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Dual Quantum Electrodynamics: Dyon-Dyon and Charge-Monopole Scattering in a High-Energy Approximation

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We develop the quantum field theory of electron-point magnetic monopole interactions and more generally, dyon-dyon interactions, based on the original string-dependent "nonlocal" action of Dirac and Schwinger. We demonstrate that a viable nonperturbative quantum field theoretic formulation can be constructed that results in a string *independent* cross section for monopole-electron and dyon-dyon scattering. Such calculations can be done only by using nonperturbative approximations such as the eikonal and not by some mutilation of lowest-order perturbation theory.

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I. INTRODUCTION

The subject of magnetic charge in quantum mechanics has been of great interest since the work of Dirac [1], who showed that electric and magnetic charge could co-exist provided the quantization condition (in rationalized units)

$$\frac{eg}{4\pi} = \frac{N}{2} \quad , \quad N \in Z \,, \tag{1.1}$$

holds, where e and g are the electric and magnetic charges, respectively, Z being the integers. In addition, the existence of topological or "extended" magnetic charge has been demonstrated in non-Abelian gauge theories [2], where they can have profound effects, most notably in the infrared regime of the QCD vacuum. For example, in the illustrative scenario of color confinement proposed by Mandelstam and 't Hooft [3] it is conjectured that the QCD vacuum behaves as a *dual* type II superconductor. Due to the condensation of magnetic monopoles which emerge from the *Abelian* projection [4] of the SU(3) color gauge group, the chromo-electric field acting between $q\bar{q}$ pairs is squeezed into dual Abrikosov flux tubes [5]. Very recent studies suggest that these flux tubes may determine the infrared properties of the gluon propagator [6]. Also it has been suggested [7] that color monopoles are sufficient to trigger the spontaneous breakdown of chiral symmetry.

The monopoles most commonly considered today as being potentially observable are those associated with the grand-unification symmetry breaking scale, monopoles having masses of order 10^{16} GeV; however, much lower symmetry-breaking scales could be relevant, giving rise to monopoles of mass of order 10 TeV [8]. [It is worthwhile mentioning that the previously-believed prohibition (*cf.* Ref. [9]) against topological monopoles in electroweak theory is apparently too restrictive [10], thus raising the possibility of the existence of monopoles at the electroweak-symmetry breaking scale.] Further, there is no reason why "elementary" Dirac monopoles should not exist; only experiment can settle this question.

Since the early 1970s there have been many experimental searches for magnetic monopoles, ranging from seeking cosmological bounds to studies of lunar samples [11]. Although none of these searches ultimately has yielded a positive signal, the arguments in favor of the existence of magnetic charge, whether elementary or "extended," remain as cogent as ever. However, the quantum field theory of elementary or pointlike magnetic charges remains poorly developed, particularly at the phenomenological level. In view of the necessity of establishing a reliable estimate for monopole production in accelerators, for example, in order to set bounds on monopole masses in terrestrial experiments [12], it is important to put the theory on a firmer foundation. (This is especially so for limits set through virtual-monopole processes [13]. For a critique of such limits see Ref. [14].) It is the purpose of this paper to initiate a complete study of monopole scattering and production in relativistic quantum electrodynamics extended to include point magnetic charges, "dual QED." Given ongoing experiments it is important to obtain reliable calculations of scattering processes.

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From the work of Dirac on the nonrelativistic and the relativistic quantum mechanics of magnetic monopoles [1,15], and the subsequent work of Schwinger [16–18] and later Zwanziger [19] on relativistic quantum field theories of magnetic monopoles, it is known that it is not possible to write down a theory of pointlike electric and magnetic currents interacting via the electromagnetic field described solely in terms of a local vector potential $A_\mu(x)$. Consistency of the Maxwell equations

$$\partial^\nu F_{\mu\nu} = j_\mu \quad \text{and} \quad \partial^\nu\, {}^*F_{\mu\nu} = {}^*j_\mu\ , \tag{1.2}$$

where ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}$, which imply the dual conservation of electric and magnetic currents, j_μ and ${}^*j_\mu$, respectively,

$$\partial_\mu j^\mu = 0 \quad \text{and} \quad \partial_\mu\, {}^*j^\mu = 0\,, \tag{1.3}$$

necessitates the introduction of the Dirac string [1,16] or a multi-valued potential [20]. In this paper we will follow the string-dependent formulation, where the singular electromagnetic potential is consistent with nonrelativistic quantum mechanics provided the Dirac quantization condition (1.1) holds, or, for particles (labeled by a, b, etc.) carrying both electric and magnetic charge (dyons), the Schwinger generalization

$$\frac{e_a g_b - e_b g_a}{4\pi} = \left\{ \begin{array}{l} \frac{N}{2}\,, \text{unsymmetric} \\ N\,, \text{symmetric} \end{array} \right\}, \quad N \in Z \tag{1.4}$$

is invoked. ("Symmetric" and "unsymmetric" refer to the presence or absence of dual symmetry in the solutions of Maxwell's equations.) That is, consistency conditions, Eqs. (1.1) and (1.4), must necessarily be satisfied if physical observables are to be rendered independent of a gauge artifact, the Dirac string singularity. *Formally,* rotational invariance of nonrelativistic quantum mechanics with monopoles follows directly from the Schrödinger equation where re-orienting the string is equivalent to a gauge transformation on the wavefunction which in turn is well-defined given (1.1) or (1.4). In addition, in early work on the nonrelativistic electron-monopole scattering by Goldhaber [21], Schwinger *et al.* [22], and Milton and DeRaad [23], not surprisingly, it was found that the resulting scattering cross section is gauge-string-independent if and only if the quantization condition is satisfied.[1]

On the other hand, attempts to incorporate monopoles consistently into relativistic quantum field theory have met with mixed success. Weinberg [25] and, somehwat thereafter, Rabl [26] demonstrated that electron–monopole scattering calculated in the one-photon-exchange approximation was described by a string-dependent scattering amplitude. A straightforward calculation begins, for example, with the interaction given by Schwinger [18,27]

$$W(j, {}^*j) = \int (dx)(dx')(dx'')\, {}^*j^\mu(x)\epsilon_{\mu\nu\sigma\tau}\partial^\nu f^\sigma\,(x-x')\, D_+\,(x'-x'')\, j^\tau\,(x'')\ . \tag{1.5}$$

Here the electric and magnetic currents are $j_\mu = e\bar\psi\gamma_\mu\psi$ and ${}^*j_\mu = g\bar\chi\gamma_\mu\chi$, for example, for spin-1/2 particles. The photon propagator is denoted by $D_+(x-x')$ and $f_\mu(x)$ is the Dirac string function which satisfies the differential equation

$$\partial_\mu f^\mu(x) = \delta(x)\,, \tag{1.6}$$

a formal solution of which is given by

$$f^\mu(x) = n^\mu\,(n\cdot\partial)^{-1}\,\delta(x)\,, \tag{1.7}$$

where n^μ is an arbitrary vector. Therefore, the lowest-order scattering amplitude, representing, for example, the interaction of a spin-1/2 electron with a spin-0 monopole is

$$T^{ss'} = 2ieg\left.\frac{(2\pi)^4\delta\,(p-p'+k-k')\,\epsilon_{\mu\nu\lambda\sigma}n^\nu\bar v^{s'}\,(p')\,\gamma^\mu u^s(p)q^\lambda\,(k+k')^\sigma}{\sqrt{2E_p 2E_{p'}\,2\omega_k 2\omega_{k'}}\,(q^2-i\epsilon)\,(n\cdot q+i\delta)}\right|_{q=p-p'}\ . \tag{1.8}$$

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[1] Upon separation of variables, the angular differential operator contains a contribution from the "intrinsic" field angular momentum carried by the interaction of the monopole with the photon field, which is naturally quantized upon using the Dirac-Schwinger condition (1.4); that is, there is an additional effective magnetic quantum number $m' = \frac{e_a g_b - e_a g_b}{4\pi}$, a fact already implicit in the classical analysis of Poincaré and Thomson [24].

Here the incoming momenta are p, k, and the outgoing momenta are p', k', respectively, while the initial and final electron spin projections are s and s', and where the different symbols for the energies of the electron and monopole, E and ω, respectively, refer to the different masses of the corresponding particles. Further, note that in this paper we use a metric with signature $(-,+,+,+)$, so that $q^2 > 0$ represents a spacelike momentum transfer. The explicit dependence on the covariant string vector, n_μ, *does not disappear* upon squaring the amplitude and multiplying by the appropriate phase space factors. To make matters worse, the value of the magnetic charge implied by (1.1), $\alpha_g = g^2/4\pi \approx 34N^2$, calls into question any approach based on a badly divergent perturbative expansion in α_g. Although these earliest efforts using a Feynman-rule perturbation theory in one of a number of approaches to dual QED (see below) resulted in string-dependent cross–sections [25,26], subsequently *ad hoc* assumptions were invoked to render the resulting cross–sections string independent (*e.g.* Refs. [28] and [29]).

