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Perturbative expansions in the inclusive decay of the tau-lepton

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Abstract

A comparative analysis is performed for various forms of perturbative expansions in spacelike and timelike regions. As applied to the inclusive decay of the τ-lepton, comparison is given for the results derived within the framework of the standard perturbation theory and the analytic approach that modernizes perturbative expansions so that the new approximations reflect basic principles of the theory: renormalization invariance, spectrality, and causality. Advantages and self-consistency of the analytic approach in describing τ-lepton decay are demonstrated.

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I. INTRODUCTION

Perturbation theory (PT) is a basic tool of calculation in quantum field theory. When combined with the renormalization procedure and the renormalization-group method (RG), it is an unquestionable success in quantum electrodynamics, in the theory of electroweak interactions, and in the perturbative region of quantum chromodynamics (QCD). As is well known, nonperturbative effects play a significant role in QCD; however, in the description of most quark–gluon systems and hadronic processes, there is always present a perturbative component. It is clear that the reliability of extracting information on nonperturbative effects is connected to the indeterminacy in the description of the perturbative component. This indeterminacy arises from the inevitable truncation of PT series. Owing to the property of asymptotic freedom in QCD, the perturbative description becomes more reliable at high energy scales. Here, unknown contributions of higher order perturbative effects do not result in large errors in the theoretical analysis. However, the high-energy region where the characteristic scale of a process exceeds tens of GeVs provides scant information concerning essentially nonperturbative processes. Therefore, it seems important to develop methods of improving perturbative expansions so as to lower theoretical uncertainties in the description of physical processes. The point is that the initial PT series, or more precisely, its finite part after renormalization, is not the final product of the theory but admits of a considerable modification. In particular, it is well known that the RG method [1] allows one to modify a perturbative expansion in accordance with the general principle of renormalization invariance, thus improving the properties of the series in the ultraviolet region. As to the infrared region, where the perturbative invariant charge possesses unphysical singularities (a ghost pole in the one-loop approximation), the RG-modified PT series remains unstable.

In the late 1950s, in the context of quantum electrodynamics, a method of elimination of the unphysical singularity from the invariant charge was proposed [2,3]. Recently, in Ref. [4], this idea was applied to the case of QCD. The analytic approach formulated there combines the RG method and Q^2-analyticity (reflecting the general principles of local quantum field theory such as spectrality and causality) and results in a number of new interesting properties of the expansion [5]. Further developments and applications of the analytic method have been considered in many works, among which we mention Refs. [6–11]. The analytic coupling in exclusive processes was applied in Refs. [12,13]. It has been established that within the analytic approach one can consistently determine the effective charge in the timelike region [6]. This opens the possibility of a self-consistent description of processes with characteristic spacelike ($q^2 < 0$) and timelike ($q^2 > 0$) momenta [7], in particular, such basic processes in QCD as inelastic lepton-hadron scattering and e^+e^- annihilation into hadrons. The method developed, called analytic perturbation theory (APT) [8], preserves the correct analytic properties of such important objects as the two-point correlation function and also provides a well-defined algorithm for calculating higher loop corrections. In the framework of APT, the theoretical ambiguity associated with higher-loop corrections and with the choice of renormalization scheme is diminished. Using the APT method to describe the perturbative component of the QCD description can change the values of nonperturbative parameters extracted from experimental data. For example, the additional APT terms beyond the standard PT prediction for the Gross–Llewellyn Smith sum rule have a sign opposite to that of the typically-used higher-twist term [14] resulting in a numerical cancellation between these two corrections [9]. At high Q^2-scales, the PT and APT results agree closely with each other, whether or not the nonperturbative effects are included.

In this paper, we concentrate on the description of the process of inclusive decay of the τ-lepton into hadrons. Discovered in 1975, the τ-lepton is the only lepton known at present whose mass is sufficiently large to possess hadronic decay modes. On the other hand, its mass, $M_\tau \simeq 1.78$ GeV, is low on the QCD scale, which allows for low-energy tests of the strong-interaction theory. This is a unique system, since first, the inclusive character of the decay permits one, in principle, to base the description on standard methods of quantum field theory without any serious model assumptions. Second, measurements of characteristics of the decay have been carried out with record accuracy for hadronic processes, with an uncertainty of less than 1% [15–17]. Thus, the high accuracy of experimental data on τ-lepton decay and the fact that there is still room for improvements in theoretical methods [18], stimulate further intensive studies along lines associated with both the perturbative description (see, e.g., Refs. [19–22]) and with nonperturbative effects (see, e.g., Refs. [23–25]). The aim of this paper is to reveal features of the application of PT and APT expansions in studying the process of τ-lepton decay into hadrons for which the perturbative contribution is not merely of theoretical interest, but which also adequately represents real physical data.

The outline of this paper is as follows. In Section 2, we collect some basic relations. Various commonly used forms of perturbative expansions are described in Section 3. A similar analysis within the APT approach is given in Section 4. The comparative analysis of PT and APT methods in describing inclusive τ-decay is presented in Section 5. Summarizing comments are presented in the last section.

II. BASIC RELATIONS

The inclusive decay of the τ-lepton is described in terms of the correlator of quark currents

$$\Pi_{\mu\nu}(q^2) = i \int d^4x\, e^{iqx} \left\langle 0 \left| T J_\mu(x) J_\nu(0)^\dagger \right| 0 \right\rangle \propto (q_\mu q_\nu - g_{\mu\nu}\, q^2)\, \Pi(q^2)\,. \tag{1}$$

For a theoretical analysis, it is convenient to use the RG invariant Adler function [26], which is connected to the correlator $\Pi(q^2)$ by the formula

$$D(Q^2) = -Q^2 \frac{d\Pi(-Q^2)}{dQ^2}\,. \tag{2}$$

Here, we use the standard convention $Q^2 = -q^2 > 0$ in the Euclidean region. The integral representation for the D-function is given in terms of the function $R(s)$, the discontinuity of the correlator across the cut,

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} R(s)\,. \tag{3}$$

The representation (3) defines the function $D(Q^2)$ as an analytic function in the complex Q^2-plane with the cut along the negative real axis.

