Anomalies in $\mathcal{PT}$-Symmetric Quantum Field Theory

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It is shown that a version of $\mathcal{PT}$-symmetric electrodynamics based on an axial-vector current coupling massless fermions to the photon possesses anomalies and so is rendered nonrenormalizable. An alternative theory is proposed based on the conventional vector current constructed from massive Dirac fields, but in which the $\mathcal{PT}$ transformation properties of electromagnetic fields are reversed. Such a theory seems to possess many attractive features.

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1 Introduction

In 1996 we proposed [1] that a new class of quantum field theories might exist, in which the Lagrangian need not be Hermitian, yet the theory still might possess a positive spectrum. In these theories both parity $\mathcal{P}$ and time-reversal $\mathcal{T}$ invariance were violated, but the product $\mathcal{PT}$ symmetry was unbroken. Apparently, it is the presence of $\mathcal{PT}$ symmetry that replaces Hermiticity in guaranteeing positivity of the spectrum.

In early papers we examined parity violation in scalar field theories with interaction [1]

$$\mathcal{L}_{\rm int} = -g(\mathrm{i}\phi)^N, \tag{1}$$

proved that the supersymmetry of theories possessing the superpotential [2]

$$\mathcal{S} = -\mathrm{i}g(\mathrm{i}\phi)^N \tag{2}$$

was not broken by (nonperturbative) quantum corrections, suggested that a stable eigenvalue condition held in massless electrodynamics defined by an axial vector current [3]

$$j_5^\mu = e\frac{1}{2}\psi\gamma^0\gamma^\mu\gamma^5\psi, \tag{3}$$

and showed that the Schwinger-Dyson equations for the theory [4]

$$\mathcal{L}_{\rm int} = -g\phi^4 \tag{4}$$

possessed both perturbative and nonperturbative solutions.

2 $\mathcal{PT}$-symmetric electrodynamics

In this talk we will reconsider the $\mathcal{PT}$-symmetric version of massless electrodynamics [3], which is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\psi\gamma^0\gamma^\mu\frac{1}{\mathrm{i}}\partial_\mu\psi + e\frac{1}{2}\psi\gamma^0\gamma^\mu\gamma^5 A_\mu\psi. \tag{5}$$

Our conventions are the following: γ^0 is antisymmetric and pure imaginary, $\gamma^0\gamma^\mu$ is symmetric and real, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is antisymmetric and real, and $(\gamma^5)^2 = -1$. The fermion field ψ is expected to be complex.

$\mathcal{L}$ possesses gauge invariance:

$$A^\mu \to A^\mu + \partial^\mu\lambda, \quad \psi \to \mathrm{e}^{\mathrm{i}e\gamma^5\lambda}\psi. \tag{6}$$

The gauge transformation on the fermion field is not a phase transformation, but a scale transformation; nevertheless it leaves invariant the fermion bilinears in the Lagrangian, and in the energy-momentum tensor. Note that a mass term $\frac{1}{2}m\psi\gamma^0\psi$ breaks this gauge symmetry.

We had expected (erroneously as we shall see) the weak-coupling Feynman rules, for graphs with even numbers of γ_5s to be the same as those in ordinary QED, except that $\alpha \to -\alpha$. This is very intriguing, for it suggests that the program of Johnson, Baker and Willey [5] might succeed, for now their eigenvalue condition for the fine structure constant reads, in terms of the "quenched" beta function,

$$0 = F_1(\alpha) = -\frac{4}{3}\left(\frac{\alpha}{4\pi}\right) + 4\left(\frac{\alpha}{4\pi}\right)^2 + 2\left(\frac{\alpha}{4\pi}\right)^3 - 46\left(\frac{\alpha}{4\pi}\right)^4 + \ldots, \tag{7}$$

which displays all the coefficients calculated to date (remarkably integers). Keeping two, three, and four terms in this series gives a sequence of quite stable positive roots:

$$\alpha_2 = 4.189, \tag{8}$$

$$\alpha_3 = 3.657, \quad (2,1) \text{ Padé: } 3.590, \tag{9}$$

$$\alpha_4 = 4.110, (3,1) \text{ Padé}. \tag{10}$$

2.1 Erratum

This is an appropriate point to acknowledge an error in Ref. [3]. There it was stated that the eigenfunction condition in conventional QED, obtained by replacing $\alpha \to -\alpha$ in Eq. (7), possesses only the following successive positive roots of F_1:

$$\alpha_3 = 13.872, \tag{11}$$

$$\alpha_4 = 3.969, \quad (1,2) \text{ Padé: } 0.814, \quad (2,1) \text{ Padé: } 0.545. \tag{12}$$

