2 - V(x) \right] g(\xi, \xi') = \frac{\delta(\xi - \xi')}{\xi}. \tag{2.11}$$

We recognize this equation as defining the semitransparent cylinder problem [22], with the replacements

$$m \to \zeta = -i\omega, \quad \kappa \to k = k_\perp, \tag{2.12}$$

so that we may immediately write down the solution in terms of modified Bessel functions,

$$\begin{aligned} g(\xi,\xi') &= I_\zeta(k\xi_<)K_\zeta(k\xi_>) - \frac{\lambda\xi_1 K_\zeta^2(k\xi_1)I_\zeta(k\xi)I_\zeta(k\xi')}{1+\lambda\xi_1 I_\zeta(k\xi_1)K_\zeta(k\xi_1)}, \qquad \xi,\xi' < \xi_1, & \text{(2.13a)}\\ &= I_\zeta(k\xi_<)K_\zeta(k\xi_>) - \frac{\lambda\xi_1 I_\zeta^2(k\xi_1)K_\zeta(k\xi)K_\zeta(k\xi')}{1+\lambda\xi_1 I_\zeta(k\xi_1)K_\zeta(k\xi_1)}, \qquad \xi,\xi' > \xi_1. & \text{(2.13b)}\end{aligned}$$

Note that in the strong-coupling limit, $\lambda \to \infty$, this reduces to the Green's function satisfying Dirichlet boundary conditions at $\xi = \xi_1$.

B. Minkowski-space limit

To recover the Minkowski-space Green's function for the semitransparent plate, we use the uniform asymptotic expansion (Debye expansion), based on the limit

$$\xi \to \infty, \qquad \xi_1 \to \infty, \quad \xi - \xi_1 \text{ finite }, \quad \zeta \to \infty, \quad \zeta/\xi_1 \text{ finite }. \tag{2.14}$$

For large ζ

$$I_\zeta(\zeta z) \sim \sqrt{\frac{t}{2\pi\zeta}}\, e^{\zeta\eta(z)} \sum_{n=0}^\infty \frac{1}{\zeta^n} u_n(t), \quad K_\zeta(\zeta z) \sim \sqrt{\frac{\pi t}{2\zeta}}\, e^{-\zeta\eta(z)} \sum_{n=0}^\infty \frac{(-1)^n}{\zeta^n} u_n(t), \tag{2.15}$$

where

$$t = \frac{1}{\sqrt{1+z^2}} \quad \text{and} \quad \eta(z) = \sqrt{1+z^2} + \ln\left[\frac{z}{1+\sqrt{1+z^2}}\right], \tag{2.16}$$

and $u_n(t)$ are polynomials of order $3n$ in t [23]. Here $z\zeta = k\xi$, for example. Expanding the above expressions around some arbitrary point ξ_0, chosen such that the differences $\xi - \xi_0$, $\xi' - \xi_0$, and $\xi_1 - \xi_0$ are finite, we find for the leading term, for example,

$$\sqrt{\xi\xi'}\, I_\zeta(k\xi)K_\zeta(k\xi') \sim \frac{1}{2\kappa}\, e^{\kappa(\xi-\xi')}, \tag{2.17}$$

where $\kappa^2 = k^2 + \hat\zeta^2$, $\hat\zeta = \zeta/\xi_0$. In this way, taking for simplicity $\xi_0 = \xi_1$, we find the Green's function for a single plate in Minkowski space,

$$\xi_1 g(\xi,\xi') \to g^{(0)}(\xi,\xi') = \frac{1}{2\kappa}\, e^{-\kappa|\xi-\xi'|} - \frac{\lambda}{\lambda+2\kappa}\frac{1}{2\kappa}\, e^{-\kappa|\xi-\xi_1|} e^{-\kappa|\xi'-\xi_1|}. \tag{2.18}$$

C. Green's function for two parallel plates

For two semitransparent plates perpendicular to the ξ-axis and located at ξ_1, ξ_2, with couplings λ_1 and λ_2, respectively, we find the following form for the Green's function:

$$\begin{aligned} g(\xi,\xi') &= I_<K_> - \frac{\lambda_1\xi_1 K_1^2 + \lambda_2\xi_2 K_2^2 - \lambda_1\lambda_2\xi_1\xi_2 K_1K_2(K_2I_1 - K_1I_2)}{\Delta} II\prime, \quad \xi,\xi' < \xi_1, & \text{(2.19a)}\\ &= I_<K_> - \frac{\lambda_1\xi_1 I_1^2 + \lambda_2\xi_2 I_2^2 + \lambda_1\lambda_2\xi_1\xi_2 I_1I_2(I_2K_1 - I_1K_2)}{\Delta} KK\prime, \quad \xi,\xi' > \xi_2, & \text{(2.19b)}\\ &= I_<K_> - \frac{\lambda_2\xi_2 K_2^2(1+\lambda_1\xi_1K_1I_1)}{\Delta} II\prime \\ &\quad - \frac{\lambda_1\xi_1 I_1^2(1+\lambda_2\xi_2K_2I_2)}{\Delta} KK\prime + \frac{\lambda_1\lambda_2\xi_1\xi_2 I_1^2K_2^2}{\Delta}(IK\prime + KI\prime), \quad \xi_1 < \xi,\xi' < \xi_2, & \text{(2.19c)}\end{aligned}$$

where

$$\Delta = (1+\lambda_1\xi_1 K_1 I_1)(1+\lambda_2\xi_2 K_2 I_2) - \lambda_1\lambda_2\xi_1\xi_2 I_1^2 K_2^2, \tag{2.20}$$

and we have used the abbreviations $I_1 = I_\zeta(k\xi_1)$, $I = I_\zeta(k\xi)$, $I_\prime = I_\zeta(k\xi')$, etc.

Again we can check that these formulas reduce to the well-known Minkowski-space limits. In the $\xi_0 \to \infty$ limit, the uniform asymptotic expansion (2.15) gives, for $\xi_1 < \xi, \xi' < \xi_2$

$$\begin{aligned}\xi_0 g(\xi,\xi') \to g^{(0)}(\xi,\xi') \;=\; & \frac{1}{2\kappa}e^{-\kappa|\xi-\xi'|} + \frac{1}{2\kappa\tilde\Delta}\left[\frac{\lambda_1\lambda_2}{4\kappa^2}2\cosh\kappa(\xi-\xi')\right.\\ & \left. -\frac{\lambda_1}{2\kappa}\left(1+\frac{\lambda_2}{2\kappa}\right)e^{-\kappa(\xi+\xi'-2\xi_2)} - \frac{\lambda_2}{2\kappa}\left(1+\frac{\lambda_1}{2\kappa}\right)e^{\kappa(\xi+\xi'-2\xi_1)}\right],\end{aligned} \tag{2.21}$$

where $(a = \xi_2 - \xi_1)$

$$\tilde\Delta = \left(1+\frac{\lambda_1}{2\kappa}\right)\left(1+\frac{\lambda_2}{2\kappa}\right)e^{2\kappa a} - \frac{\lambda_1\lambda_2}{4\kappa^2}, \tag{2.22}$$

which is exactly the expected result [14]. The correct limit is also obtained in the other two regions.

III. GRAVITATIONAL ACCELERATION OF CASIMIR APPARATUS

We next consider the situation when the plates are forced to "move rigidly" [24] in such a way that the proper distance between the plates is preserved. This is achieved if the two plates move with different but constant proper accelerations.

The canonical energy-momentum or stress tensor derived from the action (2.4) is

$$T_{\alpha\beta}(x) = \partial_\alpha\phi(x)\partial_\beta\phi(x) + g_{\alpha\beta}(x)\mathcal{L}(\phi(x)), \tag{3.1}$$

where the Lagrange density includes the δ-function potential. The components referring to the pressure and the energy density are

$$T_{33}(x) \;=\; \frac{1}{2}\frac{1}{\xi^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla}_\perp\phi\right)^2 - \frac{1}{2}V(x)\phi^2 \tag{3.2a}$$

$$\frac{1}{\xi^2}T_{00}(x) \;=\; \frac{1}{2}\frac{1}{\xi^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2 + \frac{1}{2}\left(\boldsymbol{\nabla}_\perp\phi\right)^2 + \frac{1}{2}V(x)\phi^2. \tag{3.2b}$$

The latter may be written in an alternative convenient form using the equations of motion (2.8):

$$T_{00} = \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\phi\frac{\partial^2}{\partial\tau^2}\phi + \frac{\xi}{2}\frac{\partial}{\partial\xi}\left(\phi\xi\frac{\partial}{\partial\xi}\phi\right) + \frac{\xi^2}{2}\boldsymbol{\nabla}_\perp\cdot(\phi\boldsymbol{\nabla}_\perp\phi). \tag{3.3}$$