It is convenient to separate out the parton level contribution, leaving the QCD corrections, $d(Q^2)$ and $r(s)$, in the functions $D \propto 1+d$ and $R \propto 1+r$, respectively, which are related by the formulas

$$d(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} r(s), \tag{4}$$

$$r(s) = -\frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z} d(-z) \,. \tag{5}$$

The integration contour in Eq. (5) lies in the region of analyticity of the integrand and encircles the cut of $d(-z)$ on the positive real z axis.

Expressions (4) and (5) can be rewritten in terms of an effective spectral function $\rho^{\rm eff}(\sigma)$ [6],

$$d(Q^2) = \frac{1}{\pi} \int_0^\infty d\sigma \frac{\rho^{\rm eff}(\sigma)}{\sigma + Q^2} \,, \tag{6}$$

$$r(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \, \rho^{\rm eff}(\sigma) \,, \tag{7}$$

which is determined from the discontinuity of the function $d(Q^2)$ across the cut.

The experimentally measured R_τ-ratio of hadronic and leptonic widths of the τ-lepton can be presented as follows

$$R_\tau = 3 \left(|V_{ud}|^2 + |V_{us}|^2 \right) S_{\rm EW} \left(1 + \delta_\tau \right) \,, \tag{8}$$

where V_{ud} and V_{us} are elements of the CKM quark mixing matrix [14], $S_{\rm EW}$ is the electroweak factor (see Ref. [27] for details), and the contribution of strong interactions δ_τ is expressed via $r(s)$:

$$\delta_\tau \, = \, 2 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^2 \left(1 + 2 \frac{s}{M_\tau^2} \right) r(s) \,. \tag{9}$$

The relations between the functions $r(s)$ and $d(Q^2)$ allow us to represent δ_τ as a contour integral in the complex z plane by choosing the contour to be a circle of radius $|z| = M_\tau^2$ [28]:

$$\delta_\tau \, = \, \frac{1}{2\pi i} \oint\limits_{|z|=M_\tau^2} \frac{dz}{z} \left(1 - \frac{z}{M_\tau^2} \right)^3 \left(1 + \frac{z}{M_\tau^2} \right) d(-z) \,. \tag{10}$$

The expression (9) can also be written in terms of the moments of $r(s)$

$$\delta_\tau = 2 \, \mathrm{m}_0(M_\tau^2) - 2 \, \mathrm{m}_2(M_\tau^2) + \mathrm{m}_3(M_\tau^2) \,, \tag{11}$$

which are defined by the relation

$$\mathrm{m}_k(s_0) \equiv (k+1) \int_0^{s_0} \frac{ds}{s_0} \left(\frac{s}{s_0}\right)^k r(s)\,. \tag{12}$$

Using the expression of $r(s)$ in terms of $d(Q^2)$ [see Eq. (4)], we obtain the contour representation for the moments

$$\mathrm{m}_k(s_0) = \frac{1}{2\pi i} \oint_{|z|=s_0} \frac{dz}{z} \left[1 - \left(\frac{z}{s_0}\right)^{k+1}\right] d(-z)\,, \tag{13}$$

which can be rewritten in the form

$$\mathrm{m}_k(s_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \left[1 + (-1)^k e^{i(k+1)\varphi}\right] d(-s_0 e^{i\varphi})\,. \tag{14}$$

convenient for numerical calculations. Note that expressions (9) and (10) as well as expressions (12) and (13) are equivalent only when the above-mentioned analytic properties are maintained.

III. PERTURBATION THEORY

Consider the functions $d(Q^2)$ and $r(s)$ within the framework of perturbation theory for massless quarks. We can write down Nth-order perturbative expansions as double series in the coupling $a(\mu^2) = \alpha_S(\mu^2)/\pi$

$$d^{(N)}(Q^2) = \sum_{n=1}^{N} a^n(\mu^2) \sum_{k=0}^{n} d_{n,k} \ln^k \left(\frac{Q^2}{\mu^2}\right)\,, \tag{15}$$

$$r^{(N)}(s) = \sum_{n=1}^{N} a^n(\mu^2) \sum_{k=0}^{n} r_{n,k} \ln^k \left(\frac{s}{\mu^2}\right)\,. \tag{16}$$

Then using relations (4) and (5), we derive the connection between the coefficients of these expansions

$$d_{n,k} = \sum_{m=0}^{n-k} \frac{(k+m)!}{k!\,m!} r_{n,k+m} J_m\,, \tag{17}$$

$$r_{n,k} = \sum_{m=0}^{n-k} \frac{(k+m)!}{k!\,m!} d_{n,k+m} I_m\,, \tag{18}$$

where $J_m = I_m = 0$ for odd m and

$$J_m = 2m!\,\zeta(m)\left(1 - 2^{1-m}\right)\,, \tag{19}$$

$$I_m = (-1)^{m/2} \frac{\pi^m}{m+1} \tag{20}$$

for even m. Here $\zeta(m)$ is the Riemann ζ-function. The distinction between the coefficients $d_{n,k}$ and $r_{n,k}$ starts at the three-loop level,

$$d_{3,0} = r_{3,0} + r_{3,2}\frac{\pi^2}{3} = r_{3,0} + \beta_0^2\frac{\pi^2}{3}\,, \tag{21}$$

where β_0 is the first coefficient of the QCD β-function, and is connected with the so-called π^2-terms. This usual "π^2-terminology" is in accordance with expression (19), if it is recalled that $\zeta(2m) \propto (\pi^2)^m$, and also with expression (20).

