Gravitational and Inertial Mass of Casimir Energy

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Abstract. It has been demonstrated, using variational methods, that quantum vacuum energy gravitates according to the equivalence principle, at least for the finite Casimir energies associated with perfectly conducting parallel plates. This conclusion holds independently of the orientation of the plates. We review these arguments and add further support to this conclusion by considering parallel semitransparent plates, that is, δ-function potentials, acting on a massless scalar field, in a spacetime defined by Rindler coordinates. We calculate the force on systems consisting of one or two such plates undergoing acceleration perpendicular to the plates. In the limit of small acceleration we recover (via the equivalence principle) the situation of weak gravity, and find that the gravitational force on the system is just $M\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration and M is the total mass of the system, consisting of the mass of the plates renormalized by the Casimir energy of each plate separately, plus the energy of the Casimir interaction between the plates. This reproduces the previous result in the limit as the coupling to the δ-function potential approaches infinity. Extension of this latter work to arbitrary orientation of the plates, and to general compact quantum vacuum energy configurations, is under development.

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1. Introduction

The subject of Quantum Vacuum Energy (the Casimir effect) dates from the same year as the discovery of renormalized quantum electrodynamics, 1948, and suggests that the assertion that zero-point energy is not observable is invalid. (For a contrary viewpoint see Ref. [1].) On the other hand, because of the severe divergence structure of the theory, controversy has surrounded it from the beginning. Sharp boundaries give rise to divergences in the local energy density near the surface, which may make it impossible to extract meaningful self-energies of single objects, such as the perfectly conducting sphere considered by Boyer [2]. These objections have recently been most forcefully presented by Graham, et al. [3] and Barton [4], but they date back to Deutsch and

Candelas [5, 6]. In fact, it now appears that these surface divergences can be dealt with successfully in a process of renormalization, and that finite self-energies in the sense of Boyer, may be extracted [7, 8].

But the most troubling aspect of local energy divergences is in the coupling to gravity. The source of gravity is the local energy-momentum tensor, and such surface divergences promise serious difficulties. As a prolegomenon to studying such questions, we here address in section 2 a simpler question: How does the completely finite Casimir energy of a pair of parallel conducting plates respond to gravity? (We'll address divergences in section 3.) The question, and its answer, turn out to be surprisingly less straightforward than the reader might suspect! (For a complementary view on the gravitational effects of Casimir energy, see the contribution to these Proceedings by S A Fulling et al.)

2. Variational method

2.1. Casimir stress tensor for parallel plates

Brown and Maclay [9] showed that, for parallel perfectly conducting plates separated by a distance a in the z-direction, the electromagnetic stress tensor acquires the vacuum expectation value between the plates

$$\langle T^{\mu\nu}\rangle = \frac{\mathcal{E}_c}{a}\mathrm{diag}(1,-1,-1,3), \quad \mathcal{E}_c = -\frac{\pi^2}{720a^3}\hbar c. \tag{1}$$

Outside the plates the value of $\langle T^{\mu\nu}\rangle = 0$. Because there are some subtleties here, let us review the argument for the case of a conformally coupled scalar (the electromagnetic case differs by a factor of two). Actually, the result between the plates, $0 < z < a$ is given in great detail in Ref. [10] (γ is the conformal parameter):

$$\langle T^{\mu\nu}\rangle = (u_0 + u)\mathrm{diag}(1,-1,-1,3) + (1-6\gamma)g(z)\mathrm{diag}(1,-1,-1,0), \tag{2}$$

where

$$u_0 = -\frac{1}{12\pi^2}\int_0^\infty \mathrm{d}\kappa\,\kappa^3, \quad u = -\frac{\pi^2}{1440a^4}. \tag{3}$$

Note that u_0 is a divergent constant, independent of a, and is present (as we shall see) both inside and outside the plates, so it does not contribute to any observable force or energy (the force on the plates is given by the discontinuity of $\langle T_{zz}\rangle$), and so may be simply disregarded (as long as we are not concerned with dark energy). But see below! Similarly, the term involving the Hurwitz zeta function,

$$g(z) = -\frac{1}{16\pi^2a^4}[\zeta(4,z/a) + \zeta(4,1-z/a)], \tag{4}$$

which exhibits the universal surface divergence near the plates,

$$g(z) \sim -\frac{1}{16\pi^2 z^4}, \quad z \to 0+, \tag{5}$$

is also unobservable (if we disregard gravity) because it does not contribute to the force on the plates, nor does it contribute to the total energy, since the integral over $g(z)$

between the plates is independent of the plate separation. Of course, the best way to eliminate that term is to choose the conformal value $\gamma = 1/6$.