These were nearly completely misstated. The correct positive roots are

$$\alpha_3 = 28.79, \tag{13}$$

$$\alpha_4 = 4.804, \quad (2,1) \text{ Padé of } F_1/\alpha : 0.545. \tag{14}$$

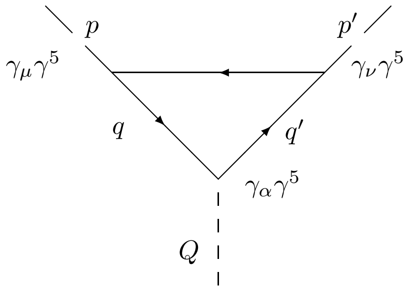

Fig. 1. Triangle graph occurring in the axial-vector $\mathcal{PT}$-symmetric quantum electrodynamics.

The conclusion, however, that the conventional QED shows no sign of stability of the eigenvalue, is of course unaltered.

2.2 Difficulties with this version of electrodynamics

However, there is no Furry's theorem for this $\mathcal{PT}$-symmetric electrodynamics, because the antisymmetrical charge matrix q of ordinary QED is replaced by the antisymmetrical γ^5 matrix: (ψ is a Grassmann element)

$$j_\mu = \frac{1}{2}\psi\gamma^0\gamma_\mu eq\psi \;\rightarrow \frac{1}{2}\psi\gamma^0\gamma_\mu e\gamma^5\psi. \tag{15}$$

Thus there is a three-photon triangle graph as shown in Fig. 1. This would seem to completely change the weak coupling expansion from that in normal QED. In particular, F_1 is not simply obtained from that in ordinary QED!

It is well-known that the AAA triangle graph possesses an axial-vector anomaly. This is usually seen as a consequence of enforcing Bose symmetry, which thereby resolves the ambiguity associated with a superficially linearly divergent loop integration [6]. This is in contrast with the more familiar AVV graph, where the axial anomaly arises from enforcement of vector current conservation [7].

2.3 Triangle Anomaly

It should be instructive to see how this comes about in a method in which all quantities are explicitly finite. This is the "causal" or "dispersive" approach, advocated by Schwinger's source-theory group in 1970s. As it happens, I have an unpublished manuscript [8] which presents the calculation of precisely the above graph in spectral form, for the general situation in which all particles have masses:

$$\begin{aligned} I_{\mu\nu\alpha} &= Q_\alpha\epsilon_{\mu\nu\lambda\sigma}p^\lambda p'^\sigma \int_{4m^2}^\infty \frac{dM^2}{2\pi\mathrm{i}}\frac{A_2(M^2)}{M^2+Q^2-i\epsilon} \\ &\quad + \epsilon_{\mu\nu\lambda\alpha}(p-p')^\lambda \int_{4m^2}^\infty \frac{dM^2}{2\pi\mathrm{i}}\frac{A_1(M^2)}{M^2+Q^2-i\epsilon}. \end{aligned} \tag{16}$$

Here m is the mass of the fermion, the outgoing "photon" momenta are on the mass shell, $p^2 = p'^2 = -\mu^2$, and we have set $p^\mu \to 0$ and $p'^\nu \to 0$ appropriate for real, outgoing vector particles. This is a consequence of the property of the three polarization vectors for a massive, spin-1 particle:

$$e^\mu_{p\lambda}: \quad p_\mu e^\mu_{p\lambda} = 0, \quad \sum_{\lambda=1}^{3} e^\mu_{p\lambda} e^\nu_{p\lambda}{}^* = g^{\mu\nu} + \frac{p^\mu p^\nu}{\mu^2}. \tag{17}$$