Therefore, the π^2-terms appear prior to the renormalization-group summation due to the requirement of correspondence between the initial perturbative expansion (15) given in the Euclidean (spacelike) region and the expansion (16) in the physical (timelike) region.[1] The π^2-terms play an important role in analyzing various hadronic processes (see, e.g., [31–34]). A large value of the π^2-contribution that distinguishes the third perturbative coefficients in the expansion of spacelike and timelike quantities can result in different signs for the second renormalization-scheme invariant ω_2 [35] for these quantities. In particular, this takes place for the process of e^+e^- annihilation into hadrons. In this case, signs of the second invariant for the timelike quantity, the well-known $R_{e^+e^-}$-ratio, and the corresponding D-function, the spacelike quantity, are different. A negative sign for the second scheme invariant for $R_{e^+e^-}$ leads to infrared freezing of the effective charge that arises in optimization of the scheme dependence [36,37] on the basis of the principle of minimal sensitivity (PMS) [35] or in the method of effective charge (ECH) [38,39]. At the same time, the sign of the second scheme invariant for the D-function remains positive, and the perturbative analysis based on either the PMS and ECH method of optimization does not lead to charge freezing in the Euclidean region (for more details, see Refs. [40,41]). Thus, there arises a discrepancy between the result of direct application of the PMS or ECH optimizations for the D-function, leading to a singular infrared behavior, and its calculation in terms of the dispersion relation (3) with the use of the PMS/ECH optimized function $R(s)$, which leads to a regular function on the whole interval of momentum transfer. This discrepancy is not solely caused by truncation of the perturbative expansion.

The initial series (15) possesses the important positive feature that any finite order preserves the correct analytic properties of the d-function. At the same time, the series (15) has the following negative features: a) its behavior in the ultraviolet and infrared regions is incorrect owing to large logarithms; b) its partial sum is not a renormalization-group invariant and turns out to be μ-dependent. Fortunately, in quantum field theory, initial perturbative expansions of the type (15) are not the final products of the theory and admit an essential modification that can be realized on the basis of some general properties of the theory. The renormalization-group method allows one to remove the μ-dependence of a perturbative series and improve its behavior in the ultraviolet region by accumulating the large logarithms in Eq. (15) into the running coupling. The RG-version of Eq. (15) is usually written as

$$d^{(N)}(Q^2) = \sum_{n=1}^{N} d_{n-1}\,\bar{a}^n(Q^2)\,. \tag{22}$$

[1] π^2-terms can also be derived as a result of analytic continuation of the running coupling from the Euclidean region into the physical one [29–31].

The expansion coefficients, $d_{n-1} \equiv d_{n,0}$, depend (with the exception of d_0) on the scheme of renormalization and on the number of active quark flavors. The running coupling $\bar{a}(Q^2)$ is determined by the renormalization-group equation

$$\ln \frac{Q^2}{Q_0^2} = \int_{\bar{a}(Q_0^2)}^{\bar{a}(Q^2)} \frac{d\,a}{\beta(a)} \,. \tag{23}$$

Thus, the general principle of RG invariance allows one to improve the ultraviolet behavior and RG properties of perturbative expansions. However, the series so derived are, as before, ill-defined in the infrared region, and moreover, the sole positive property of the initial expansion (15) is lost. Now, the correct analytic properties of the series in the complex Q^2-plane are violated due to unphysical singularities of the running coupling $\bar{a}(Q^2)$, and thus the connection between d and r given by relations (4) and (5) is destroyed.

In this connection, we note that, in principle, it is possible to use the d-function in the form (22) in Eq. (5) and to represent the function $r(s)$ as an expansion in the perturbative running coupling $\bar{a}(Q^2)$. (Asymmetry of the d- and r-functions arises owing to the difference of coefficients, $d_n \neq r_n$ for $n \geq 3$, caused by the π^2-terms [30]). However, this trick, which is often used for a description of processes with characteristic timelike momenta, is not self-consistent. Indeed, if the function $r(s)$ thus derived is substituted into equation (4), the initial function $d(Q^2)$ used in finding $r(s)$ cannot be reproduced. And moreover, the integral in Eq. (4) will be divergent due to unphysical singularities of the perturbative invariant charge.

Parametrization of $r(s)$ as an expansion in the perturbative running coupling also leads to the problem of direct calculation of the QCD contribution to R_τ-ratio according to Eq. (9), since the region of integration covers the infrared region. It would seem that the transformation to the contour representation (10) allows one to avoid this difficulty, since in this case unphysical singularities of the running coupling lie outside of the contour, and the procedure of integration can formally be easily accomplished. However, in our opinion, this trick ("sweeping the difficulty under the rug") does by no means solve the problem (see also Refs. [42,43]). Actually, incorrect analytic properties of the running coupling result in Eqs. (9) and (10) for δ_τ being no longer equivalent [8]. For instance, in the leading order, by comparing the two expressions (12) and (13) for the moments, we obtain the following equality ($s_0 > \Lambda^2$, $k \geq 0$):

$$\begin{aligned}
&\frac{1}{2\pi i} \oint_{|z|=s_0} \frac{dz}{z} \left[1 - \left(\frac{z}{s_0} \right)^{k+1} \right] \frac{1}{\ln(-z/\Lambda^2)} \\
&= (k+1) \int_0^{s_0} \frac{ds}{s_0} \left(\frac{s}{s_0} \right)^k \left[\frac{1}{2} - \frac{1}{\pi} \arctan \frac{\ln(s/\Lambda^2)}{\pi} + (-1)^k \Theta(\Lambda^2 - s) \right] ,
\end{aligned} \tag{24}$$

where Θ is the Heaviside step function. Eq. (24) shows that if the running coupling is taken to be given by the contour representation (13), one cannot reproduce any analog of this running coupling in the initial expression (12) defined in the timelike region, since the expression in brackets on the right-hand side of Eq. (24) contains k-dependence that should, naturally, be absent in the running coupling.

The general origin of the above difficulties lies in the fact that the conventional RG improvement leads to unphysical singularities of the invariant charge, violating the definite analytic properties required by the fundamental principles of the theory. Therefore, perturbative expansions require further modification to restore the analytic properties lost upon the RG improvement. This step, made in the analytic approach in QCD [4], combines the renormalization-group method with Q^2-analyticity. (An analytic charge slightly different from that proposed in Ref. [4] was considered in Refs. [44,45]; the method of a-expansion [46] also ensures correct analytic properties.) In this paper, we use analytic perturbation theory [8], in which not only are the above-mentioned analytic properties not violated, but also the algorithm for calculating higher-loop approximations is well defined.