Since the exterior calculation does not appear to be referred to in Ref. [10], let us sketch the calculation here: Consider parallel Dirichlet plates at $z = 0$ and $z = a$. The reduced Green's function satisfies

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \kappa^2\right) g(z, z') = \delta(z - z'), \tag{6}$$

where $\kappa^2 = k^2 - \omega^2 = k^2 + \zeta^2$. The solution for $z, z' < 0$ is

$$g(z, z') = -\frac{1}{\kappa}\mathrm{e}^{\kappa z_<} \sinh \kappa z_>. \tag{7}$$

It is very straightforward to calculate the one-loop expectation value of the stress tensor from

$$\mathrm{i}\langle T^{\mu\nu}\rangle = \left(\partial^\mu \partial'^\nu - \frac{1}{2} g^{\mu\nu} \partial^\lambda \partial'_\lambda\right) G(x, x')\bigg|_{x'=x} - \gamma(\partial^\mu \partial^\nu - g^{\mu\nu}\partial^2)G(x, x). \tag{8}$$

After integrating over $\omega = \mathrm{i}\zeta$ and $\mathbf{k}$, we find the result ($z < 0$)

$$\langle T^{\mu\nu}\rangle = u_0 \mathrm{diag}(1, -1, -1, 3) - \frac{(1 - 6\gamma)}{16\pi^2|z|^4} \mathrm{diag}(1, -1, -1, 0). \tag{9}$$

This is exactly as expected. The u_0 term is the same as inside the box, so is just the vacuum value, and the second term is the universal surface divergence (independent of plate separation), which can be eliminated by setting $\gamma = 1/6$.

Thus, we conclude that the physical stress tensor VEV is just that found by Brown and Maclay:

$$\langle T^{\mu\nu}\rangle = u\, \mathrm{diag}(1, -1, -1, 3)\theta(z)\theta(a - z). \tag{10}$$

in terms of the usual step function.

2.2. Variational principle

Now we address the question of the gravitational interaction of this Casimir apparatus [11]. It seems this question can be most simply answered through use of the gravitational definition of the energy-momentum tensor,

$$\delta W_m \equiv -\frac{1}{2}\int (\mathrm{d}x)\sqrt{-g}\, \delta g^{\mu\nu} T_{\mu\nu} = \frac{1}{2}\int (\mathrm{d}x)\sqrt{-g}\, \delta g_{\mu\nu} T^{\mu\nu}. \tag{11}$$

For a weak field, $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$ (Schwinger's definition [12] of $h_{\mu\nu}$). So if we think of turning on the gravitational field as the perturbation, we can ignore $\sqrt{-g}$. The gravitational energy, for a static situation, is therefore given by ($\delta W = -\int dt\, \delta E$)

$$E_g = -\int (\mathrm{d}\mathbf{x}) h_{\mu\nu} T^{\mu\nu}. \tag{12}$$

We can use the gravity-free electromagnetic Casimir stress tensor (10), with u now replaced by $\mathcal{E}_c/a$ for the electromagnetic situation.

We now use the metric [13, 14]

$$g_{00} = -\left(1 + 2gz\right), \quad g_{ij} = \delta_{ij}. \tag{13}$$

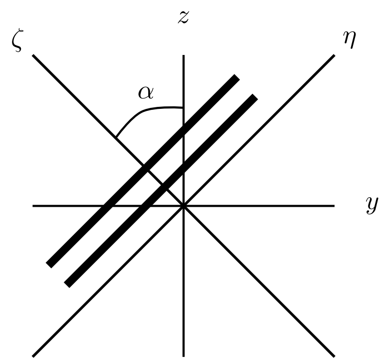

Figure 1. Relation between two Cartesian coordinate frames: One attached to the earth (x, y, z), where $-z$ is the direction of gravity, and one attached to the parallel-plate Casimir apparatus (ζ, η, χ), where ζ is in the direction normal to the plates. The parallel plates are indicated by the heavy lines parallel to the η axis. The $x = \chi$ axis is perpendicular to the page.

This is appropriate for a constant gravitational field. (But see below.) Let us consider a Casimir apparatus of parallel plates separated by a distance a, with transverse dimensions $L \gg a$. Let the apparatus be oriented at an angle α with respect to the direction of gravity, as shown in figure 1. Let us take the Cartesian coordinate system attached to the earth to be (x, y, z), where, as noted above, z is the direction of $-\mathbf{g}$. Let the Cartesian coordinates associated with the Casimir apparatus be (ζ, η, χ), where ζ is normal to the plates, and η and χ are parallel to the plates. The relation between the two sets of coordinates is

$$z = \zeta \cos\alpha + \eta \sin\alpha, \quad y = \eta \cos\alpha - \zeta \sin\alpha, \quad x = \chi. \tag{14}$$

Let the center of the apparatus be located at $(\zeta_0, \eta = 0, \chi = 0)$.

