$\mathcal{PT}$-Symmetric Quantum Field Theory

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In the context of the $\mathcal{PT}$-symmetric version of quantum electrodynamics, it is argued that the $\mathcal{C}$ operator introduced in order to define a unitary inner product has nothing to do with charge conjugation.

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1 Introduction

A new approach to quantum theories was proposed in 1996 in the context of the so-called δ-expansion [1]. The idea was to study non-Hermitian theories which nevertheless possessed positive spectra. Parity symmetry $\mathcal{P}$ was broken in these theories, as well as time-reversal invariance $\mathcal{T}$. However, $\mathcal{PT}$ symmetry was unbroken. In early papers we examined scalar field theories with interaction [1]

$$\mathcal{L}_{\rm int} = -g(\mathrm{i}\phi)^N, \tag{1}$$

supersymmetric theories possessing the superpotential [2]

$$\mathcal{S} = -\mathrm{i}g(\mathrm{i}\phi)^N, \tag{2}$$

and massless electrodynamics with an axial vector current [3]

$$j_5^\mu = e\frac{1}{2}\psi\gamma^0\gamma^\mu\gamma^5\psi, \tag{3}$$

as well as studied the Schwinger-Dyson equations for such theories as [4]

$$\mathcal{L}_{\rm int} = -g\phi^4. \tag{4}$$

Many remarkable features were discovered in such investigations, such as perturbative parity violation, stability of eigenvalue conditions for coupling constants, and asymptotic freedom. The idea is essentially nonperturbative in concept, because the path integrals defining the theory, in general, must be defined as nontrivial contours in the complex plane.

However, most of what is known about $\mathcal{PT}$-symmetric theories comes from examples in quantum mechanics, that is, $d = 1$ quantum field theory. For example, it has been proved that the spectrum of

$$H = p^2 + x^2(\mathrm{i}x)^\nu, \quad \nu \geq 0, \tag{5}$$

is real and positive [5].

2 Unitarity

The most troubling aspect of $\mathcal{PT}$-symmetric theories has been that of the apparent violation of unitarity. In a remarkable development last year it was discovered how to define a unitary norm, at least for quantum mechanical theories like that described by the Hamiltonian (5). Let ϕ_n by the nth eigenfunction of H,

$$H\phi_n(x) = E_n\phi_n(x), \tag{6}$$

which is a differential equation imposed on a complex contour C. The eigenfunctions can be chosen to have eigenvalue 1 of the $\mathcal{PT}$ operator:

$$\mathcal{PT}\phi_n(x) = \phi_n(x). \tag{7}$$

The eigenfunctions are complete in the sense that

$$\sum_n (-1)^n\phi_n(x)\phi_n(y) = \delta(x-y), \tag{8}$$

which means that the $\mathcal{PT}$ inner product,

$$(f,g) \equiv \int_C \mathrm{d}x\,[\mathcal{PT}f(x)]g(x), \quad \mathcal{PT}f(x) = f^*(-x), \tag{9}$$

(the complex conjugation is to be applied to quantities in the functional form, not to the coordinate x) defines a metric which is not definite:

$$(\phi_n, \phi_m) = (-1)^n\delta_{mn}. \tag{10}$$

Because of the severe interpretational issues associated with an indefinite metric, it is fortunate that last year Bender, Brody, and Jones [6] discovered how to define a positive metric. In terms of the eigenfunctions, they defined a (dynamical) $\mathcal{C}$ operator:

$$\mathcal{C}(x,y) = \sum_n \phi_n(x)\phi_n(y), \tag{11}$$

which has square unity:

$$\int \mathrm{d}y\,\mathcal{C}(x,y)\mathcal{C}(y,z) = \delta(x-z), \tag{12}$$

but is distinct from the parity operator:

$$\mathcal{P}(x,y) = \delta(x+y) = \sum_n (-1)^n\phi_n(x)\phi_n(-y), \tag{13}$$

The positive-definite inner product is now defined by

$$\langle f|g\rangle = \int_C \mathrm{d}x\,[\mathcal{CPT}f(x)]g(x). \tag{14}$$

This defines a nontrivial extension of quantum mechanics. Of course, the physical interpretation of $\mathcal{C}$ is far from clear.

3 Electrodynamics

The notation suggests that $\mathcal{C}$ is some sort of charge-conjugation operator. The place to examine such an idea would seem to be the fermion sector, since it was from the Dirac equation that the concept of antiparticles emerged. Consider the massless Dirac Lagrangian, written in term of the Majorana representation:

$$\mathcal{L} = -\frac{1}{2}\psi\gamma^0\gamma^\mu\frac{1}{\mathrm{i}}\partial_\mu\psi. \tag{15}$$

The symmetry of $\gamma^0\gamma^\mu$, combined with the antisymmetry of the derivative operator, requires that the Dirac field ψ be a Grassmann variable.

There are thus two ways to introduce interactions. In either one, one starts from a global transformation that leaves the Dirac Lagrangian invariant:

$$\psi \to \mathrm{e}^{\mathrm{i}\theta\lambda}\psi, \tag{16}$$

where λ is a constant, and θ is an antisymmetric matrix that commutes with $\gamma^0\gamma^\mu$.

- The first choice is $\theta = eq$,

  $$q = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{17}$$

  the antisymmetrical charge matrix, living in an independent two-dimensional space. When λ is now promoted to a space-time dependent function, we are led to the usual current of electrodynamics:

  $$j^\mu = \frac{1}{2}\psi\gamma^0\gamma^\mu eq\psi, \tag{18}$$

  because the gauge transformation of A_μ, $A_\mu \to A_\mu + \partial_\mu\lambda$ in the interaction $j^\mu A_\mu$ cancels that of the free Lagrangian. It is the additional two-fold multiplicity of the Dirac field that corresponds to the presence of antiparticles.

  Schwinger sharpened this argument. He insisted on the Euclidean postulate—that the extension of the theory to Euclidean space bear no memory of the original timelike direction—and showed therefore that every spin 1/2 particle must have a chargelike attribute [7].

- The second possibility is that $\theta = e\gamma^5$; this leads to the axial-current interaction $j_5^\mu A_\mu$, with

  $$j_5^\mu = \frac{1}{2}\psi\gamma^0\gamma^\mu e\gamma^5\psi. \tag{19}$$

  For a massless theory, such a current is conserved, barring anomalies, so we still have a consistent theory. This is the $\mathcal{PT}$-symmetric QED referred to above. However, there is now no additional two-fold multiplicity of the Dirac field, and therefore, apparently, no antiparticles. We therefore suspect that referring to the dynamical $\mathcal{C}$ operator as charge conjugation may be misleading.

4 Discussion

It is not yet clear if there exists a consistent treatment of fermions in the $\mathcal{PT}$-symmetric framework. The exploration of the Dirac equation, both at the classical and second-quantized level, promises to shed light on the connection of charge-conjugation, the $\mathcal{C}$ operator, and unitarity in this exciting extension of quantum mechanical ideas. I hope that this summary will provoke substantial developments. Further developments will be indicated in my contribution to the Workshop on Pseudo-Hermitian Hamiltonians.

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