The spectral functions are determined by taking a cut across the q-q' lines in the graph: ($M^2 = -Q^2$)

$$\begin{aligned}\tilde{I}_{\mu\nu\alpha} &= \int d\omega_q\, d\omega_{q'}\, (2\pi)^4 \delta(Q - q - q') \frac{1}{m^2 + (p-q)^2} \\ &\quad \times \mathrm{Tr}\left[\gamma_5 \gamma_\mu [m + \gamma(p-q)] \gamma_5 \gamma_\nu (m - \gamma q') \gamma_5 \gamma_\alpha (m + \gamma q)\right] \\ &= Q_\alpha \epsilon_{\mu\nu\lambda\sigma} p^\lambda p'^\sigma A_2(M^2) + \epsilon_{\mu\nu\lambda\alpha} (p - p')^\lambda A_1(M^2). \end{aligned} \tag{18}$$

Here the invariant phase-space measure is

$$d\omega_p = \frac{(d\mathbf{p})}{(2\pi)^3} \frac{1}{2p^0}. \tag{19}$$

Explicit formula may be straightforwardly worked out for the spectral functions for this general mass case:

$$\begin{aligned} A_1(M^2) &= -\frac{1}{16\pi} \frac{v}{\zeta^4} \Bigg\{ 3 - 4\zeta^2 + \zeta^4 \\ &\quad - \left[3 - \zeta^2 - 3\zeta^4 + \zeta^6 + 4\zeta^2 v^2 (1 - \zeta^2)\right] \frac{1}{4v\zeta} \ln \phi \Bigg\}, \end{aligned} \tag{20}$$

$$\begin{aligned} A_2(M^2) &= -\frac{1}{8\pi M^2} \frac{v}{\zeta^4} \Bigg\{ 3 - 4\zeta^2 + \zeta^4 \\ &\quad - \left[3 - \zeta^2 + 5\zeta^4 + \zeta^6 - 4\zeta^2 v^2 (1 + \zeta^2)\right] \frac{1}{4v\zeta} \ln \phi, \Bigg\}. \end{aligned} \tag{21}$$

Here

$$v = \left(1 - \frac{4m^2}{M^2}\right)^{1/2}, \quad \zeta = \left(1 - \frac{4\mu^2}{M^2}\right)^{1/2}, \tag{22}$$

and

$$\phi = \frac{1 + \zeta^2 + 2v\zeta}{1 + \zeta^2 - 2v\zeta}. \tag{23}$$

The anomaly is obtained by taking the divergence with respect to the unrestricted vertex α, that is, by multiplying by Q^α.

$$\mathrm{i} Q^\alpha I_{\mu\nu\alpha} = \epsilon_{\mu\nu\lambda\sigma} p^\alpha p'^\sigma \left\{ \int_{4m^2}^{\infty} \frac{dM^2}{2\pi} \frac{2A_1(M^2) - M^2 A_2(M^2)}{M^2 + Q^2 - i\epsilon} + a \right\}, \tag{24}$$

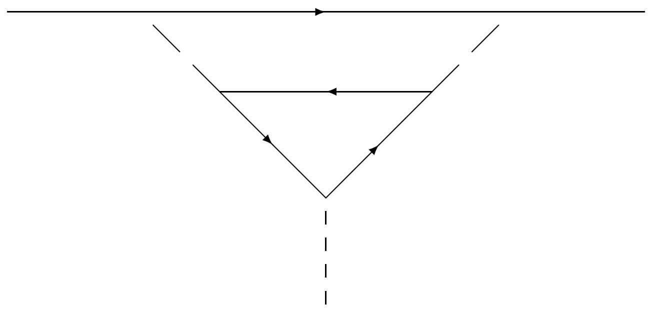

Fig. 2. Divergent graph which cannot be renormalized in axial-vector electrodynamics.

where the spectral function in the integral is just the Q^α contraction with the cut amplitude (18). It has a rather simple form:

$$2A_1 - M^2 A_2 = -\frac{1}{4\pi\zeta^3}(\zeta^2 - v^2)\ln\phi. \tag{25}$$

The integral vanishes as $m \to 0$. The anomaly arises because now $Q^2 \neq -M^2$:

$$a = \int_{4m^2}^{\infty} \frac{dM^2}{2\pi} A_2(M^2) = \frac{1}{(2\pi)^2}. \tag{26}$$

The evaluation, independent of μ^2/m^2, may be straightforwardly verified. Of course, the case of direct interest is much simpler, because $\mu = 0$ for a photon:

$$a = \frac{1}{4\pi^2}\int_0^1 v\, dv\, \ln\left(\frac{1+v}{1-v}\right) = \frac{1}{4\pi^2}. \tag{27}$$

It might appear that the correct limit here is to first set $m = 0$, then take $\mu \to 0$. However this is an anomalous threshold situation, best handled by analytic continuation from the $m > \mu$ case.