Now we calculate the gravitational energy

$$\begin{aligned} E_g &= \int (\mathrm{d}\mathbf{x}) g z T^{00} = \frac{\mathcal{E}_c}{a} g L \int_{-L/2}^{L/2} \mathrm{d}\eta \int_{\zeta_0 - a/2}^{\zeta_0 + a/2} \mathrm{d}\zeta (\zeta \cos\alpha + \eta \sin\alpha) \\ &= \frac{g\mathcal{E}_c}{a} L^2 \cos\alpha a \zeta_0 + K, \end{aligned} \tag{15}$$

where K is a constant, independent of ζ_0. Thus, the gravitational force per area on the apparatus is independent of orientation

$$\frac{F}{A} = -\frac{\partial E_g}{A \partial z_0} = -\frac{\epsilon}{2a}\mathcal{E}_c = -g\mathcal{E}_c, \quad z_0 = \zeta_0 \cos\alpha, \tag{16}$$

a small upward push. Here $\epsilon = 2ga$ is a measure of the gravitational force relative to the Casimir force. Note that on the earth's surface, the dimensionless number ϵ is very small. For a plate separation of 1μm,

$$\epsilon = \frac{2ga}{c^2} = 2.2 \times 10^{-22}, \tag{17}$$

so the considerations here would appear to be only of theoretical interest. The effect is far smaller than the Casimir forces between the plates.

It is somewhat simpler to use the energy formula to calculate the force by considering the variation in the gravitational energy directly, as we can illustrate by considering a mass point at the origin:

$$T^{\mu\nu} = m\delta(\mathbf{r})\delta^{\mu 0}\delta^{\nu 0}. \tag{18}$$

If we displace the particle rigidly upward by an amount δz_0, the change in the metric is $\delta h_{00} = -g\delta z_0$. This implies a change in the energy, exactly as expected:

$$\delta E_g = -mc^2\left(-g\delta z_0\right) = mg\delta z_0. \tag{19}$$

Now we repeat this calculation for the Casimir apparatus. The gravitational force per area on the rigid apparatus is $\frac{F}{A} = -\frac{\delta\mathcal{E}_g}{\delta z_0} = -g\mathcal{E}_c$, again the same result found in (16), which agrees with the second result found by Calloni et al. [13] but is 1/4 that found by Bimonte et al. [14], who reproduce the first result of Ref. [13]. Our result is consistent with the principle of equivalence, and with one result of Jaekel and Reynaud [15].

2.3. Alternative calculation

As in electrodynamics, we should be able to proceed, starting from the definition of the field

$$\delta W = \int (\mathrm{d}x)\delta T^{\mu\nu}h_{\mu\nu}. \tag{20}$$

Again, check this for the force on a mass point, with stress tensor given by (18), so

$$\delta E_g = -\int (\mathrm{d}\mathbf{r})m\left(-\delta\mathbf{r}\cdot\boldsymbol{\nabla}\right)\delta(\mathbf{r})h^{00} = -m\delta\mathbf{r}\cdot\boldsymbol{\nabla}h^{00}. \tag{21}$$

Since $h^{00} = -gz$, we conclude $-\frac{\delta E}{\delta\mathbf{r}} = \mathbf{F} = -mg\hat{z}$.

For the constant field the force on a Casimir apparatus is obtained from the change in the energy density

$$T^{00} = \frac{\mathcal{E}_c}{a}\theta(a/2-\zeta+\zeta_0)\theta(\zeta-\zeta_0+a/2), \tag{22}$$

that is, recalling that $z_0 = \zeta_0\cos\alpha$,

$$\delta T^{00} = \frac{\mathcal{E}_c}{a}\delta z_0\frac{1}{\cos\alpha}\left[\delta(\zeta-\zeta_0-a/2)-\delta(\zeta-\zeta_0+a/2)\right], \tag{23}$$

which yields a result identical to (16) $[h^{00} = -g(\zeta\cos\alpha+\eta\sin\alpha)]$

$$-\frac{\delta E_g}{A\delta z_0} = \frac{F}{A} = \frac{\mathcal{E}_c}{a}\frac{1}{\cos\alpha}h^{00}\Big|_{\zeta=\zeta_0-a/2}^{\zeta=\zeta_0+a/2} = -g\mathcal{E}_c. \tag{24}$$