Recently, in the context of calculating virtual monopole processes, De Rújula [30] has advocated the notion that single photon-mediated monopole-antimonopole Drell-Yan production amplitudes are rendered string independent through the implausible argument of Deans [28], which amounts to dropping the pole terms in the photon propagator![2] Further, in a series of papers, Ignatiev and Joshi [31], while arguing that the prescription of Deans lacks believability, propose (as did Rabl [26]) averaging the scattering amplitude over all possible directions of the string in order to eliminate the string dependence. In contrast to Deans, however, they arrive at a null result for the amplitude in lowest order. Consequently, they argue that general principles of quantum field theory which do not rely upon the use of perturbation theory must be invoked; that is, they consider the constraints placed upon the scattering amplitude due to the requirements of discrete C and P symmetries (see also Tolkachev et al. [32]).

In contrast, by studying the *formal* behavior of Green's functions in the relativistic quantum field theory of electrons and monopoles, both Schwinger [18,27] and Brandt, Neri and Zwanziger [33] demonstrated Lorentz and gauge invariance. In essence these demonstrations are nonperturbative and rely on the use of pointlike particle trajectories for the matter fields. In the Schwinger approach, however, classical particle currents

$$\begin{aligned} j^\mu(x) &= \sum_e e \int dx_e^\mu \delta\left(x - x_e\right) \\ {}^*j^\mu(x) &= \sum_g g \int dx_g^\mu \delta\left(x - x_g\right) \end{aligned} \tag{1.9}$$

are substituted for the field theoretic sources, whereupon a change in the string trajectory gives rise to a change in the action which is a multiple of 2π because of the quantization conditions (1.1) or (1.4). In the second approach, putting aside the issue of renormalization, Brandt *et al.* succeed in demonstrating the string independence of correlation functions of gauge- and Lorentz-invariant quantities using a functional path integral formalism, once again as a consequence of Eqs. (1.1) and (1.4), by converting the path integral over fields to one over closed particle trajectories (see Refs. [34–36]).

However, while Lorentz invariance and string independence were *formally* demonstrated in dual QED (albeit with the above caveats), such demonstrations have been conspicuously absent in practical calculations.[3] This deficiency stems from the fact that in most phenomenological treatments of electron-monopole processes the "string independence" of the quantum field theory and the strength of the coupling are treated as separate issues. This viewpoint (see *e.g.* Refs. [31,29]) could not be more misleading. In fact, these two points are intimately related.[4] If any lesson is to be learned from the above-mentioned demonstrations of Lorentz and string invariance, it is this: The quantization conditions (1.1) and (1.4), being intimately tied to the demonstration of Lorentz invariance, are intrinsically nonperturbative statements (and for that matter are obtained independently of any perturbative approximation). As such, attempts to demonstrate Lorentz invariance, whether formally or phenomenologically, which are based on a perturbative expansion in the coupling will most assuredly fail. While this is not a novel viewpoint (see Refs. [18,33]), it seems to have been overlooked in the more recent studies of point monopole processes (see also Ref. [29]).[5]

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[2] Essentially it amounts to discarding string-dependent contributions on the basis of the argument that they cannot contribute to any gauge invariant quantity if the theory is to be viable.

[3] This is surprising because one expects that the invariant nonrelativistic scattering result (see Ref. [22] and references therein) corresponds in a certain kinematic regime to a infinite summation of a particular *subclass* of Feynman diagrams. Moreover, the relativistic Dirac equation also yields a string-independent cross section [37].

[4] In this regard we remind the reader of the following viewpoint of Schwinger: "Relativistic invariance will appear to be violated in any treatment based on a perturbation expansion. Field theory is more than a set of 'Feynman's rules'." [16]

[5] A very recent note [38] exemplifying the failure to connect the issue of the strength of the magnetic coupling with that of

In fact, there exists one instance in the literature of a successful calculation demonstrating string independence for a relativistic scattering cross section within dual QED. Utilizing Schwinger's functional source theory [39,18] in the context of a high energy, low momentum transfer (zeroth order eikonal) approximation (*e.g.* Refs. [40–44] and references therein), Urrutia [45] demonstrated string independence of the charge-monopole scattering cross section, although again in his treatment the currents were approximated by those of classical point particles as in Eq. (1.9).

The outline of this paper is as follows. In Section II we establish the string-dependent dual-electrodynamic action. In Section III we quantize the dual potential action by solving the Schwinger-Dyson equations for the vacuum persistence amplitude for electron-monopole (and dyon-dyon) processes. Then, by taking the functional Fourier transform of this solution, we are led unambiguously to the dual QED functional path integral that is compatible with the source theory formalism developed by Schwinger [18,27]. In the process of this development, we enforce a gauge fixing condition that suggests that using the Dirac string formulation is not only entirely natural and consistent with the underlying gauge symmetry of the charge-monopole system, but in fact is preferable to the multi-valued potential of Ref. [20]. Having accomplished this, in Section IV we calculate the dyon-dyon scattering cross section in the context of this relativistic *string dependent* version of dual QED and demonstrate that a nonperturbative approach in conjunction with (1.1) or (1.4) is in fact necessary to demonstrate Lorentz and gauge (string) invariance of physical observables. To accomplish this we use the nonperturbative functional field-theory formalism of Schwinger [35] and Symanzik [46].[6] Within this formulation we are able to generalize the string-independent eikonal result of Urrutia [45] while treating the currents as constructed from quantum fields by invoking the quantization condition (1.4). Finally in Section V we draw some conclusions concerning the phenomenology of dual QED and outline our continuing efforts in this direction while in addition proposing future work. Appendices deal with some aspects of the path-integral formalism in dual QED.

II. DUAL ELECTRODYNAMICS

Decades after Dirac's early quantum-mechanical treatment (both nonrelativistic [1] and relativistic [15]) of the electron-monopole system, attempts to establish a consistent interacting quantum field theory of (point) electrons and monopoles were carried out in two different approaches by Schwinger [16] and by Zwanziger [19].

Schwinger's approach is based upon the construction of a *nonlocal* Hamiltonian which is a function of two transverse, string-dependent potentials, $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{B}^{\mathrm{T}}$, in addition to, say, spin-1/2 electron and monopole fields. Second quantization is established by postulating the commutation rules between the fields and their conjugate field momenta. In this formulation the two gauge fields have a nonvanishing commutator, and in fact are not *linearly independent.* Although this operator method is noncovariant, it has the merit of explicitly displaying the dual symmetry between the Dirac equations for the spin-$\frac{1}{2}$ electron and monopole fields, as well as of the Maxwell equations for the magnetic and electric fields. In turn Schwinger [18,27] developed and advocated a covariant formalism derived from his source theory approach to quantum field theory. This *first-order* formulation of dual QED is written in terms of the field strength tensor $F_{\mu\nu}$ and one independent vector potential A_μ. Alternatively, one can trade the dependence on the field strength for that on an auxiliary dual vector potential B_μ. The latter field is introduced to identify unambiguously the photon field coupling to the dual monopole current, $^*j$, through the effective "skeletal" action (1.5), coupling electric and magnetic currents. This latter formulation proves helpful in calculating scattering processes.

A second *local* covariant Lagrangian formulation approach, developed by Zwanziger, utilizes two four-potentials A_μ and B_μ, which again are not independent. In addition, this formalism satisfies duality symmetry both at the level of the action as well as between the equations of motion for A_μ and B_μ and between the equations for the spin-$\frac{1}{2}$ electron and monopole fields. The equivalence between these formalisms is shown in Refs. [48,49].

However, the former covariant formalism of Schwinger is a natural generalization of Dirac's (relativistic) quantum mechanics in the form of a covariant Lagrangian field theoretic description of electron-monopole interactions. In this context it should come as no surprise that Schwinger's source theory approach [27] laid the groundwork for the first consistent, gauge-invariant nonperturbative calculation of electron-monopole scattering [45].

We also note that a "one potential" Hamiltonian formulation was developed later by Blagojević [49] where a *explicit* set of Feynman rules was displayed. Finally we mention in passing the work of Deans [28], where a formalism based on Dirac's monopole theory was quantized within the framework of Mandelstam's gauge-invariant quantum field theory [50]. In Deans' paper, while a set of *string-dependent* Feynman rules was developed, they were used without

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string independence deduces the inverse renormalization of electric and magnetic charges in contradistinction to the identity of these renormalizations shown by Schwinger [17].