IV. ANALYTIC PERTURBATION THEORY

In the analytic approach [4], the basic object is a spectral function which enters into some integral representation. In particular, for two-point functions, it is the Källén–Lehmann representation; whereas for structure functions for inelastic lepton–hadron scattering, the integral representation is that of Jost–Lehmann–Dyson. The spectral function for the objects under consideration here can be obtained by using the perturbative series (22) as a initial approach. Truncated at the three-loop level, the perturbative d-function is[2]

$$d_{\rm pt}(Q^2) = a_{\rm pt}(Q^2) + d_1 a^2_{\rm pt}(Q^2) + d_2 a^3_{\rm pt}(Q^2)\,. \tag{25}$$

In the $\overline{\rm MS}$ scheme for three active quarks ($n_f = 3$) relevant in τ decay, the expansion coefficients are $d_1^{\overline{\rm MS}} = 1.6398$ and $d_2^{\overline{\rm MS}} = 6.3710$ [33,47].

The expansion (25) generates the following approximation to the spectral function:

$$\rho(\sigma) = \varrho_0(\sigma) + d_1\varrho_1(\sigma) + d_2\varrho_2(\sigma)\,, \tag{26}$$

where the coefficients d_1 and d_2 are the same as in the PT series (25) and the expansion functions are determined by the discontinuity of the corresponding power of the perturbative coupling, $\varrho_n(\sigma) = {\rm Im}[a^{n+1}_{\rm pt}(-\sigma - i\epsilon)]$. By using the spectral function (26) in Eq. (6), we obtain the d-function in the form of the expansion (not a power series in a)

$$d_{\rm an}(Q^2) = a_{\rm an}(Q^2) + d_1\Delta^{(2)}_{\rm an}(Q^2) + d_2\,\Delta^{(3)}_{\rm an}(Q^2)\,, \tag{27}$$

where the $\Delta^{(n)}_{\rm an}$ are analytic functions. The Euclidean running coupling $a_{\rm an}(Q^2)$ and the running coupling $\tilde{a}_{\rm an}(s)$ determined in the physical region are expressed through the function $\varrho_0(\sigma)$ as

$$a_{\rm an}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + Q^2}\,\varrho_0(\sigma)\,, \tag{28}$$

and

$$\tilde{a}_{\rm an}(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\,\varrho_0(\sigma)\,. \tag{29}$$

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[2]Hereafter, to distinguish the description in PT and APT approaches, we use subscripts "pt" and "an". We omit the bar over the running coupling a.

It is sometimes suggested that the behavior of the invariant charge is symmetric between the Euclidean ($Q^2 > 0$) and physical ($Q^2 = -s < 0$) regions. In other words, when the spacelike and timelike arguments coincide in magnitude, $Q^2 = s$, the corresponding functions also coincide in magnitude. However, as was shown in Ref. [48] on the basis of general principles of the theory, this assumption is not correct. Actually, these functions have equal infrared limiting values and the same ultraviolet behavior (reflecting the property of asymptotic freedom), but in the intermediate region they do not coincide.

In the leading order, the spectral function has the form

$$\varrho_0^{(1)}(\sigma) = \frac{1}{\beta_0} \frac{\pi}{\ln^2(\sigma/\Lambda^2) + \pi^2} \, , \tag{30}$$

where $\beta_0 = (11 - 2n_f/3)/4$. Substituting this expression into Eqs. (28) and (29), we obtain explicit expressions for the invariant charges in the spacelike and timelike regions [4–7]

$$a_{\rm an}^{(1)}(Q^2) = \frac{1}{\beta_0} \left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right] \, , \tag{31}$$

$$\tilde{a}_{\rm an}^{(1)}(s) \; = \; \frac{1}{\beta_0} \left[\frac{1}{2} - \frac{1}{\pi} \arctan \frac{\ln(s/\Lambda^2)}{\pi} \right] \, . \tag{32}$$

The expression in the Euclidean region (31) contains the usual logarithmic term that coincides with the perturbation expression containing the ghost pole at $Q^2 = \Lambda^2$. The contribution of this pole is compensated by the second term in Eq. (31) of a power character in Q^2. Written in terms of the initial $a_{\rm pt}$, this term is of the structure of $\exp(-1/a_{\rm pt})$ and therefore makes no contribution to the power series expansion in the coupling $a_{\rm pt}$. That is, the Q^2-power contribution in the Euclidean running coupling (31), invisible in perturbation theory, is restored automatically on the basis of the analyticity principle. In contradistinction to the perturbative running coupling $a_{\rm pt}(Q^2)$, the analytic function $a_{\rm an}(Q^2)$ has no unphysical singularities: the ghost pole and corresponding branch points (which appear in higher order) are absent.

Regularity of the running coupling in the timelike region has another origin.[3] When the regular function (32) is expanded in a series in the singular perturbative coupling $a_{\rm pt}(Q^2) \propto 1/\ln(Q^2/\Lambda^2)$, the expansion will contain π^2-terms. Thus, in the APT method the timelike running coupling (32) sums up the π^2-terms into a regular function. It is important to note that the functions (31) and (32), in agreement with Eqs. (4) and (5), satisfy the same relations as the spacelike and timelike functions $d(Q^2)$ and $r(s)$:

$$Q^2 \int_0^\infty \frac{ds}{(s + Q^2)^2} \, \tilde{a}_{\rm an}(s) \; = \; a_{\rm an}(Q^2) \; , \tag{33}$$

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[3]The invariant charge in the timelike region should not be an analytic function, and the word "regularity" does not mean analyticity (for a discussion, see Ref. [41]).

$$-\frac{1}{2\pi i}\int_{s-i\epsilon}^{s+i\epsilon}\frac{dz}{z}\,a_{\rm an}(-z) \;=\; \tilde{a}_{\rm an}(s)\,. \tag{34}$$