2.4. Metric near the surface of the earth

However, the above metric (13), while sufficing for massive Newtonian objects, might seem inappropriate for photons. Rather, shouldn't we use the perturbation of the Schwarzschild metric, which for weak fields ($GM/r \ll 1$) is in isotropic coordinates [16]:

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}\right)c^2\mathrm{d}t^2 + \left(1+\frac{2GM}{r}\right)\mathrm{d}\mathbf{r}^2? \tag{25}$$

If we expand this a short distance z above the earth's surface, of radius R, we find

$$g_{00} \approx -\left(1 - \frac{2GM}{R} + 2gz\right), \quad g_{ij} \approx \delta_{ij}\left(1 + \frac{2GM}{R} - 2gz\right). \tag{26}$$

Now, for our Casimir apparatus shown in figure 1, each component of the Casimir stress tensor contributes with equal weight:

$$-\frac{\delta E_g}{A\delta z_0} = -ga\left(T^{00} + T^{11} + T^{22} + T^{33}\right) = -2g\mathcal{E}_c, \tag{27}$$

since $T = T^\lambda{}_\lambda = 0$, which is twice the previous result. Note that again the result is independent of α. If instead, we use the second method we have

$$\delta T^{\mu\nu} = -\delta z\frac{\mathcal{E}_c}{a}(1,-1,-1,3)\frac{1}{\cos\alpha}\left[\delta\left(\zeta - \zeta_0 + \frac{a}{2}\right) - \delta\left(\zeta - \zeta_0 - \frac{a}{2}\right)\right], \tag{28}$$

so, again we get the same result:

$$-\frac{\delta E_g}{\delta z_0} = -\frac{\mathcal{E}_c}{a}\int\frac{(\mathrm{d}\mathbf{r})}{\cos\alpha}\left[\delta\left(\zeta - \zeta_0 + \frac{a}{2}\right) - \delta\left(\zeta - \zeta_0 - \frac{a}{2}\right)\right](-2gz) = F = -A\,2g\mathcal{E}_c, \tag{29}$$

where $z = \zeta\cos\alpha + \eta\sin\alpha$.

We might think we would be able to obtain the same result using the original Schwarzschild coordinates, where $h_{00} = -gz, \quad h_{\rho\rho} = -gz$, and all other components of $h_{\mu\nu}$ are zero. However, now if we use the first method above, the result is proportional to $T^{00} + T^{\rho\rho} = \frac{\mathcal{E}_c}{a}4\cos^2\alpha$, which implies (fortuitously) Bimonte et al.'s earlier result [14] for $\alpha = 0$: $\frac{F}{A} = -4g\mathcal{E}_c\cos^2\alpha$.

2.5. Gauge noninvariance

The reason we get different answers in different coordinate systems is that our starting point is not gauge invariant. Under a coordinate redefinition, which for weak fields is a gauge transformation of $h_{\mu\nu}$ [12], $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, where ξ_μ is a vector field, the interaction W is invariant only if the stress tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$. Otherwise, there is a change in the action, $\Delta W = -2\int(\mathrm{d}x)\xi_\nu\partial_\mu T^{\mu\nu}$.

Now in our case (where we make the finite size of the plate explicit, but ignore edge effects because $L \gg a$)

$$T^{\mu\nu} = \frac{\mathcal{E}_c}{a}\mathrm{diag}(1,-1,-1,3)\theta\left(\zeta - \zeta_0 + \frac{a}{2}\right)\theta\left(\frac{a}{2} - \zeta + \zeta_0\right) \times\theta(\eta + L/2)\theta(L/2 - \eta)\theta(\chi + L/2)\theta(L/2 - \chi). \tag{30}$$

Thus the nonzero components of $\partial_\mu T^{\mu\nu}$ are

$$\partial_\mu T^{\mu\zeta} = \frac{3\mathcal{E}_c}{a}\left[\delta(\zeta - \zeta_0 + a/2) - \delta(\zeta - \zeta_0 - a/2)\right]\theta\ldots, \tag{31a}$$

$$\delta_\mu T^{\mu\eta} = -\frac{\mathcal{E}_c}{a}\left[\delta(\eta + L/2) - \delta(\eta - L/2)\right]\theta\ldots, \tag{31b}$$

$$\delta_\mu T^{\mu\chi} = -\frac{\mathcal{E}_c}{a}\left[\delta(\chi + L/2) - \delta(\chi - L/2)\right]\theta\ldots, \tag{31c}$$

where $\theta\ldots$ refer to the remaining step functions. Therefore, the change in the energy obtained from ΔW is