[6]For and excellent review of these techniques see [47].

criticism of the badly divergent perturbation series; in addition, as we have seen, Lorentz-frame (string) dependence is argued away in a highly questionable fashion.

A. The "Dual-Potential" String Dependent Action

While the completely general formalism for charge-monopole quantum field theory was developed by Schwinger in the context of source theory, for the sake of accessibility here we develop a string-dependent dual QED of monopole-electron interactions in a more familiar functional formalism.[7]

In order to facilitate the construction of the dual-QED formalism we recognize that the well-known continuous global U(1) *dual* symmetry [16,18,27] implied by Eqs. (1.2), (1.3), given by

$$\begin{pmatrix} j' \\ {}^*j' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} j \\ {}^*j \end{pmatrix}, \tag{2.1a}$$

$$\begin{pmatrix} F' \\ {}^*F' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} F \\ {}^*F \end{pmatrix}, \tag{2.1b}$$

suggests the introduction of an auxiliary vector potential $B_\mu(x)$ dual to $A_\mu(x)$. In order to satisfy the Maxwell and charge conservation equations, Dirac modified the field strength tensor according to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + {}^*G_{\mu\nu}\,, \tag{2.2}$$

where now Eqs. (1.2) and (1.3) give rise to the consistency condition on $G_{\mu\nu}(x) = -G_{\nu\mu}(x)$

$$\partial^\nu\, {}^*F_{\mu\nu} = -\partial^\nu G_{\mu\nu} = {}^*j_\mu\,. \tag{2.3}$$

We then obtain the following inhomogeneous solution to the dual Maxwell's equation (2.3) for the tensor $G_{\mu\nu}(x)$ in terms of the string function f_μ and the magnetic current, which for a spin-1/2 monopole represented by a Dirac field χ is ${}^*j^\mu(x) = g\bar\chi(x)\gamma^\mu\chi(x)$:

$$\begin{aligned} G_{\mu\nu}(x) &= (n\cdot\partial)^{-1}\left(n^\mu\, {}^*j^\nu(x) - n^\nu\, {}^*j^\mu(x)\right) \\ &= \int (dy)\left(f_\mu(x-y)\, {}^*j_\nu(y) - f_\nu(x-y)\, {}^*j_\mu(y)\right)\,, \end{aligned} \tag{2.4}$$

where use is made of Eqs. (1.3), (1.6), and (1.7). A minimal generalization of the QED Lagrangian including electron-monopole interactions reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi\left(i\gamma\partial + e\gamma A - m_\psi\right)\psi + \bar\chi\left(i\gamma\partial - m_\chi\right)\chi\,, \tag{2.5}$$

where the coupling of the monopole field $\chi(x)$ to the electromagnetic field occurs through the quadratic field strength term according to Eq. (2.2). We now rewrite the Lagrangian (2.5) to display more clearly that interaction by introducing the auxiliary potential $B_\mu(x)$.

Variation of Eq. (2.5) with respect to the field variables, ψ, χ and A_μ, yields in addition to the Maxwell equations for the field strength,[8] $F_{\mu\nu}$, Eq. (1.2) where $j^\mu(x) = e\bar\psi(x)\gamma^\mu\psi(x)$, the equation of motion for the electron field

$$\left(i\gamma\partial + e\gamma A(x) - m_e\right)\psi(x) = 0\,, \tag{2.6}$$

and the nonlocal equation of motion for the monopole field,

$$\left(i\gamma\partial - m_\chi\right)\chi(x) - \frac{1}{2}\int (dy)\ {}^*F^{\mu\nu}(y)\frac{\delta G_{\mu\nu}(y)}{\delta\bar\chi(x)} = 0\,. \tag{2.7}$$

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[7] However, we emphasize that the source theory approach is in fact just that, the "source" of these ideas.

[8] We regard $G_{\mu\nu}(x)$ as dependent on $\bar\chi$, χ but not A_μ. Thus, the dual Maxwell equation is given by the subsidiary condition (2.3).

It is straightforward to see from the Dirac equation for the monopole (2.7) and the construction (2.4) that introducing the auxiliary dual field (which is a functional of $F_{\mu\nu}$ and depends on the string function f_μ)

$$B_\mu(x) = -\int (dy) f^\nu\,(x-y)\ {}^*F_{\mu\nu}(y)\,, \tag{2.8}$$

results in the following Dirac equation for the monopole field

$$(i\gamma\partial + g\gamma B(x) - m_g)\,\chi(x) = 0\,. \tag{2.9}$$

Here we have chosen the string to satisfy the oddness condition (this is the "symmetric" solution)

$$f^\mu(x) = -f^\mu(-x), \tag{2.10}$$

which is related to Schwinger's integer quantization condition [22,51]. Now (2.6) and (2.9) display the dual symmetry expressed in Maxwell's equations (1.2) and (1.3). Noting that B_μ satisfies

$$\int (dx') f^\mu(x-x') B_\mu(x') = 0\,, \tag{2.11}$$

we see that Eq. (2.8) is a gauge-fixed vector field [52,53] defined in terms of the field strength through an *inversion* formula (see Subsection III A). In terms of these fields the "dual-potential" action can be re-expressed in terms of the vector potential A_μ and field strength tensor $F_{\mu\nu}$ (where B_μ is the functional (2.8) of $F_{\mu\nu}$) in first-order formalism as

$$\begin{aligned} W = \int (dx) \Big\{ & -\frac{1}{2} F^{\mu\nu}(x)\,(\partial_\mu A_\nu\,(x) - \partial_\nu A_\mu\,(x)) + \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \\ & + \bar{\psi}(x)\,(i\gamma\partial + e\gamma A(x) - m_\psi)\,\psi(x) + \bar{\chi}(x)\,(i\gamma\partial + g\gamma B(x) - m_\chi)\,\chi(x) \Big\}\,, \end{aligned} \tag{2.12a}$$

or in terms of *dual* variables,

$$\begin{aligned} W = \int (dx) \Big\{ & -\frac{1}{2}\,{}^*F^{\mu\nu}(x)\,(\partial_\mu B_\nu\,(x) - \partial_\nu B_\mu\,(x)) + \frac{1}{4}\,{}^*F^{\mu\nu}(x)\,{}^*F_{\mu\nu}(x) \\ & + \bar{\psi}(x)\,(i\gamma\partial + e\gamma A(x) - m_\psi)\,\psi(x) + \bar{\chi}(x)\,(i\gamma\partial + g\gamma B(x) - m_\chi)\,\chi(x) \Big\}\,. \end{aligned} \tag{2.12b}$$

In Eq. (2.12a), $A_\mu(x)$ and $F_{\mu\nu}(x)$ are the independent field variables[9] and $B_\mu(x)$ is given by Eq. (2.8), while in Eq. (2.12b) the dual fields are the independent variables, in which case,

$$\begin{aligned} A_\mu(x) &= -\int (dy) f^\nu\,(x-y)\ \ F_{\mu\nu}(y) \\ &= \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} \int (dy) f^\nu(x-y)\ {}^*F^{\lambda\sigma}(y). \end{aligned} \tag{2.13}$$

[Note that Eq. (2.12b) may be obtained from the form (2.12a) by inserting Eq. (2.13) into the former and then identifying B_μ according to the construction (2.8). In this way the sign of $\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}{}^*F_{\mu\nu}{}^*F^{\mu\nu}$ is flipped.] Consequently, the field equation relating ${}^*F^{\mu\nu}$ and B^μ is

$${}^*F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - \int (dy)\,{}^*\,(f_\mu(x-y) j_\nu(y) - f_\nu(x-y) j_\mu(y))\,, \tag{2.14}$$

which is simply obtained from Eq. (2.2) by making the duality transformation $\mathcal{E} \to \mathcal{M}$, $\mathcal{M} \to -\mathcal{E}$, where $\mathcal{E}$ stands for any electric quantity and $\mathcal{M}$ for any magnetic quantity.

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[9]Using Eq. (2.8), variations of the action, Eq. (2.12a), with respect to $A_\mu(x)$ and $F_{\mu\nu}(x)$ yield Eqs. (1.2) and (2.2) where ${}^*G_{\mu\nu}(x)$ is the dual of Eq. (2.4).

III. QUANTIZATION OF DUAL QED: SCHWINGER-DYSON EQUATIONS

Although the various actions describing the interactions of point electric and magnetic poles can be described in terms of a set of Feynman rules which one conventionally uses in perturbative calculations, the large value of α_g or $eg/4\pi$ renders them useless for this purpose. In addition, as mentioned in Section I, calculations of physical processes using the perturbative approach from string-dependent actions such as Eqs. (2.12a) and (2.12b) have led only to string dependent results. In conjunction with a nonperturbative functional approach, however, the Feynman rules serve to elucidate the electron-monopole interactions. We express these interactions in terms of the "dual-potential" formalism as a quantum generalization of the relativistic classical theory of section II A. We use the Schwinger action principle [54] to quantize the electron-monopole system by solving the corresponding Schwinger-Dyson equations for the generating functional. Using a functional Fourier transform of this generating functional in terms of a path integral for the electron-monopole system, we rearrange the generating functional into a form that is well-suited for the purpose of nonperturbative calculations.