In the framework of APT, there is little sensitivity to the approximation made in solving the renormalization-group equation for the running coupling [8]. In principle, in constructing APT, one can use the standard PT parametrization of the running coupling as an expansion in inverse powers of $L \equiv \ln(Q^2/\Lambda^2)$ [14]. Then the derived expression for the APT running coupling in the timelike region, $\tilde{a}_{\rm an}^{(3)}(s)$, has a rather simple form [49]. The APT result for running couplings up to the third order can be written in terms of the Lambert function [50,51]. In Fig. 1, the evolution of the QCD running coupling obtained in the PT and APT approaches is compared. Having solved the three-loop RG-equation, we have obtained the exact PT running coupling (dashed line), which was used for obtaining the function $\varrho_0(\sigma)$. The analytic running coupling $a_{\rm an}(Q^2)$, Eq. (28), is shown as a solid line and the APT timelike running coupling, Eq. (29), is shown as a dotted line. One can see that at low energy scales the difference between the PT and APT Euclidean running coupling becomes significant: instead of a rapidly changing function as occurs in the PT case, we get a slowly changing function in the APT case (see Refs. [7,9] for more details). In the region of sufficiently large timelike momenta, the well-known approximate formula with the π^2-term,

$$\tilde{a} = a - \beta_0^2\,\frac{\pi^2}{3}a^3\,, \tag{35}$$

works well both for PT and APT approaches. In Fig. 1, the dash-dotted line represents the three-loop PT running coupling calculated by using this formula.

V. τ-DECAY

The present experimental value of R_τ is known with an accuracy unprecedented for low-energy hadronic processes: The averaged fit of the Particle Data Group 2000 [14] is $R_\tau = 3.646 \pm 0.022$ and the combined experimental result of the TAU 2000 Workshop is $R_\tau = 3.640 \pm 0.010$ [17]. However, the value of α_s extracted from those data at the τ mass scale has a quite large error dominated by the theoretical uncertainty of QCD calculations. For example, the ALEPH Collaboration analysis gives the value $\alpha_s(M_\tau) = 0.334 \pm 0.007_{\rm expt} \pm 0.021_{\rm theor}$ [15] in which the theoretical uncertainty is three times larger than the experimental one. It might be supposed that the large theoretical uncertainty is connected to a poorly known nonperturbative contribution. However, estimates show that the nonperturbative contribution is rather small and compatible with zero: $\delta_{\rm np} = -0.003 \pm 0.004$ [15,18]. The chief uncertainty arises from the use of the conventional perturbative approximation.

The application of perturbation theory to the description of inclusive τ-decay was developed in two directions. The approach given in Ref. [28], usually called FOPT (fixed-order perturbation theory), leads to a representation of the physical quantity R_τ as an expansion in powers of the running coupling at $Q^2 = M_\tau^2$. The three-loop perturbative approximation to δ_τ in Eq. (8) reads as follows:

$$\delta_\tau = a_{\rm pt}\left(M_\tau^2\right) + K_1\,a_{\rm pt}^2\left(M_\tau^2\right) + K_2\,a_{\rm pt}^3\left(M_\tau^2\right)\,. \tag{36}$$

The coefficients calculated in the $\overline{\text{MS}}$ scheme for three active quarks are $K_1 = 5.2023$ and $K_2 = 26.366$ [27,33]. Note here that the π^2-contribution dominates in the second coefficient K_2.

If the function $d_{\rm pt}$ in the form of the expansion (25) is substituted into the contour integral (10), one obtains the following expression [52], which is not a power series expansion in $a_{\rm pt}$:

$$\delta_\tau = A^{(1)}\left(M_\tau^2\right) + d_1\, A^{(2)}\left(M_\tau^2\right) + d_2\, A^{(3)}\left(M_\tau^2\right)\,, \tag{37}$$

where

$$A^{(n)}\left(M_\tau^2\right) = \frac{1}{2\pi i}\oint_{|z|=M_\tau^2}\frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right)a_{\rm pt}^n(-z)\,. \tag{38}$$

This result is called the contour-improved fixed-order perturbation theory, FOPT_{CI}.

Both these approaches, FOPT and FOPT_{CI}, are widely used in the analysis of τ-decay data and this ambiguity in the perturbative description has an influence on the accuracy of theoretical predictions. Let us discuss the status of both approaches. First of all, we will emphasize that if the method of calculation is consistent with the correct analytic properties of the function $d(Q^2)$, both representations (9) and (10) are equivalent. This means that if we use the initial perturbative expansions of the functions $r(s)$ and $d(Q^2)$, Eqs. (16) and (15) respectively, before the renormalization-group resummation [that is, their representations in perturbation theory with the expansion parameter $a(\mu^2)$, containing the correct analytic properties of the function $d(Q^2)$ in any partial sum of the series], we find that the initial expression (9) and the contour representation (10) give the same result

$$\begin{aligned}\delta_\tau &= a_{\rm pt}(\mu^2)\, r_{1,0} + a_{\rm pt}^2(\mu^2)\left[\left(r_{2,0}-\frac{19}{12}r_{2,1}\right) + r_{2,1}\ln\left(\frac{M_\tau^2}{\mu^2}\right)\right] \\ &+ a_{\rm pt}^3(\mu^2)\left[\left(r_{3,0}-\frac{19}{12}r_{3,1}+\frac{265}{72}r_{3,2}\right)+\left(r_{3,1}-\frac{19}{6}r_{3,2}\right)\ln\left(\frac{M_\tau^2}{\mu^2}\right)+r_{3,2}\ln^2\left(\frac{M_\tau^2}{\mu^2}\right)\right]\,.\end{aligned} \tag{39}$$

The representation δ_τ as an expansion in powers of $a(M_\tau^2)$ can be derived from Eq. (39), if the renormalization group is applied, replacing there $\mu^2 = M_\tau^2$. The contour representation, Eq. (37), in which the perturbative running coupling enters the complex contour integration, is not admissible, since unphysical singularities of the invariant charge break the connection between the contour representation and the initial expression (9). This can be observed from the expression δ_τ in terms of the moments (11) and the equality (24) in which to preserve the sign of the equality, one cannot neglect the contribution from the Θ function on the right-hand side. Note that it may appear that the contour representation can be obtained in the following way. As a first step, one passes to a representation in terms of a contour integral using approximations with the expansion parameter $a(\mu^2)$ that do not destroy the analytic properties necessary for this transformation. In the next step, instead of the above procedure in which we put $\mu^2 = M_\tau^2$ (taking the value of the subtraction point equal to a fixed external parameter M_τ) we take, prior to the evaluation of the contour integral, the quantity μ^2 to be a complex variable over which integration is carried out. But one must recall here that the procedure of renormalization where the idea of the subtraction arises is such that the value of μ^2 cannot be arbitrary. Complex values of this parameter spoil the