$$\begin{aligned}\Delta E_g = \frac{6\mathcal{E}_c}{a}\int \mathrm{d}\eta\,\mathrm{d}\chi\,[\xi_\zeta(\zeta_0 - a/2,\eta,\chi) - \xi_\zeta(\zeta_0 + a/2,\eta,\chi)] \\ - \frac{2\mathcal{E}_c}{a}\int \mathrm{d}\zeta\,\mathrm{d}\chi\,[\xi_\eta(\zeta,-L/2,\chi) - \xi_\eta(\zeta,L/2,\chi)] \\ - \frac{2\mathcal{E}_c}{a}\int \mathrm{d}\zeta\,\mathrm{d}\eta\,[\xi_\chi(\zeta,\eta,-L/2) - \xi_\chi(\zeta,\eta,L/2)]\,.\end{aligned} \tag{32}$$

2.6. Fermi coordinates

Since we have demonstrated that the gravitational force on a Casimir apparatus is not a gauge-invariant concept, we must ask if there is any way to extract a physically meaningful result. There seem to be two possible ways to proceed. Either we add another interaction, say a fluid exerting a pressure on the plates, resulting in a total stress tensor that is conserved, or we find a physical basis for believing that one coordinate system is more realistic than another. The former procedure is undoubtedly more physical, but will yield model dependent results. The latter apparently has a natural solution.

A Fermi coordinate system is the general relativistic generalization of an inertial coordinate frame. Such a system has been given by Marzlin [17] for a resting observer in the field of a static mass distribution. It is actually *a priori* obvious that in such a system g_{ij} is quadratic in the distance from the observer. Thus the "constant field metric" is simply the Fermi coordinate metric for a gravitating body,

$$\mathrm{d}s^2 = -(1+2gz)\mathrm{d}t^2 + \mathrm{d}\mathbf{r}'^2. \tag{33}$$

Thus, coordinate lengths don't depend on z. The metric (13) is indeed appropriate, and the corresponding gravitational force is therefore given by the result found in that case, $F/A = -g\mathcal{E}_c$, as in (16).

2.7. Gauge transformation

Now we can use the method described in (32) to transform the energy in isotropic coordinates to that in Fermi coordinates. We compute the additional gravitational energy, in terms of the gauge field ξ_μ, which carries us from isotropic coordinates to Fermi coordinates, $h^F_{\mu\nu} = h^I_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. Here from (26) and (33)

$$h^I_{00} = -gz, \quad h^I_{ij} = -gz\delta_{ij}, \quad h^F_{00} = -gz, \quad h^F_{ij} = 0. \tag{34}$$

The gauge field turns out to be

$$\xi_\zeta = \frac{1}{2}g\left(\frac{1}{2}\zeta^2\cos\alpha + \zeta\eta\sin\alpha\right) + f(\eta,\chi), \tag{35a}$$

$$\xi_\eta = \frac{1}{2}g\left(\zeta\eta\cos\alpha + \frac{1}{2}\eta^2\sin\alpha\right) + g(\zeta,\chi), \tag{35b}$$

$$\xi_\chi = \frac{1}{2}g\,(\zeta\cos\alpha + \eta\sin\alpha)\,\chi + h(\zeta,\eta), \tag{35c}$$

where the functions f, g, and h are irrelevant. Substituting this into the expression for ΔE_g, (32), we obtain

$$\Delta E_g = \frac{6\mathcal{E}_c}{a}\int_{-L/2}^{L/2}\mathrm{d}\eta\int_{-L/2}^{L/2}\mathrm{d}\chi\frac{1}{4}g\cos\alpha\,(-2\zeta_0 a) - \frac{2\mathcal{E}_c}{a}\int_{\zeta_0-a/2}^{\zeta_0+a/2}\mathrm{d}\zeta\int_{-L/2}^{L/2}\mathrm{d}\chi\frac{1}{2}g\cos\alpha(-L)\zeta$$
$$- \frac{2\mathcal{E}_c}{a}\int_{\zeta_0-a/2}^{\zeta_0+a/2}\mathrm{d}\zeta\int_{-L/2}^{L/2}\mathrm{d}\eta\frac{1}{2}g(\zeta\cos\alpha+\eta\sin\alpha)(-L)$$
$$= -\,Ag\mathcal{E}_c\zeta_0\cos\alpha = -Ag\mathcal{E}_c z_0, \tag{36}$$

which when differentiated with respect to z_0 gives an additional force, $-\frac{\delta\Delta E_g}{A\delta z_0} = \frac{\Delta F}{A} = g\mathcal{E}_c$. When this is added to the isotropic force (27), we obtain the Fermi force,

$$\frac{F^I+\Delta F}{A} = -2g\mathcal{E}_c + g\mathcal{E}_c = -g\mathcal{E}_c = \frac{F^F}{A}, \tag{37}$$

as given in (16). This answer is the second one given in Calloni et al. [13], but is not referred to in the 2006 Bimonte et al. paper [14]. Those authors have now modified their analysis to agree with ours [18].