The force density is given by

$$f_\lambda = -\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}T^\nu{}_\lambda) + \frac{1}{2}T^{\mu\nu}\partial_\lambda g_{\mu\nu}, \tag{3.4}$$

or in Rindler coordinates

$$f_\xi = -\frac{1}{\xi}\partial_\xi(\xi T^{\xi\xi}) - \xi T^{00}. \tag{3.5}$$

When we integrate over all space to get the force, the first term is a surface term which does not contribute:[2]

$$\mathcal{F} = \int d\xi\,\xi f_\xi = -\int\frac{d\xi}{\xi^2}T_{00}. \tag{3.6}$$

[2] Note that in previous works, such as Refs. [14, 15], the surface term was included, because the integration was carried out only over the interior and exterior regions. Here we integrate over the surface as well, so the additional so-called surface energy is automatically included.

This could be termed the Rindler coordinate force per area, defined as the change in momentum per unit Rindler coordinate time τ per unit cross-sectional area. If we multiply $\mathcal{F}$ by the gravitational acceleration g we obtain the gravitational force per area on the Casimir energy. This result (3.6) seems entirely consistent with the equivalence principle, since $\xi^{-2}T_{00}$ is the energy density. Using the expression (3.3) for the energy density, taking the vacuum expectation value, and rescaling $\zeta = \hat{\zeta}\xi$, we see that the gravitational force per cross sectional area is merely

$$\mathcal{F} = \int d\xi\, \xi \int \frac{d\hat{\zeta}\, d^2\mathbf{k}}{(2\pi)^3} \hat{\zeta}^2 g(\xi, \xi). \tag{3.7}$$

This result for the energy contained in the force equation (3.7) is an immediate consequence of the general formula for the Casimir energy [25]

$$E_c = -\frac{1}{2i} \int (d\mathbf{r}) \int \frac{d\omega}{2\pi} 2\omega^2 \mathcal{G}(\mathbf{r}, \mathbf{r}), \tag{3.8}$$

in terms of the frequency transform of the Green's function,

$$G(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \mathcal{G}(\mathbf{r}, \mathbf{r}'). \tag{3.9}$$

Alternatively, we can start from the following formula for the force density for a single semitransparent plate, following directly from the equations of motion (2.8),

$$f_\xi = \frac{1}{2}\phi^2 \partial_\xi \lambda \delta(\xi - \xi_1). \tag{3.10}$$

The vacuum expectation value of this yields the force in terms of the Green's function,

$$\mathcal{F} = -\lambda \frac{1}{2} \int \frac{d\zeta\, d^2\mathbf{k}}{(2\pi)^3} \partial_{\xi_1} [\xi_1 g(\xi_1, \xi_1)]. \tag{3.11}$$

A. Gravitational force on a single plate

For example, the force on a single plate at ξ_1 is given by

$$\mathcal{F} = -\partial_{\xi_1} \frac{1}{2} \int \frac{d\zeta\, d^2\mathbf{k}}{(2\pi)^3} \ln[1 + \lambda \xi_1 I_\zeta(k\xi_1) K_\zeta(k\xi_1)], \tag{3.12}$$

Expanding this about some arbitrary point ξ_0, with $\zeta = \hat{\zeta}\xi_0$, using the uniform asymptotic expansion (2.15), we get

$$\xi_1 I_\zeta(k\xi_1) K_\zeta(k\xi_1) \sim \frac{\xi_1}{2\zeta} \frac{1}{\sqrt{1 + (k\xi_1/\zeta)^2}} \approx \frac{\xi_1}{2\kappa\xi_0} \left(1 - \frac{k^2}{\kappa^2} \frac{\xi_1 - \xi_0}{\xi_0}\right). \tag{3.13}$$

From this, if we introduce polar coordinates for the $\mathbf{k}$-$\hat{\zeta}$ integration ($\kappa^2 = k^2 + \hat{\zeta}^2$), the coordinate force is

$$\begin{aligned}
\mathcal{F} &= -\frac{1}{2}\partial_{\xi_1} \frac{\xi_0}{2\pi^2} \int_0^\infty d\kappa\, \kappa^2 \frac{\lambda}{2\kappa + \lambda} \left(1 + \frac{\xi_1 - \xi_0}{\xi_0}\right) \left(1 - \frac{\langle k^2 \rangle}{\kappa^2} \frac{\xi_1 - \xi_0}{\xi_0}\right) \\
&= -\frac{\lambda}{4\pi^2} \partial_{\xi_1} (\xi_1 - \xi_0) \int_0^\infty \frac{d\kappa}{2\kappa + \lambda} \langle \hat{\zeta}^2 \rangle \\
&= -\frac{1}{96\pi^2 a^3} \int_0^\infty \frac{dy\, y^2}{1 + y/\lambda a},
\end{aligned} \tag{3.14}$$

where for example

$$\langle \hat{\zeta}^2 \rangle = \frac{1}{2} \int_{-1}^1 d\cos\theta\, \cos^2\theta\, \kappa^2 = \frac{1}{3}\kappa^2. \tag{3.15}$$

The divergent expression (3.14) is just the negative of the quantum vacuum energy of a single plate.