A. Gauge Symmetry

In order to construct the generating functional for Green's functions in the electron-monopole system we must restrict the gauge freedom resulting from the local gauge invariance of the action (2.12a). The *inversion* formulae for A_μ and B_μ, Eqs. (2.13) and (2.8) respectively, might suggest using the technique of gauge-fixed fields [52,50] as was adopted in [28]. However, we use the technique of gauge fixing according to methods outlined by Zumino [55] and generalized by Zinn-Justin [56] in the language of stochastic quantization.

The gauge fields are obtained in terms of the string and the gauge invariant field strength, by contracting the field strength (2.2), (2.4) with the Dirac string, $f^\mu(x)$, in conjunction with Eq. (1.6), yielding the following inversion formula for the equation of motion,

$$A_\mu(x) = -\int (dx')f^\nu(x-x')F_{\mu\nu}(x') + \partial_\mu\tilde{\Lambda}_e(x) \ , \tag{3.1}$$

where we use the suggestive notation, $\tilde{\Lambda}_e(x)$

$$\tilde{\Lambda}_e(x) = \int (dx')f^\nu(x-x')A_\nu(x') \ . \tag{3.2}$$

It is evident that Eq. (3.1) transforms consistently under gauge transformation

$$A_\nu(x) \longrightarrow A_\nu(x) + \partial_\nu\Lambda_e(x) \, , \tag{3.3}$$

while in addition we note that the Lagrangian (2.12a) is invariant under the gauge transformation,

$$\psi \rightarrow \exp\left[ie\Lambda_e\right]\psi \, , \quad A_\mu \rightarrow A_\mu + \partial_\mu\Lambda_e \, , \tag{3.4a}$$

as is the dual action (2.12b) under

$$\chi \rightarrow \exp\left[ig\Lambda_g\right]\chi \, , \quad B_\mu \rightarrow B_\mu + \partial_\mu\Lambda_g \, . \tag{3.4b}$$

Assuming the freedom to choose $\tilde{\Lambda}_e(x) = -\Lambda_e(x)$, we bring the vector potential into gauge-fixed form,[10] coinciding with Eq. (2.13),

$$A_\mu(x) = -\int (dy)f^\nu\,(x-y)\ \ F_{\mu\nu}(y) \tag{3.5}$$

[10] It is worth noting the similarity of this condition to the Schwinger-Fock gauge in ordinary QED, $x\cdot\mathcal{A}(x)=0$ which yields the gauge-fixed photon field, $\mathcal{A}_\mu(x) = -x^\nu\int_0^1 ds\, sF_{\mu\nu}(xs)$.

where the gauge choice is equivalent to a *string-gauge* condition[11]

$$\int (dx')f^\mu(x-x')A_\mu(x') = 0\,. \tag{3.7}$$

More generally, the fact that a gauge function exists, such that $\Lambda_e(x) = -\tilde{\Lambda}_e(x)$ [*cf.* Eq. (3.2)], implying that we have the freedom to consistently fix the gauge, is in fact not a trivial claim. If this were not true, it would certainly derail the consistency of incorporating monopoles into QED while utilizing the Dirac string formalism. On the contrary, the *string gauge condition*, Eq. (3.7), is in fact a class of possible consistent gauge conditions characterized by the symbolic operator function (1.7) depending on a unit vector n^μ (which may be either spacelike or timelike). In a similar manner, given the dual field strength (2.14) the dual vector potential takes the following form [*cf.* Eq. (2.8)]

$$B_\mu(x) = -\int (dx')f^\nu(x-x')\,{}^*F_{\mu\nu}(x') + \partial_\mu\tilde{\Lambda}_g\,, \tag{3.8}$$

where

$$\tilde{\Lambda}_g(x) = \int (dx')f^\mu(x-x')B_\mu(x')\,. \tag{3.9}$$

In order to quantize this system we must divide out the equivalence class of field values defined by a gauge trajectory in field space; in this sense the gauge condition restricts the vector potential to a hypersurface of field space which is embodied in the generalization of Eq. (3.7)

$$\int (dx')f^\mu(x-x')A_\mu(x') = \Lambda_e(x)\,, \tag{3.10}$$

where here Λ_e is any function defining a unique gauge fixing hypersurface in field space.[12]

In a path integral formalism, we enforce the condition (3.10) by introducing a δ function, symbolically written as

$$\delta(f^\mu A_\mu - \Lambda_e) = \int [d\lambda_e]\exp\left[i\int (dx)\lambda_e(x)\left(\int (dx')f^\mu(x-x')A_\mu(x') - \Lambda_e(x)\right)\right]\,, \tag{3.11}$$

or by introducing a Gaussian functional integral

$$\begin{aligned}\Phi(f^\mu A_\mu - \Lambda_e) = \int [d\lambda_e]\exp\bigg[&-\frac{i}{2}\int (dx)(dx')\lambda_e(x)M(x,x')\lambda_e(x') \\ &+i\int (dx)\lambda_e(x)\left(\int (dx')f^\mu(x-x')A_\mu(x') - \Lambda_e(x)\right)\bigg]\,,\end{aligned} \tag{3.12}$$

where the symmetric matrix $M(x,x') = \kappa^{-1}\delta(x-x')$ describes the spread of the integral $\int (dx')f^\mu(x-x')A_\mu(x')$ about the gauge function, $\Lambda_e(x)$. That is, we enforce the gauge fixing condition (3.10) by adding the quadratic form appearing here to the action (2.12a) and in turn eliminating λ_e by its "equation of motion"

$$\lambda_e(x) = \kappa\left(\int (dy)\,f^\mu(x-y)A_\mu(y) - \Lambda_e(x)\right)\,. \tag{3.13}$$

Now the equations of motion (1.2) take the form,

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[11]Taking the divergence of Eq. (3.5) and using Eq. (1.2), the gauge-fixed condition (3.5) can be written as

$$\partial_\mu \mathcal{A}^\mu = \int (dy)f^\mu\,(x-y)\,j_\mu(y)\,, \tag{3.6}$$

which is nothing other than the gauge-fixed condition of Zwanziger in the two-potential formalism [19].

[12]Choosing a different function Λ_e (which the gauge freedom permits us to do) merely yields a different section of field space under the restriction that it cut each equivalence class of field values once.

$$\partial^\nu F_{\mu\nu}(x) - \int (dx')\lambda_e(x') f_\mu(x'-x) = j_\mu(x), \tag{3.14a}$$

$$\partial^\nu\, {}^*F_{\mu\nu}(x) - \int (dx')\lambda_g(x') f_\mu(x'-x) = {}^*j_\mu(x). \tag{3.14b}$$

where the second equation refers to a similar gauge fixing in the dual sector. Taking the divergence of Eqs. (3.14a) implies $\lambda_e = 0$ from Eqs. (1.6) and (1.3), which consistently yields the gauge condition (3.10). Using our freedom to make a transformation to the gauge-fixed condition (3.5), $\Lambda_e = 0$, the equation of motion (3.14a) for the potential becomes

$$\left[- g_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu + \kappa\, n_\mu (n\cdot\partial)^{-2} n_\nu \right] A^\nu(x) = j_\mu(x) + \epsilon_{\mu\nu\sigma\tau}\frac{n^\nu}{(n\cdot\partial)}\partial^\sigma\, {}^*j^\tau(x)\,, \quad n^\mu A_\mu = 0, \tag{3.15}$$

where we now have used the symbolic form of the string function (1.7). We have retained the term proportional to $n_\mu n_\nu$ in the kernel, scaled by the arbitrary parameter κ,

$$K_{\mu\nu} = \left[- g_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu + \kappa\, n_\mu (n\cdot\partial)^{-2} n_\nu \right] \tag{3.16}$$

so that $K_{\mu\nu}$ possesses an inverse

$$D_{\mu\nu}(x) = \left[g_{\mu\nu} - \frac{n_\mu\partial_\nu + n_\nu\partial_\mu}{(n\cdot\partial)} + n^2\left(1 - \frac{1}{\kappa}\frac{(n\cdot\partial)^2\partial^2}{n^2}\right)\frac{\partial_\mu\partial_\nu}{(n\cdot\partial)^2}\right] D_+(x)\,, \tag{3.17}$$

that is, $\int (dx') K_{\mu\alpha}(x-x') D^{\alpha\nu}(x'-x'') = g_\mu^\nu \delta(x-x'')$, where $D_+(x)$ is the massless scalar propagator,

$$D_+(x) = \frac{1}{-\partial^2 - i\epsilon}\delta(x)\,. \tag{3.18}$$

This in turn enables us to rewrite Eq. (3.15) as an integral equation, expressing the vector potential in terms of the electron and monopole currents,

$$\begin{aligned} A_\mu(x) = &\int (dx') D_{\mu\nu}(x-x') j^\nu(x') \\ &+ \epsilon^{\nu\lambda\sigma\tau}\int (dx')(dx'') D_{\mu\nu}(x-x') f_\lambda(x'-x'')\partial''_\sigma\, {}^*j_\tau(x'')\,. \end{aligned} \tag{3.19}$$

The steps for $B_\mu(x)$ are analogous.