3. Rindler coordinates

We now turn to the consideration of the Casimir apparatus undergoing uniform acceleration [19]. Relativistically, uniform acceleration is described by hyperbolic motion

$$t = \xi\sinh\tau, \quad z = \xi\cosh\tau, \tag{38}$$

where ξ^{-1} is the proper acceleration, which corresponds to the metric

$$\mathrm{d}t^2 - \mathrm{d}z^2 = \xi^2\mathrm{d}\tau^2 - \mathrm{d}\xi^2. \tag{39}$$

The d'Alembertian operator takes on cylindrical form

$$-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 = -\frac{1}{\xi^2}\left(\frac{\partial}{\partial\tau}\right)^2 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right). \tag{40}$$

3.1. Single accelerated plate

For a single semitransparent plate at ξ_1, the Green's function can be written as

$$G(x,x') = \int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^2k_\perp}{(2\pi)^2}\mathrm{e}^{-i\omega(\tau-\tau')}\mathrm{e}^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp}g(\xi,\xi'), \tag{41}$$

where the reduced Green's function satisfies ($k = |\mathbf{k}_\perp|$)

$$\left[-\frac{\omega^2}{\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + k^2 + \mu\delta(\xi-\xi_1)\right]g = \frac{1}{\xi}\delta(\xi-\xi'), \tag{42}$$

which we recognize as just the semitransparent cylinder problem with $m \to \zeta = -\mathrm{i}\omega$ and $\kappa \to k$. Thus, the Green's function for a single plate is

$$g(\xi,\xi') = I_\zeta(k\xi_<)K_\zeta(k\xi_>) - \frac{\mu\xi_1 K_\zeta^2(k\xi_1)I_\zeta(k\xi)I_\zeta(k\xi')}{1+\mu\xi_1 I_\zeta(k\xi_1)K_\zeta(k\xi_1)}, \quad \xi,\xi' < \xi_1, \tag{43a}$$

$$= I_\zeta(k\xi_<)K_\zeta(k\xi_>) - \frac{\mu\xi_1 I_\zeta^2(k\xi_1)K_\zeta(k\xi)K_\zeta(k\xi')}{1+\mu\xi_1 I_\zeta(k\xi_1)K_\zeta(k\xi_1)}, \quad \xi,\xi' > \xi_1. \tag{43b}$$

where the strong coupling limit, $\mu \to \infty$, corresponds to Dirichlet boundary conditions.

3.2. Minkowski-space limit

If we use the uniform asymptotic expansion (UAE), based on the limit

$$\xi \to \infty, \quad \xi_1 \to \infty, \quad \xi - \xi_1 \text{ finite }, \quad \zeta = \hat{\zeta}\xi_1 \to \infty, \quad \hat{\zeta} \text{ finite }, \tag{44}$$

we recover the Green's function for a single plate in Minkowski space,

$$\xi_1 g(\xi, \xi') \to \frac{\mathrm{e}^{-\kappa|\xi-\xi'|}}{2\kappa} - \frac{\mu}{2\kappa} \frac{\mathrm{e}^{-\kappa(|\xi-\xi_1|+|\xi'-\xi_1|)}}{\mu + 2\kappa}, \tag{45}$$

where $\kappa = \sqrt{k^2 + \zeta^2}$, $\omega = \mathrm{i}\zeta$.

3.3. Energy-momentum tensor

The canonical energy-momentum for a scalar field is given by $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \frac{1}{\sqrt{-g}} \mathcal{L}$, where the Lagrange density includes the δ-function potential. Using the equations of motion the energy density is

$$T_{00} = \frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \phi \frac{\partial^2}{\partial \tau^2} \phi + \frac{\xi}{2} \frac{\partial}{\partial \xi} \left(\phi \xi \frac{\partial}{\partial \xi} \phi \right) + \frac{\xi^2}{2} \boldsymbol{\nabla}_\perp \cdot (\phi \boldsymbol{\nabla}_\perp \phi). \tag{46}$$