B. Parallel plates falling in a constant gravitational field

In general, we have two alternative forms for the gravitational force on the two-plate system:

$$\mathcal{F} = -(\partial_{\xi_1} + \partial_{\xi_2})\frac{1}{2}\int \frac{d\zeta\, d^2\mathbf{k}}{(2\pi)^3}\ln\Delta, \tag{3.16}$$

Δ given in Eq. (2.20), which is equivalent to (3.7). (In the latter, however, bulk energy, present if no plates are present, must be omitted.) From either of the above two methods, we find the coordinate force is given by

$$\mathcal{F} = -\frac{1}{4\pi^2}\int_0^\infty d\kappa\,\kappa^2\ln\Delta_0, \tag{3.17}$$

where $\Delta_0 = e^{-2\kappa a}\tilde{\Delta}$, $\tilde{\Delta}$ given in Eq. (2.22). The integral may be easily shown to be

$$\begin{aligned}\mathcal{F} &= \frac{1}{96\pi^2 a^3}\int_0^\infty dy\, y^3 \frac{1+\frac{1}{y+\lambda_1 a}+\frac{1}{y+\lambda_2 a}}{\left(\frac{y}{\lambda_1 a}+1\right)\left(\frac{y}{\lambda_2 a}+1\right)e^y - 1} - \frac{1}{96\pi^2 a^3}\int_0^\infty dy\, y^2\left[\frac{1}{\frac{y}{\lambda_1 a}+1}+\frac{1}{\frac{y}{\lambda_2 a}+1}\right] &\text{(3.18a)}\\ &= -(\mathcal{E}_c + \mathcal{E}_{d1} + \mathcal{E}_{d2}), &\text{(3.18b)}\end{aligned}$$

which is just the negative of the Casimir energy of the two semitransparent plates including the divergent pieces [14, 15]. Note that $\mathcal{E}_{di}$, $i = 1, 2$, are simply the divergent energies (3.14) associated with a single plate.

C. Renormalization

The divergent terms in Eq. (3.18) simply renormalize the masses (per unit area) of each plate:

$$\begin{aligned}E_{\text{total}} &= m_1 + m_2 + \mathcal{E}_{d1} + \mathcal{E}_{d2} + \mathcal{E}_c \\ &= M_1 + M_2 + \mathcal{E}_c,\end{aligned} \tag{3.19}$$

where m_i is the bare mass of each plate, and the renormalized mass is $M_i = m_i + \mathcal{E}_{di}$. Thus the gravitational force on the entire apparatus obeys the equivalence principle

$$g\mathcal{F} = -g(M_1 + M_2 + \mathcal{E}_c). \tag{3.20}$$

The minus sign reflects the downward acceleration of gravity on the surface of the earth. Note here that the Casimir interaction energy $\mathcal{E}_c$ is negative, so it reduces the gravitational attraction of the system.

IV. CONCLUSIONS

We have found, in conformation with the result given in Ref. [10], an extremely simple answer to the question of how Casimir energy accelerates in a weak gravitational field: Just like any other form of energy, the gravitational force F divided by the area of the plates is

$$\frac{F}{A} = -g\mathcal{E}_c. \tag{4.1}$$

This is the result expected by the equivalence principle, but is in contradiction to some earlier disparate claims in the literature [5–9]. This result exactly agrees with that found by Saharian et al. [16] for Dirichlet, Neumann, and perfectly conducting plates for the finite Casimir interaction energy. The acceleration of Dirichlet plates follows from our result when the strong coupling limit $\lambda \to \infty$ is taken. What makes our conclusion particularly interesting is that it refers not only to the finite part of the Casimir interaction energy between semitransparent plates, but to the divergent parts as well, which are seen to simply renormalize the gravitational mass of each plate, as they would the inertial mass. The reader may object that by equating gravitational force with uniform acceleration we have built in the equivalence principle, and so does any procedure based on Einstein's equations; but the real nontriviality here is that quantum fluctuations obey the same universal law.

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