Hermiticity of the renormalized Lagrangian, leading to serious difficulties, and should be eliminated from further consideration. Thus, the complex representation of δ_τ in the form $\mathrm{FOPT_{CI}}$ (37) with the standard running coupling [14] containing unphysical singularities is self-contradictory. Applying it in the analysis of experimental data on the τ-lepton decay, which is a low energy timelike process, is not valid and cannot provide reliable information on perturbative and nonperturbative QCD parameters.[4] Despite the above difficulties, due to its more stable behavior with respect to higher-loop terms and scheme dependence, the $\mathrm{FOPT_{CI}}$ series has been widely used in probing nonperturbative effects.

The analytic approach discussed in the previous section provides the analytic properties necessary for the equivalence of the initial and contour representations, Eqs. (9) and (10), respectively. Within the framework of the APT approach, both forms can be rewritten in terms of the spectral function $\rho(\sigma)$ as [8]

$$\delta_\tau = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma}\rho(\sigma) - \frac{1}{\pi}\int_0^{M_\tau^2} \frac{d\sigma}{\sigma}\left(1-\frac{\sigma}{M_\tau^2}\right)^3\left(1+\frac{\sigma}{M_\tau^2}\right)\rho(\sigma)\,. \tag{40}$$

Besides yielding the correct analytic properties, the APT approach provides a remarkable loop stability not only in the ultraviolet region but also at low-energy scales. This stability is connected to the stability of the analytic running coupling, for which a maximal difference between the one- and two-loop approximations at small Q^2 does not exceed $8\,\%$; whereas the three-loop approximation differs from the two-loop one by less than $1\,\%$ [4,5].

In Fig. 2, we illustrate the dependence of R_τ on the running coupling $\alpha = \pi a(M_\tau^2)$ in the FOPT and APT approaches comparing the convergence properties in the one-loop (dotted lines), two-loop (dashed lines), and three-loop (solid lines) approximations. Numbers above the curves specify the order of the PT approximation. The shaded area shows the corridor of experimental errors and corresponds to $R_\tau^{\mathrm{expt}} = 3.646 \pm 0.022$ [14]. Fig. 2 demonstrates the loop stability of APT results as compared to PT.

A similar analysis of the perturbative contribution to inclusive τ-decay has been presented in Ref. [53]. The authors of that paper claimed that to describe the experimental data for τ-decay the value of $\alpha_{\mathrm{an}}(M_\tau^2)$ should be taken in the range of 1.5–2.0. This conclusion contradicts our result. Fig. 2 clearly demonstrates that in order to reproduce the experimental data such large values of $\alpha_{\mathrm{an}}(M_\tau^2)$ are not required. Moreover, in the APT approach, the value of the analytic running coupling $\alpha_{\mathrm{an}}(Q^2)$ is bounded from above and cannot exceed the infrared limiting value $\alpha_{\mathrm{an}} \le \pi/\beta_0 \simeq 1.4$ [4]. Beyond this, in Ref. [53], it was noted that this impossibly large value $\alpha_{\mathrm{an}}(M_\tau^2)$ corresponds to $\alpha_s(M_Z^2) \simeq 0.15$. We remark that in order to obtain the value of α_s at the Z-boson mass scale, the region of five active quarks should be approached by applying a special procedure of matching from the three-quark region [7,10]. Corresponding estimates have been given in Ref. [21].

In Ref. [53] it was emphasized that the merit of the contour representation is that it produces expressions for quantities in the physical region that are not expansions in the

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[4] In this connection, we note that the conclusions of the authors of Ref. [53] obtained on the basis of the contour representation are far from being justified.

parameter $\pi/\ln(s/\Lambda^2)$, which is not a small quantity in the intermediate energy region. In particular, one can write the formula (see Eq. (11) from [53]):

$$2\pi\, \mathrm{Im}\Pi(s+i0) = 1 + \frac{1}{\pi\beta_0}\left[\frac{\pi}{2} - \arctan\left(\frac{1}{\pi}\ln\frac{s}{\Lambda^2}\right)\right]. \tag{41}$$

Although we agree with the authors of Ref. [53] that it is important to sum up the π^2-terms in the region of intermediate energies,[5] and that it is preferable to use the expression which sums up the π^2-contributions rather than the asymptotic expression $\propto 1/\ln(s/\Lambda^2)$ for the bracketed quantity in Eq. (41), we note the following: Expression (41) unambiguously leads to an analytic charge that corresponds to the first QCD contribution in the D-function expansion. Indeed, substituting Eq. (41) into Eq. (3), we obtain $D(Q^2) \propto 1 + a_{\rm an}^{(1)}(Q^2)$, where the one-loop analytic coupling is defined by Eq. (31). So, we conclude that the analytic approach provides a consistent form of expressions of the type (41) in the timelike region, whose evident advantage is the summation of the π^2-contributions into a regular function, and provides the resulting corresponding analytic expressions for the D-function in the spacelike region, where unphysical singularities are cancelled by functions of a non-logarithmic type vanishing in a perturbative expansion. In other words, summation of the π^2-contributions in the s-channel produces power (nonlogarithmic in Q^2) contributions in the t-channel that regulate the analytic properties by compensating unphysical singularities in the logarithmic terms.

A significant source of theoretical uncertainty arises from the renormalization scheme (RS) dependence of the results obtained due to the inevitable inclusion of only a finite number of terms in the PT series. In QCD, that uncertainty is the greater, the smaller the value of the energy typical of the process. There are no general principles that give preference to a particular renormalization scheme, and in this sense, all such schemes are equivalent. The APT method improves this situation and gives very stable results over a wide range of renormalization schemes, and at the three-loop level (the level of theoretical calculation for many physical processes at present) can give results with a theoretical uncertainty connected with a finite approximation order significantly smaller than in the standard method.