B. Vacuum Persistence Amplitude and the Path Integral

Given the gauge-fixed but string-dependent action we are prepared to quantize this theory of dual QED. Quantization using a path integral formulation of such a string dependent action is by no means straightforward; therefore we will develop the generating functional making use of a functional approach. Using the quantum action principle (*cf.* Ref. [54]) we write the generating functional for Green functions (or the vacuum persistence amplitude) in the presence of external sources $\mathcal{J}$

$$Z(\mathcal{J}) = \langle 0_+ \mid 0_- \rangle^{\mathcal{J}}\,, \tag{3.20}$$

for the electron-monopole system. Schwinger's action principle states that under an arbitrary variation

$$\delta\langle 0_+ \mid 0_- \rangle^{\mathcal{J}} = i\langle 0_+ | \delta W(\mathcal{J}) | 0_- \rangle^{\mathcal{J}}\,, \tag{3.21}$$

where $W(\mathcal{J})$ is the action given in Eq. (2.12a) externally driven by the sources, $\mathcal{J}$, which for the present case are given by the set $\{J, {}^*J, \bar\eta, \eta, \bar\xi, \xi\}$:

$$W(\mathcal{J}) = W + \int (dx) \left\{ J^\mu A_\mu + {}^*J^\mu B_\mu + \bar\eta\psi + \bar\psi\eta + \bar\xi\chi + \bar\chi\xi \right\}. \tag{3.22}$$

The one-point functions are then given by

$$\begin{aligned}
\frac{\delta}{i\delta J^\mu(x)} \log Z(\mathcal{J}) &= \frac{\langle 0_+|A_\mu(x)|0_-\rangle^{\mathcal{J}}}{\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}}, \qquad & \frac{\delta}{i\delta\, {}^*J^\mu(x)} \log Z(\mathcal{J}) &= \frac{\langle 0_+|B_\mu(x)|0_-\rangle^{\mathcal{J}}}{\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}}, \\
\frac{\delta}{i\delta\bar\eta(x)} \log Z(\mathcal{J}) &= \frac{\langle 0_+|\psi(x)|0_-\rangle^{\mathcal{J}}}{\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}}, \qquad & \frac{\delta}{i\delta\bar\xi(x)} \log Z(\mathcal{J}) &= \frac{\langle 0_+|\chi(x)|0_-\rangle^{\mathcal{J}}}{\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}}.
\end{aligned} \tag{3.23}$$

Using Eqs. (3.23) we can write down derivatives with respect to the charges[13] in terms of functional derivatives [57–59] with respect to the external sources;

$$\begin{aligned}
\frac{\partial}{\partial e}\langle 0_+|0_-\rangle^{\mathcal{J}} &= i\langle 0_+| \int (dx) j^\mu(x) A_\mu(x) |0_-\rangle^{\mathcal{J}} \\
&= -i \int (dx) \left(\frac{\delta}{\delta \tilde A_\mu(x)} \frac{\delta}{\delta J^\mu(x)} \right) \langle 0_+|0_-\rangle^{\mathcal{J}}, \\
\frac{\partial}{\partial g}\langle 0_+|0_-\rangle^{\mathcal{J}} &= i\langle 0_+| \int (dx)\, {}^*j^\mu(x) B_\mu(x) |0_-\rangle^{\mathcal{J}} \\
&= -i \int (dx) \left(\frac{\delta}{\delta \tilde B_\mu(x)} \frac{\delta}{\delta\, {}^*J^\mu(x)} \right) \langle 0_+|0_-\rangle^{\mathcal{J}}.
\end{aligned} \tag{3.24}$$

Here we have introduced an effective source to bring down the electron and monopole currents,

$$\frac{\delta}{\delta \tilde A_\mu} \equiv \frac{1}{i}\frac{\delta}{\delta\eta}\gamma^\mu \frac{\delta}{\delta\bar\eta}, \qquad \frac{\delta}{\delta \tilde B_\mu} \equiv \frac{1}{i}\frac{\delta}{\delta\xi}\gamma^\mu \frac{\delta}{\delta\bar\xi}. \tag{3.25}$$

These first order differential equations can be integrated with the result

$$\langle 0_+ \mid 0_-\rangle^{\mathcal{J}} = \exp\left[-ig \int (dx) \left(\frac{\delta}{\delta \tilde B_\nu(x)} \frac{\delta}{\delta\, {}^*J^\nu(x)} \right) - ie \int (dx) \left(\frac{\delta}{\delta \tilde A_\mu(x)} \frac{\delta}{\delta J^\mu(x)} \right) \right] \langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}}, \tag{3.26}$$

where $\langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}}$ is the vacuum amplitude in the absence of interactions. By construction, the vacuum amplitude and Green's functions for the coupled problem are determined by functional derivatives with respect to the external sources $\mathcal{J}$ of the uncoupled vacuum amplitude, where $\langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}}$ is the product of the separate amplitudes for the quantized electromagnetic and Dirac fields since they constitute completely independent systems in the absence of coupling, that is,

$$\langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}} = \langle 0_+ \mid 0_-\rangle_0^{(\bar\eta,\eta,\bar\xi,\xi)} \langle 0_+ \mid 0_-\rangle_0^{(J,\,{}^*J)}. \tag{3.27}$$

First we consider $\langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}}$ as a function of J and ${}^*J$

$$\frac{\delta}{i\delta J^\mu(x)} \langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}} = \langle 0_+|A_\mu(x)|0_-\rangle_0^{\mathcal{J}}. \tag{3.28}$$

Taking the matrix element of the integral equation (3.19) but now with external sources rather than dynamical currents we find

$$\langle 0_+|A_\mu(x)|0_-\rangle_0^{\mathcal{J}} = \int (dx') D_{\mu\nu}(x-x') \left(J^\nu(x') + \epsilon^{\nu\lambda\sigma\tau} \int (dx'') f_\lambda(x'-x'') \partial''_\sigma\, {}^*J_\tau(x'') \right) \langle 0_+|0_-\rangle_0^{\mathcal{J}}. \tag{3.29}$$

[13]Here we redefine the electric and magnetic currents $j \to ej$ and ${}^*j \to g\,{}^*j$. Note that the changes in the action due to induced changes in the fields vanish by virtue of the stationary principle.

Using Eq. (3.15) we arrive at the equivalent gauge-fixed functional equation,

$$\left[-g_{\mu\nu}\partial^2+\partial_\mu\partial_\nu+\kappa n_\mu(n\cdot\partial)^{-2}n_\nu\right]\frac{\delta}{i\delta J^\nu(x)}\langle 0_+\mid 0_-\rangle_0^{\mathcal{J}}$$
$$=\left(J_\mu(x)+\epsilon_{\mu\nu\sigma\tau}\int(dx')f^\nu(x-x')\partial'^\sigma\,{}^*J^\tau(x')\right)\langle 0_+\mid 0_-\rangle_0^{\mathcal{J}}\,, \tag{3.30}$$

which is subject to the gauge condition

$$n^\nu\frac{\delta}{\delta J^\nu}\langle 0_+|0_-\rangle_0^{\mathcal{J}}=0, \tag{3.31a}$$

or

$$\int(dx')f^\nu(x-x')\frac{\delta}{\delta J^\nu(x')}\langle 0_+|0_-\rangle_0^{\mathcal{J}}=0. \tag{3.31b}$$