The force density is given by

$$f_\lambda = -\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} T^\nu{}_\lambda) + \frac{1}{2} T^{\mu\nu} \partial_\lambda g_{\mu\nu}, \tag{47}$$

or

$$f_\xi = -\frac{1}{\xi} \partial_\xi (\xi T^{\xi\xi}) - \xi T^{00}. \tag{48}$$

When we integrate over all space to get the ("coordinate") force (per area), the first term is a surface term which does not contribute:

$$\mathcal{F} = \int \mathrm{d}\xi \, \xi \, f_\xi = -\int \frac{\mathrm{d}\xi}{\xi^2} T_{00}, \tag{49}$$

which when multiplied by the gravitational acceleration g is the gravitational force/area on the Casimir energy. Using the expression (46) for the energy density, and rescaling $\zeta = \hat{\zeta}\xi$, we see that the gravitational force is merely

$$\mathcal{F} = \int \mathrm{d}\xi \, \xi \int \frac{\mathrm{d}\hat{\zeta} \, \mathrm{d}^2 k}{(2\pi)^3} \hat{\zeta}^2 g(\xi, \xi). \tag{50}$$

This result is an immediate consequence of the general formula

$$E_c = -\frac{1}{2\mathrm{i}} \int (\mathrm{d}\mathbf{r}) \int \frac{\mathrm{d}\omega}{2\pi} 2\omega^2 \mathcal{G}(\mathbf{r}, \mathbf{r}), \tag{51}$$

in terms of the frequency transform of the Green's function,

$$G(x, x') = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(t-t')} \mathcal{G}(\mathbf{r}, \mathbf{r}'). \tag{52}$$

3.4. Force on single plate

Alternatively, we can start from the following formula for the force density for a single semitransparent plate,

$$f_\xi = \frac{1}{2}\phi^2\partial_\xi\mu\delta(\xi-\xi_1), \tag{53}$$

or, in terms of the Green's function,

$$\mathcal{F} = -\mu\frac{1}{2}\int\frac{\mathrm{d}\zeta\,\mathrm{d}^2k}{(2\pi)^3}\partial_{\xi_1}[\xi_1 g(\xi_1,\xi_1)]. \tag{54}$$

For example, the force on a single plate is given by

$$\mathcal{F} = -\partial_{\xi_1}\frac{1}{2}\int\frac{\mathrm{d}\zeta\,\mathrm{d}^2k}{(2\pi)^2}\ln[1+\mu\xi_1 I_\zeta(k\xi_1)K_\zeta(k\xi_1)], \tag{55}$$

Expanding this about some arbitrary point ξ_0, with $\zeta = \hat\zeta\xi_0$, and using the UAE, we get (a is an arbitrary scale to make y dimensionless)

$$\mathcal{F} = -\frac{1}{96\pi^2 a^3}\int_0^\infty\frac{\mathrm{d}y\,y^2}{1+y/\mu a}, \tag{56}$$

which is just the negative of the (divergent) quantum vacuum energy of a single plate.

3.5. Two accelerated plates

For two plates at ξ_1, ξ_2, for $\xi,\xi'<\xi_1$,

$$g(\xi,\xi') = I_<K_> - \frac{\mu_1\xi_1K_1^2+\mu_2\xi_2K_2^2-\mu_1\mu_2\xi_1\xi_2K_1K_2(K_2I_1-K_1I_2)}{\Delta}II', \tag{57}$$

where

$$\Delta = (1+\mu_1\xi_1K_1I_1)(1+\mu_2\xi_2K_2I_2)-\mu_1\mu_2\xi_1\xi_2I_1^2K_2^2, \tag{58}$$

and where we have used the abbreviations $I_a = I_\zeta(k\xi_a)$, $I = I_\zeta(k\xi)$, $I' = I_\zeta(k\xi')$, etc. For $\xi,\xi'>\xi_2$,

$$g(\xi,\xi') = I_<K_> - \frac{\mu_1\xi_1I_1^2+\mu_2\xi_2I_2^2+\mu_1\mu_2\xi_1\xi_2I_1I_2(I_2K_1-I_1K_2)}{\Delta}KK', \tag{59}$$

and for ξ_1, ξ_2, for $\xi_1<\xi,\xi'<\xi_2$,

$$\begin{aligned} g(\xi,\xi') = I_<K_> &- \frac{\mu_2\xi_2K_2^2(1+\mu_1\xi_1K_1I_1)}{\Delta}II' \\ &- \frac{\mu_1\xi_1I_1^2(1+\mu_2\xi_2K_2I_2)}{\Delta}KK' + \frac{\mu_1\mu_2\xi_1\xi_2I_1^2K_2^2}{\Delta}(IK'+KI'). \end{aligned} \tag{60}$$