So the $\mathcal{PT}$-symmetric electrodynamics proposed by us in Ref. [3] seems to possess a serious difficulty: Because $\mathcal{P}$-violating Green's functions occur, containing an odd number of $\gamma^5\gamma^\mu$ vertices—that is, there is no Furry's theorem—an axial vector anomaly can occur in the theory. We have explicitly calculated such an anomaly. Therefore, it appears that the theory is rendered nonrenormalizable. The nonrenormalizable divergent graphs appear when, for example, the two photons of the triangle graph are attached to a fermion line, as shown in Fig. 2.

There are a number of caveats that must be noted concerning this conclusion:

- The Feynman rules, and the unitarity arguments that lead to the dispersion relations, may have to be modified in the $\mathcal{PT}$ theory because the contours that define the theory do not lie along the real axis in general.

– There may be a subtlety associated with massless fermions.

– Although this particular version of $\mathcal{PT}$QED may be flawed, it may be possible to find other variants. (We shall suggest such a possibility in the next section.)

– Of course, it must always be acknowledged that the criterion of renormalizability is not the final arbiter; nonrenormalizable theories can still be useful, effective ones.

3 Alternative $\mathcal{PT}$-symmetric electrodynamics

The axial-vector current theory seems to be fatally flawed. Fortunately, there is an alternative $\mathcal{PT}$-symmetric theory which seems to be, in fact, closer to the spirit of the general development. Instead, I propose using the usual current,

$$j^\mu = \frac{1}{2}\psi\gamma^0\gamma^\mu eq\psi, \tag{28}$$

but couple it to an *axial-vector* photon field A_μ:

$$\mathcal{L}_{\rm int} = {\rm i} j^\mu A_\mu. \tag{29}$$

Note that the factor of i is inserted to ensure $\mathcal{PT}$ invariance. Under either $\mathcal{P}$ or $\mathcal{T}$ separately, the time component of j^μ does not change, while the space component reverses sign, hence

$$\mathcal{PT}: \quad j^\mu \to j^\mu. \tag{30}$$

This is consistent with the modified Maxwell equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = {\rm i}\rho, \quad \boldsymbol{\nabla}\times\mathbf{B} = \frac{\partial}{\partial t}\mathbf{E} + {\rm i}\mathbf{j}, \tag{31}$$

provided under $\mathcal{PT}$:[1])

$$\mathcal{PT}: \quad \mathbf{E}\to\mathbf{E}, \quad \mathbf{B}\to\mathbf{B}, \quad A^\mu \to -A^\mu. \tag{32}$$

$\mathcal{PT}$ invariance of the theory is thus assured:

$$\mathcal{L}_{\rm int} \to \mathcal{L}_{\rm int}. \tag{33}$$

This theory seems to be a $\mathcal{PT}$QED with

– Furry's theorem holding (no odd Green's functions)

– No axial-vector anomaly

[1]) There actually seem to be two possible theories: Either **E** is an axial vector and **B** is a polar vector, if the scalar potential but not the vector potential, changes sign under parity (this smells a bit like magnetic charge), or **B** is an axial vector and **E** is a polar vector if the vector potential not the scalar potential changes sign under parity. The latter is more a theory of electric charge.

– Usual perturbation theory with $\alpha \to -\alpha$.

Everything we erroneously had said about the $\mathrm{i}j_5^\mu A_\mu$ theory does seems to hold for the $\mathrm{i}j^\mu A_\mu$ theory.

This new theory would seem to be asymptotically free, since the sign of the beta function reverses from that in ordinary QED. An antiscreening effect may occur here because of the attraction of like charges. Quantum-mechanically, the sign of the vacuum polarization reverses. Whether this implies confinement is under investigation.

We are also examining questions of the stability of the vacuum, in the presence of a strong electric field E. The Schwinger mechanism says that the probability of pair creation in ordinary QED is proportional to

$$P(0 \to e^+e^-) \sim \mathrm{e}^{-\pi m^2/eE}; \tag{34}$$

what happens in our case?

We will be examining this new "conventional $\mathcal{PT}$-symmetric" QED for consistency, and we ask whether somewhere might it be realized in nature?

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