In turn, from Eq. (3.26) we obtain the full functional equation for $\langle 0_+\mid 0_-\rangle^{\mathcal{J}}$:

$$\left[-g_{\mu\nu}\partial^2+\partial_\mu\partial_\nu+\kappa\, n_\mu(n\cdot\partial)^{-2}n_\nu\right]\frac{\delta}{i\delta J^\nu(x)}\langle 0_+\mid 0_-\rangle^{\mathcal{J}}$$
$$=\exp\left[-ig\int(dy)\left(\frac{\delta}{\delta\tilde{B}_\alpha(y)}\frac{\delta}{\delta\,{}^*J^\alpha(y)}\right)-ie\int(dy)\left(\frac{\delta}{\delta\tilde{A}_\alpha(y)}\frac{\delta}{\delta J^\alpha(y)}\right)\right]$$
$$\times\left(J_\mu(x)+\epsilon_{\mu\nu\sigma\tau}\int(dx')f^\nu(x-x')\partial'^\sigma\,{}^*J^\tau(x')\right)\langle 0_+\mid 0_-\rangle_0^{\mathcal{J}}\,. \tag{3.32}$$

Commuting the external currents to the left of the exponential on the right side of Eq. (3.32) and using Eqs. (3.23), we are led to the Schwinger-Dyson equation for the vacuum amplitude, where we have restored the meaning of the functional derivatives with respect to $\tilde{A}$, $\tilde{B}$ given in Eq. (3.25),

$$\Bigg\{\left[-g_{\mu\nu}\partial^2+\partial_\mu\partial_\nu+\kappa n_\mu(n\cdot\partial)^{-2}n_\nu\right]\frac{\delta}{i\delta J_\nu(x)}$$
$$-e\frac{\delta}{i\delta\eta(x)}\gamma_\mu\frac{\delta}{i\delta\bar{\eta}(x)}-\epsilon_{\mu\nu\sigma\tau}\int(dx')f^\nu(x-x')\partial'^\sigma g\frac{\delta}{i\delta\xi(x')}\gamma^\tau\frac{\delta}{i\delta\bar{\xi}(x')}\Bigg\}\langle 0_+\mid 0_-\rangle^{\mathcal{J}}$$
$$=\left(J_\mu(x)+\epsilon_{\mu\nu\sigma\tau}\int(dx')f^\nu(x-x')\partial'^{\sigma*}J^\tau(x')\right)\langle 0_+\mid 0_-\rangle^{\mathcal{J}}\,. \tag{3.33}$$

In an analogous manner, using

$$\frac{\delta}{i\delta\,{}^*J^\mu(x)}\langle 0_+\mid 0_-\rangle_0^{\mathcal{J}}=\langle 0_+|B_\mu(x)|0_-\rangle_0^{\mathcal{J}}\,, \tag{3.34}$$

we obtain the functional equation (which is consistent with duality)

$$\Bigg\{\left[-g_{\mu\nu}\partial^2+\partial_\mu\partial_\nu+\kappa n_\mu(n\cdot\partial)^{-2}n_\nu\right]\frac{\delta}{i\delta\,{}^*J_\nu(x)}$$
$$-g\frac{\delta}{i\delta\xi(x)}\gamma_\mu\frac{\delta}{i\delta\bar{\xi}(x)}+\epsilon_{\mu\nu\sigma\tau}\int(dx')f^\nu(x-x')\partial'^\sigma e\frac{\delta}{i\delta\eta(x')}\gamma^\tau\frac{\delta}{i\delta\bar{\eta}(x')}\Bigg\}\langle 0_+\mid 0_-\rangle^{\mathcal{J}}$$
$$=\left({}^*J_\mu(x)-\epsilon_{\mu\nu\sigma\tau}\int(dx')f^\nu(x-x')\partial'^\sigma J^\tau(x')\right)\langle 0_+\mid 0_-\rangle^{\mathcal{J}}\,, \tag{3.35}$$

which is subject to the gauge condition

$$\int(dx')f^\mu(x-x')\frac{\delta}{\delta^*J_\mu(x')}\langle 0_+|0_-\rangle^{\mathcal{J}}=0. \tag{3.36}$$

In a straightforward manner we obtain the functional Dirac equations

$$\left\{i\gamma\partial + e\gamma^\mu \frac{\delta}{i\delta J^\mu(x)} - m_\psi\right\} \frac{\delta}{i\delta\bar\eta(x)}\langle 0_+ \mid 0_-\rangle^{\mathcal{J}} = -\eta(x)\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}\,, \tag{3.37a}$$

$$\left\{i\gamma\partial + g\gamma^\mu \frac{\delta}{i\delta\, {}^*J^\mu(x)} - m_\chi\right\} \frac{\delta}{i\delta\bar\xi(x)}\langle 0_+ \mid 0_-\rangle^{\mathcal{J}} = -\xi(x)\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}\,. \tag{3.37b}$$

In order to obtain a generating functional for Green's functions we must solve the set of equations (3.33), (3.35), (3.37a), (3.37b) subject to Eqs. (3.31b) and (3.36) for $\langle 0_+ \mid 0_-\rangle^{\mathcal{J}}$. In the absence of interactions, we can immediately integrate the Schwinger-Dyson equations; in particular, (3.35) then integrates to

$$\begin{aligned}\langle 0_+ \mid 0_-\rangle_0^{\mathcal{J}} = \mathcal{N}(J) \exp\Bigg\{ &\frac{i}{2}\int (dx)(dx')\, {}^*J_\mu(x)\, D^{\mu\nu}(x-x')\, {}^*J_\nu(x') \\ &+ i\epsilon_{\mu\nu\sigma\tau}\int (dx)(dx')(dx'')\, {}^*J_\beta(x) D^{\beta\mu}(x-x')\partial'^\nu f^\sigma(x'-x'')J^\tau(x'')\Bigg\}\,.\end{aligned} \tag{3.38}$$

We determine $\mathcal{N}$, which depends only on J, by inserting Eq. (3.38) into Eq. (3.33) or (3.30):

$$\ln \mathcal{N}(J) = \frac{i}{2}\int (dx)(dx') J_\mu(x) D^{\mu\nu}(x-x') J_\nu(x'), \tag{3.39}$$

resulting in the generating functional for the photonic sector

$$\begin{aligned}\langle 0_+ \mid 0_-\rangle_0^{(J,\,{}^*J)} = \exp\Bigg\{ &\frac{i}{2}\int (dx)(dx') J_\mu(x) D^{\mu\nu}(x-x') J_\nu(x') \\ &+ \frac{i}{2}\int (dx)(dx')\, {}^*J_\mu(x) D^{\mu\nu}(x-x')\, {}^*J_\nu(x') \\ &- i\int (dx)(dx') J_\mu(x)\tilde{D}^{\mu\nu}(x-x')\, {}^*J_\nu(x'')\Bigg\}\,,\end{aligned} \tag{3.40}$$

where we use the shorthand notation for the "dual propagator" that couples magnetic to electric charge

$$\tilde{D}_{\mu\nu}(x-x') = \epsilon_{\mu\nu\sigma\tau}\int (dx'') D_+(x-x'')\, \partial''^\sigma f^\tau(x''-x')\,. \tag{3.41}$$

The term coupling electric and magnetic sources has the same form as in Eq. (1.5); here, we have replaced $D^{\kappa\mu} \to g^{\kappa\mu}D_+$, as we may because of the appearance of the Levi-Civita symbol in Eq. (3.41). In an even more straightforward manner Eqs. (3.37a), (3.37b) integrate to

$$\langle 0_+ \mid 0_-\rangle_0^{(\bar\eta,\eta,\bar\xi,\xi)} = \exp\left\{ i\int (dx)(dx')\left[\bar\eta(x) G_\psi(x-x')\eta(x') + \bar\xi(x) G_\chi(x-x')\xi(x')\right]\right\}\,, \tag{3.42}$$

where G_ψ and G_χ are the free propagators for the electrically and magnetically charged fermions, respectively,

$$\begin{aligned} G_\psi(x) &= \frac{1}{-i\gamma\partial + m_\psi}\delta(x)\,, \\ G_\chi(x) &= \frac{1}{-i\gamma\partial + m_\chi}\delta(x)\,. \end{aligned} \tag{3.43}$$

In the presence of interactions the coupled equations (3.33), (3.35), (3.37a), (3.37b) are solved by substituting Eqs. (3.40) and (3.42) into Eq. (3.26). The resulting generating function is

$$Z(\mathcal{J}) = \exp\left(-ie\int (dx) \frac{\delta}{\delta\eta(x)}\gamma^\mu \frac{\delta}{i\delta J^\mu(x)}\frac{\delta}{\delta\bar\eta(x)}\right) \exp\left(-ig\int (dy)\frac{\delta}{\delta\xi(y)}\gamma^\mu\frac{\delta}{i\delta\, {}^*J^\mu(y)}\frac{\delta}{\delta\bar\xi(y)}\right) Z_0(\mathcal{J})\,. \tag{3.44}$$

C. Nonperturbative Generating Functional

Due to the fact that any expansion in α_g or eg is not practically useful we recast the generating functional (3.44) into a functional form better suited for a nonperturbative calculation of the four-point Green's function.