In passing from one renormalization scheme to another, the coupling constant is transformed as follows

$$a' = a\,(1 + v_1 a + v_2 a^2 + \cdots)\,. \tag{42}$$

At the three-loop level, one can obtain the running coupling as a solution to the RG equation with the three-loop β-function

$$\beta(a) = \mu^2 \frac{\partial\, a}{\partial\, \mu^2} = -\beta_0\, a^2\,(1 + b_1 a + b_2 a^2)\,, \tag{43}$$

[5] Moreover, a correct use of the π^2-terms when processing the experimental data in the timelike region turns out to be important also in the traditional asymptotic region of order of several tens of GeV and higher [34].

where the three-loop coefficient b_2 is scheme-dependent. The values of the coefficients evaluated for $n_f = 3$ are $\beta_0 = 4.5$, $b_1 = 1.778$, and $b_2^{\overline{\mathrm{MS}}} = 4.471$.

The expansion coefficients d_1 and d_2 in Eq. (25) are also RS dependent. When passing to a new scheme, the coefficients b_2, d_1, and d_2 are transformed as follows:

$$\begin{aligned} b_2' &= b_2 - v_1^2 - b_1 v_1 + v_2\,, \\ d_1' &= d_1 - v_1\,, \\ d_2' &= d_2 - 2(d_1 - v_1)v_1 - v_2\,. \end{aligned} \tag{44}$$

Thus, we arrive the new d-function

$$d' = a'\,(1 + d_1' a' + d_2' a'^2)\,, \tag{45}$$

where the constant a' is computed with the new β-function in which the three-loop coefficient b_2 is replaced by the primed one b_2'.

Taking into account the transformation law of the scale parameter Λ [54] and Eqs. (44), one can obtain the following two scheme invariants (see [35])

$$\omega_1 = \beta_0 \ln \frac{Q^2}{\Lambda^2} - d_1\,, \tag{46}$$

$$\omega_2 = b_2 + d_2 - b_1 d_1 - d_1^2\,. \tag{47}$$

The first invariant contains the entire momentum dependence, and the second invariant is just a number whose value is determined by the process under consideration. Although there are no arguments of a general character that would allow us to decide between any two different RS, one can determine a class of "natural" schemes that look reasonable at the three-loop level. The corresponding criterion has been proposed in Ref. [55]. It consists in requiring that one should be restricted to those schemes for which the cancellations between terms in the second scheme-invariant (47) are not too large. Quantitatively, a criterion of that sort can be specified with the cancellation index

$$C = \frac{1}{|\omega_2|}\left(|b_2| + |d_2| + d_1^2 + |d_1|\, b_1\right)\,. \tag{48}$$

Choosing a certain maximal value of the cancellation index $C_{\max}$ that establishes the limit for the class of admissible schemes, one can investigate the stability of the result by examining all schemes with an index $C \le C_{\max}$. For instance, for $C_{\max}$ one can take the index corresponding to the PMS. In this case, we have a relatively narrow class of allowed schemes that are determined by the maximal cancellation index $C_{\mathrm{PMS}} \simeq 2$.

The RS dependence of perturbative results has been considered in numerous papers. For τ-decay it has been investigated, for example, in Refs. [36,56]. Abundant experimental data accumulated in measurements of the hadronic decay modes of the τ-lepton allow one not only to analyze such a "global" quantity as R_τ, but also to construct various characteristics of this process determined for the timelike and spacelike regions with high accuracy. In particular, for comparisons of theoretical results with experiment it turns out to be convenient to use the function $D(Q^2)$, see Refs. [57–59]. Here we discuss the problem of the RS dependence of the theoretical predictions for the light D-function [49].

In Fig. 3 we plot the behaviors of the QCD contribution to the D-function, $d(Q^2)$, in different RS. It is seen that predictions in the perturbative approach for $d(Q^2)$ obtained within different RS diverge considerably as early as the value $Q \simeq 2$ GeV (see dashed curves A and B). Note should be made of the fact that the schemes A and B are similar to each other and to the optimal PMS and ECH schemes in the sense of the cancellation index: $C_{\rm A} \simeq C_{\rm B} \simeq 2$. For the ECH method, the cancellation index is minimal, equaling unity. The cancellation index for the $\overline{\rm MS}$ scheme turns out to be somewhat bigger, $C_{\overline{\rm MS}} \simeq 3.1$. In Fig. 3, we also draw the curves representing perturbative results in the PMS, ECH, $\overline{\rm MS}$ and K schemes.[6] As seen from Fig. 3, the dispersion of the PT results obtained in the above-mentioned schemes is significant. For the same schemes, in Fig. 3 we also present results obtained in the APT approach. There the scheme arbitrariness is extremely small, and all the curves corresponding to the schemes A, B, PMS, ECH, $\overline{\rm MS}$, and K calculated in APT merge into one thick solid curve. Thus, in the APT, the scheme arbitrariness is very dramatically reduced as compared with that in analogous PT calculations.

VI. CONCLUSIONS

Perturbation theory is a basic tool of quantum field theory calculations. The structure of an initial perturbative approximation of some quantity is not a rigid construction fixed once and for all, as it is, for example, in simple classical and quantum-mechanical problems, but admits a considerable modification due to specific properties of quantum field theory. Such modification is based on further information of a general character about the sum of the series. In particular, the properties of renormalization invariance, which is lost in a finite order of the initial expansion, allow rearrangements of the perturbative series in terms of the invariant charge. In this case, the properties of the series change essentially. In distinction to the initial expression containing large logarithms, the expansion obtained within the renormalization-group method can be used for analyzing the ultraviolet region. However, in doing so, the analytic properties are lost, because the perturbative running coupling possesses unphysical singularities in the infrared region. The difficulty associated with these unphysical singularities is overcome in the APT approach. In a new modified perturbation theory the correct analytic properties, lost in the renormalization-group summation, are restored, and the property of renormalization invariance is preserved. In the analytic approach, processes with typical spacelike and timelike momenta are considered self-consistently. The expansion in APT is not a power series in the running coupling either in the spacelike or the timelike region, and its properties, compared to those of the initial perturbative expansion, change not only in the ultraviolet region but also in the infrared region. Thus, the analytic approach is the next step in the renormalization-group method: it modifies the perturbative expansion on the basis of general properties of the theory so that the new approximations reflect fundamental principles of the theory—renormalization invariance, spectrality, and causality.