In the $\xi_0\to\infty$ limit, the UAE gives, for $\xi_1<\xi,\xi'<\xi_2$ ($a=\xi_2-\xi_1$)

$$\begin{aligned} \xi_0 g(\xi,\xi') \to \frac{1}{2\kappa}\mathrm{e}^{-\kappa|\xi-\xi'|} + \frac{1}{2\kappa\tilde\Delta}\Bigg[&\frac{\mu_1\mu_2}{4\kappa^2}2\cosh\kappa(\xi-\xi') \\ &- \frac{\mu_1}{2\kappa}\left(1+\frac{\mu_2}{2\kappa}\right)\mathrm{e}^{-\kappa(\xi+\xi'-2\xi_2)} - \frac{\mu_2}{2\kappa}\left(1+\frac{\mu_1}{2\kappa}\right)\mathrm{e}^{\kappa(\xi+\xi'-2\xi_1)}\Bigg], \end{aligned} \tag{61}$$

with

$$\tilde\Delta = \left(1+\frac{\mu_1}{2\kappa}\right)\left(1+\frac{\mu_2}{2\kappa}\right)\mathrm{e}^{2\kappa a} - \frac{\mu_1\mu_2}{4\kappa^2}, \tag{62}$$

which is exactly the expected result. The same holds in the other two regions.

3.6. Force on two-plate system

In general, we have two alternative forms for the force on the two-plate system:

$$\mathcal{F} = -(\partial_{\xi_1} + \partial_{\xi_2})\frac{1}{2}\int \frac{\mathrm{d}\zeta\,\mathrm{d}^2k}{(2\pi)^3}\ln\Delta, \tag{63}$$

which is equivalent to

$$\mathcal{F} = \int \mathrm{d}\xi \int \frac{\mathrm{d}\zeta\,\mathrm{d}^2k}{(2\pi)^3}\hat{\zeta}^2 g(\xi,\xi). \tag{64}$$

From either of these two methods, we find the gravitational force on the Casimir energy to be in the $\xi \to \infty$ limit

$$\mathcal{F} = -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\kappa\,\kappa^2\ln\Delta_0, \quad \Delta_0 = \mathrm{e}^{-2\kappa a}\tilde{\Delta}. \tag{65}$$

Explicitly,

$$\mathcal{F} = \frac{1}{96\pi^2 a^3}\int_0^\infty \mathrm{d}y\,y^3 \frac{1 + \frac{1}{y+\mu_1 a} + \frac{1}{y+\mu_2 a}}{\left(\frac{y}{\mu_1 a}+1\right)\left(\frac{y}{\mu_2 a}+1\right)\mathrm{e}^y - 1}$$
$$- \frac{1}{96\pi^2 a^3}\int_0^\infty \mathrm{d}y\,y^2\left[\frac{1}{\frac{y}{\mu_1 a}+1} + \frac{1}{\frac{y}{\mu_2 a}+1}\right] = -(\mathcal{E}_c + \mathcal{E}_{d1} + \mathcal{E}_{d2}), \tag{66}$$

which is just the negative of the Casimir energy of the two semitransparent plates. The divergent terms are just the sum of the Casimir energies of each plate separately, (56), which serve to simply renormalize the mass/area of each plate:

$$E_{\text{total}} = m_1 + m_2 + \mathcal{E}_{d1} + \mathcal{E}_{d2} + \mathcal{E}_c = M_1 + M_2 + \mathcal{E}_c, \tag{67}$$

and thus the gravitational force on the entire apparatus obeys the equivalence principle

$$g\mathcal{F} = -g(M_1 + M_2 + \mathcal{E}_c). \tag{68}$$

Saharian et al. [20] earlier reached a similar conclusion, but only for the finite part of the energy.

4. Conclusions

- We have found, after a certain confusion, an extremely simple answer to how Casimir energy gravitates: just like any other form of energy,

  $$\frac{F}{A} = -g\mathcal{E}_c. \tag{69}$$

  This result is independent of the orientation of the Casimir apparatus relative to the gravitational field. This refutes the claim sometimes attributed to Feynman that virtual photons do not gravitate.
- Although gravitational energies have a certain ill-defined character, being gauge- or coordinate-variant, this result is obtained for a Fermi observer, the relativistic generalization of an inertial observer.

- This conclusion is supported by an explicit calculation in Rindler coordinates, describing a uniformly accelerated observer. This demonstrates, quite generally, that the total Casimir energy, including the divergent parts, which renormalize the masses of the plates, possesses the gravitational mass demanded by the equivalence principle.
- New developments of this work are in progress, and will be described in part in the contribution to this proceedings by K V Shajesh.

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