First we utilize the well-known Gaussian combinatoric relation [46,47]; moving the exponentials containing the interaction vertices in terms of functional derivatives with respect to fermion sources past the free fermion propagators, we obtain (coordinate labels are now suppressed)

$$Z(\mathcal{J}) = \exp\left\{ i\int \bar{\eta}\left(G_\psi \left[1 - e\gamma\cdot\frac{\delta}{i\delta J}G_\psi\right]^{-1}\right)\eta + \mathrm{Tr}\ln\left(1 - e\gamma\cdot\frac{\delta}{i\delta J}G_\psi\right)\right\}$$
$$\times \exp\left\{ i\int \bar{\xi}\left(G_\chi \left[1 - g\gamma\cdot\frac{\delta}{i\delta {}^*J}G_\chi\right]^{-1}\right)\xi + \mathrm{Tr}\ln\left(1 - g\gamma\cdot\frac{\delta}{i\delta {}^*J}G_\chi\right)\right\} Z_0(J, {}^*J)\,. \tag{3.45}$$

Now, we re-express Eq. (3.40), the non-interacting part of the generating functional of the photonic action, $Z_0(J, {}^*J)$, using a functional Fourier transform,

$$Z_0(J, {}^*J) = \int [dA]\,[dB]\,\tilde{Z}_0\,(A, B) \exp\left[i\int (J\cdot A + {}^*J\cdot B)\right]\,, \tag{3.46}$$

or

$$Z_0(J, {}^*J) = \int\, [dA]\,[dB] \exp\left(i\Gamma_0[A, B, J, {}^*J]\right)\,, \tag{3.47}$$

where (using a matrix notation for integration over coordinates)

$$\Gamma_0[A, B, J, {}^*J] = \int (J\cdot A + {}^*J\cdot B) - \frac{1}{2}\int A^\mu K_{\mu\nu} A^\nu + \frac{1}{2}\int B'^\mu \tilde{\Delta}^{-1}_{\mu\nu} B'^\nu \tag{3.48}$$

with the abbreviation

$$B'_\mu(x) = B_\mu(x) - \epsilon_{\mu\nu\sigma\tau}\int (dx')\partial^\nu f^\sigma(x - x')A^\tau(x') \tag{3.49}$$

and the string-dependent "correlator"

$$\tilde{\Delta}_{\mu\nu}(x - x') = \int (dx'')\left\{ f^\sigma(x - x'')f_\sigma\,(x'' - x')\,g_{\mu\nu} - f_\mu\,(x - x'')\,f_\nu\,(x'' - x')\right\} \tag{3.50}$$

(see Appendix A for details). Using Eq. (3.48) we recast Eq. (3.45) as

$$Z(\mathcal{J}) = \int\, [dA]\,[dB]\,F_1(A)F_2(B) \exp\left(i\Gamma_0[A, B, J, {}^*J]\right)\,. \tag{3.51}$$

Here the fermion functionals F_1 and F_2 are obtained by the replacements $\frac{\delta}{\mathrm{i}\delta J} \to A$, $\frac{\delta}{i\delta {}^*J} \to B$:

$$F_1(A) = \exp\left\{ \mathrm{Tr}\ln\left(1 - e\gamma\cdot AG_\psi\right) + i\int \bar{\eta}\left(G_\psi\left[1 - e\gamma\cdot AG_\psi\right]^{-1}\right)\eta\right\}\,,$$
$$F_2(B) = \exp\left\{ \mathrm{Tr}\ln\left(1 - g\gamma\cdot BG_\chi\right) + i\int \bar{\xi}\left(G_\chi\left[1 - g\gamma\cdot BG_\chi\right]^{-1}\right)\xi\right\}\,. \tag{3.52}$$

We perform a change of variables by shifting about the stationary configuration of the effective action, $\Gamma_0[A, B, J, {}^*J]$:

$$A_\mu(x) = \bar{A}_\mu(x) + \phi_\mu(x)\,, \qquad B'_\mu(x) = \bar{B}'_\mu(x) + \phi'_\mu(x) \tag{3.53}$$

where $\bar{A}$ and $\bar{B}$ are given by the solutions to

$$\frac{\delta\Gamma_0\,(A, B, J, {}^*J)}{\delta A^\tau} = 0\,, \qquad \frac{\delta\Gamma_0\,(A, B, J, {}^*J)}{\delta B^\tau} = 0\,, \tag{3.54}$$

namely (most easily seen by regarding A and B' as independent variables),

$$\begin{aligned}\bar{A}_\mu(x) &= \int (dx')D_{\mu\kappa}(x-x')\Big(J^\kappa(x') - \epsilon^{\kappa\nu\sigma\tau}\int (dx'')\partial'_\nu f_\sigma(x'-x'')\,{}^*J_\tau(x'')\Big),\\ \bar{B}_\mu(x) &= \int (dx')D_{\mu\kappa}(x-x')\Big({}^*J^\kappa(x') + \epsilon^{\kappa\nu\sigma\tau}\int (dx'')\partial'_\nu f_\sigma(x'-x'')J_\tau(x'')\Big),\end{aligned} \tag{3.55}$$

reflecting the form of Eq. (3.19) and its dual. Note that the solutions (3.55) respect the dual symmetry, which is not however manifest in the form of the effective action (3.48). Using the properties of Volterra expansions for functionals and performing the resulting quadratic integration over $\phi(x)$ and $\phi'(x)$ (see Appendix B), we obtain a rearrangement of the generating functional for the monopole-electron system that is well suited for nonperturbative calculations:

$$\begin{aligned}\frac{Z(\mathcal{J})}{Z_0(J,{}^*J)} = \exp\Bigg\{\frac{i}{2}\int (dx)(dx')\Bigg(&\frac{\delta}{\delta\bar{A}_\mu(x)}D_{\mu\nu}(x-x')\frac{\delta}{\delta\bar{A}_\nu(x')} + \frac{\delta}{\delta\bar{B}_\mu(x)}D_{\mu\nu}(x-x')\frac{\delta}{\delta\bar{B}_\nu(x')}\Bigg)\\ &- i\int (dx)(dx')\frac{\delta}{\delta\bar{A}_\mu(x)}\tilde{D}_{\mu\nu}\,(x-x')\,\frac{\delta}{\delta\bar{B}_\nu(x')}\Bigg\}\\ \times\exp\Bigg\{i\int (dx)(dx')\bar{\eta}\,(x)\,G(x,x'|\bar{A})\eta\,(x') &+ i\int (dx)(dx')\bar{\xi}\,(x)\,G(x,x'|\bar{B})\xi\,(x')\Bigg\}\\ \times\exp\Bigg\{-\int_0^e de'\,\mathrm{Tr}\gamma\bar{A}G(x,x|\bar{A}) &- \int_0^g dg'\,\mathrm{Tr}\gamma\bar{B}G(x,x|\bar{B})\Bigg\}.\end{aligned} \tag{3.56}$$

Here the two-point fermion Green's functions $G(x_1,y_1|\bar{A})$, and $G(x_2,y_2|\bar{B})$ in the background of the stationary photon field $\bar{A},\bar{B}$ are given by

$$\begin{aligned}G(x,x'|\bar{A}) &= \langle x|(\gamma p + m_\psi - e\,\gamma\bar{A})^{-1}|x'\rangle\,,\\ G(x,x'|\bar{B}) &= \langle x|(\gamma p + m_\chi - g\,\gamma\bar{B})^{-1}|x'\rangle\,,\end{aligned} \tag{3.57}$$

where the trace includes integration over spacetime. This result is equivalent to the functional Fourier transform given in Eq. (3.46) including the fermionic monopole-electon system:

$$\begin{aligned}Z(\mathcal{J}) = \int [dA]\,[dB]\,&\det(-i\gamma D_A + m_\psi)\det(-i\gamma D_B + m_\chi)\\ &\times\exp\Bigg\{i\int (dx)(dx')\Big(\bar{\eta}\,(x)\,G(x,x'|A)\eta\,(x') + \bar{\xi}\,(x)\,G(x,x'|B)\xi\,(x')\Big)\Bigg\}\\ &\times\exp\Bigg\{-\frac{i}{2}\int\Big(A^\mu K_{\mu\nu}A^\nu - B'^\mu\tilde{\Delta}^{-1}_{\mu\nu}B'^\nu\Big) + i\int (J\cdot A + {}^*J\cdot B)\Bigg\},\end{aligned} \tag{3.58}$$

where we have integrated over the fermion degrees of freedom.