[6]The K scheme [60] is interesting in that there is a fixed point for the three-loop running coupling ($C_{\rm K} \simeq 5.3$).

In this paper, a comparative analysis of the merits and drawbacks has been made for different forms of the perturbative expansions both in general aspects and in the context of the application to the inclusive decay of the τ-lepton. In the description of this process, the perturbative component is, despite the low-energy scale of the process, decisive. Also, arguments are given in favor of analytic perturbation theory, which is consistent with general principles of the theory. As we have emphasized, the decay of the τ-lepton can be described in terms of timelike or Euclidean variables. In the analytic approach, these two descriptions are equivalent. Moreover, in the analytic perturbation theory, the dependence of the calculated results on the choice of renormalization procedure is dramatically reduced. In the class of natural schemes, we can say that the results at third order are practically independent of the renormalization scheme. Calculations in the framework of analytic perturbation theory considerably reduce the theoretical uncertainty of the results. Use of analytic perturbation theory to describe the perturbative component of the process should increase the reliability in obtaining information about nonperturbative effects. The above properties of analytic perturbation theory are attractive also for comparison of theoretical results with experimental data on τ-decay, known at present with high accuracy.

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FIGURES

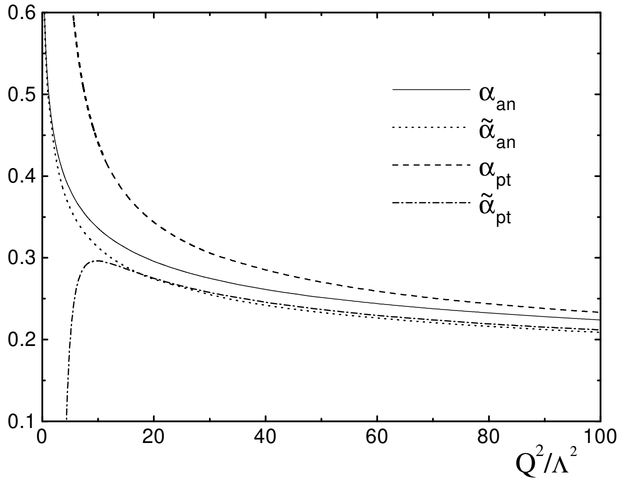

FIG. 1. *Comparison of the three-loop running coupling in the PT and APT approaches. Here the solid line shows the analytic coupling in the spacelike region and the dotted line the analytic coupling in the timelike region. The dashed line gives the perturbative coupling, while the dotted-dashed line shows the effect of including the* π^2 *correction in the PT approach.*

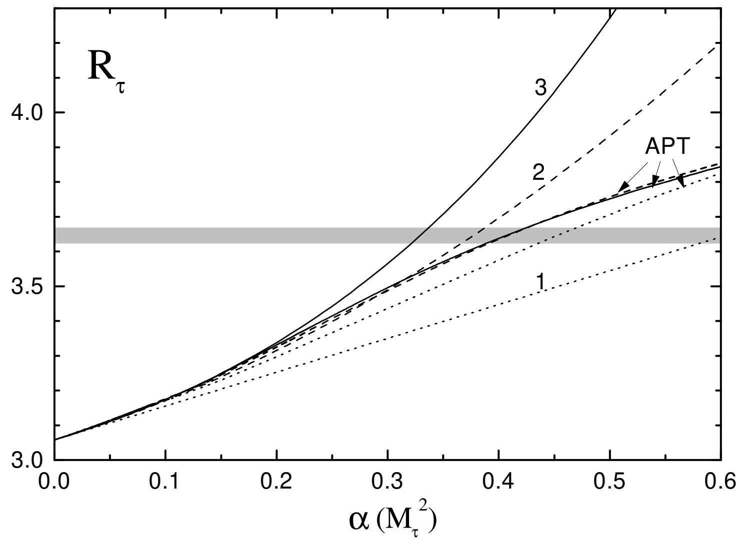

FIG. 2. *The PT and APT predictions for the* R_τ *ratio vs. the running coupling in the* $\overline{\mathrm{MS}}$ *scheme. The numbers labelling the curves denote the level of the loop expansion used.*

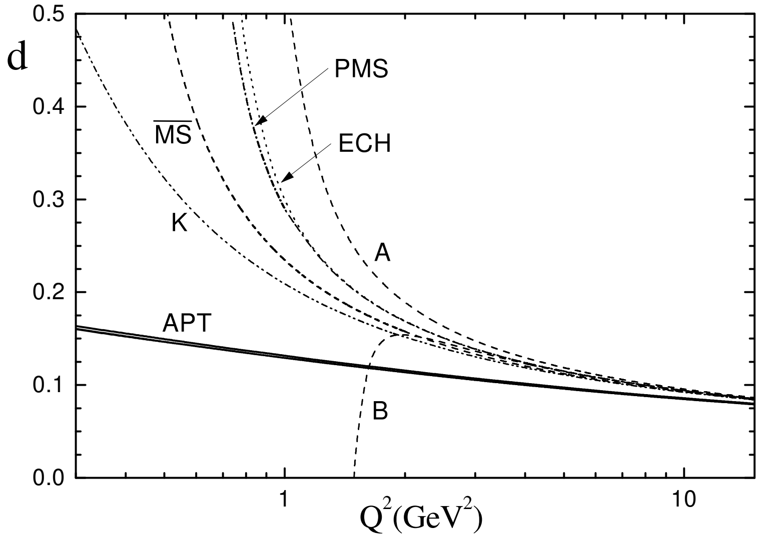

FIG. 3. *Renormalization scheme dependence of the d-function as a function of Q^2 for the PT and APT approaches. The APT results are shown as solid lines which are very close to each other and practically merge into one curve.*