Finally from our knowledge of the manner in which electric and magnetic charge couple to photons through Maxwell's equations we can immediately write the generalization of Eq. (3.56) for dyons, the different species of which are labeled by the index a:

$$\begin{aligned}Z(\mathcal{J}) = \exp&\Bigg\{\frac{i}{2}\int (dx)(dx')\mathcal{K}^\mu(x)\mathcal{D}_{\mu\nu}\,(x-x')\,\mathcal{K}^\nu\,(x')\Bigg\}\\ &\times\exp\Bigg\{\frac{i}{2}\int (dx)(dx')\frac{\delta}{\delta\bar{\mathcal{A}}_\mu(x)}\mathcal{D}_{\mu\nu}(x-x')\frac{\delta}{\delta\bar{\mathcal{A}}_\nu(x')}\Bigg\}\\ &\times\exp\Bigg\{i\sum_a\int (dx)(dx')\bar{\zeta}_a\,(x)\,G_a(x,x'|\bar{\mathcal{A}}_a)\zeta_a\,(x')\Bigg\}\\ &\times\exp\Bigg\{-\sum_a\int_0^1 dq\,\mathrm{Tr}\gamma\bar{\mathcal{A}}_a G_a(x,x|q\bar{\mathcal{A}}_a)\Bigg\}.\end{aligned} \tag{3.59}$$

where $\mathcal{A}_a = e_a A + g_a B$, ζ_a is the source for the dyon of species a, and a matrix notation is adopted,

$$\mathcal{K}^{\mu}(x) = \begin{pmatrix} J(x) \\ {}^{*}J(x) \end{pmatrix}, \qquad \frac{\delta}{\delta \bar{\mathcal{A}}_{\mu}(x)} = \begin{pmatrix} \delta/\delta \bar{A}_{\mu}(x) \\ \delta/\delta \bar{B}_{\mu}(x) \end{pmatrix}, \tag{3.60a}$$

and

$$\mathcal{D}_{\mu\nu}(x-x') = \begin{pmatrix} D_{\mu\nu}(x-x') & -\tilde{D}_{\mu\nu}(x-x') \\ \tilde{D}_{\mu\nu}(x-x') & D_{\mu\nu}(x-x') \end{pmatrix}. \tag{3.61a}$$

IV. STRING INDEPENDENCE OF THE DYON-DYON SCATTERING CROSS SECTION

In this section we demonstrate the string independence of the dyon-dyon and charge-monopole (the latter being a special case of the former) scattering cross section. We will use the generating functional (3.59) developed in the last section to calculate the scattering cross section nonperturbatively. We find that we are able to demonstrate phenomenological string invariance of the scattering cross section. It appears that in much the same manner as the Coulomb phase arises as a soft effect in high energy charge scattering, the string dependence arises from the exchange of soft photons.

To calculate the dyon-dyon scattering cross section we obtain the four–point Green's function for this process from Eq. (3.59)

$$G(x_1, y_1; x_2, y_2) = \frac{\delta}{i\delta\bar{\zeta}_1(x_1)} \frac{\delta}{i\delta\zeta_1(y_1)} \frac{\delta}{i\delta\bar{\zeta}_2(x_2)} \frac{\delta}{i\delta\zeta_2(x_2)} Z(\mathcal{J})\bigg|_{\mathcal{J}=0}. \tag{4.1}$$

The subscripts on the sources refer to the two different dyons.

Here we confront our calculational limits; these are not too dissimilar from those encountered in diffractive scattering or in the strong-coupling regime of QCD [60–63]. As a first step in analyzing the string dependence of the scattering amplitudes, we study high-energy forward scattering processes where *soft* photon contributions dominate. In diagrammatic language, in this kinematic regime it is customary to restrict attention to that subclass in which there are no closed fermion loops and the photons are exchanged between fermions [60]. In the context of Schwinger-Dyson equations this amounts to quenched or ladder approximation (see Fig. 1). In this approximation the linkage operators, $\mathbf{L}$, connect two fermion propagators via photon exchange, as we read off from Eq. (3.56):

$$\mathrm{e}^{\mathbf{L}_{12}} = \exp\left\{ i \int (dx)(dx') \frac{\delta}{\delta \bar{\mathcal{A}}_1^{\mu}(x)} \mathcal{D}^{\mu\nu}(x-x') \frac{\delta}{\delta \bar{\mathcal{A}}_2^{\nu}(x')} \right\}. \tag{4.2}$$

In this approximation Eq. (4.1) takes the form

$$G(x_1, y_1; x_2, y_2) = -\mathrm{e}^{\mathbf{L}_{12}} G_1(x_1, y_1|\bar{\mathcal{A}}_1) G_2(x_2, y_2|\bar{\mathcal{A}}_2)\Big|_{\bar{A}=\bar{B}=0}, \tag{4.3}$$

where we express the two-point function using the proper-time parameter representation of an ordered exponential

$$G_a(x, y|\bar{\mathcal{A}}_a) = i \int_0^{\infty} d\xi\, \mathrm{e}^{-i\xi(m_a - i\gamma\partial)} \exp\left\{ i \int_0^{\xi} d\xi' \mathrm{e}^{\xi'\gamma\partial} \gamma \bar{\mathcal{A}}_a \mathrm{e}^{-\xi'\gamma\partial} \right\}_+ \delta(x-y) \tag{4.4}$$

where "+" denotes path ordering in ξ'. The 12 subscripts in $\mathbf{L}_{12}$ emphasize that only photon lines that link the two fermion lines are being considered.

A. High Energy Scattering Cross Section

Adapting techniques outlined in [64,65] we consider the connected form of Eq. (4.3). We use the connected two-point function and the identities

$$\mathrm{e}^{\mathbf{L}} = 1 + \int_0^1 da\, \mathrm{e}^{a\mathbf{L}} \mathbf{L} \tag{4.5}$$

and

$$\frac{\delta}{\delta \bar{A}_\mu(x)} G(y,z|\bar{A}) = eG(y,x|\bar{A})\gamma^\mu G(x,z|\bar{A})\,. \tag{4.6}$$

Using Eqs. (4.3) and (4.4) one straightforwardly is led to the following representation of the four-point Green function,

$$\begin{aligned} G(x_1,y_1;x_2,y_2) &= -i\int_0^1 da \int (dz_1)(dz_2)\left(\mathsf{q}_1\cdot\mathsf{q}_2\, D_{\mu\nu}(z_1-z_2) - \mathsf{q}_1\times\mathsf{q}_2 \tilde{D}_{\mu\nu}(z_1-z_2)\right)) \\ &\times \mathrm{e}^{a\mathbf{L}_{12}} G_1(x_1,z_1|\bar{A}_1)\gamma_\mu G_1(z_1,y_1|\bar{A}_1) G_2(x_2,z_2|\bar{A}_2)\gamma^\nu G_2(z_2,y_2|\bar{A}_2)\Big|_{\bar{A}=\bar{B}=0}\,, \end{aligned} \tag{4.7}$$

where the charge combinations invariant under duality transformations are

$$\begin{aligned} \mathsf{q}_1\cdot\mathsf{q}_2 &= e_1e_2 + g_1g_2 \\ \mathsf{q}_1\times\mathsf{q}_2 &= e_1g_2 - g_1e_2\,. \end{aligned} \tag{4.8}$$

In order to account for the soft nonperturbative effects of the interaction between electric and magnetic charges we consider the limit in which the momentum exchanged by the photons is small compared to the mass of the fermions. This affords a substantial simplification in evaluating the path-ordered exponential in Eq. (4.4); in conjunction with the assumption of small momentum transfer compared to the incident and outgoing momenta, $q/p_{(1,2)} \ll 1$, this amounts to the Bloch-Nordsieck [66] or *eikonal approximation* (see [40–43] and for more modern applications in diffractive and strong coupling QCD processes [60–63]). In this approximation Eq. (4.4) becomes

$$G_a(x,y|\bar{A}) = i\int_0^\infty d\xi\, \mathrm{e}^{-i\xi m}\delta\left(x-y-\xi\frac{p}{m}\right)\exp\left\{i\int_0^\xi d\xi' \frac{p}{m}\cdot\bar{A}\left(x-\xi'\frac{p}{m}\right)\right\}. \tag{4.9}$$

With this simplification each propagator in Eq. (4.3) can be written as an exponential of linear function of the gauge field. Performing mass shell amputation on each external coordinate and taking the Fourier transform of Eq. (4.7) we obtain the scattering amplitude, $T(p_1,p_1';p_2,p_2')$:

$$\begin{aligned} T\left(p_1,p_1';p_2,p_2'\right) &= -i\int_0^1 da\, \mathrm{e}^{a\mathbf{L}_{12}} \int (dz_1)(dz_2)\left(\mathsf{q}_1\cdot\mathsf{q}_2\, D_{\mu\nu}\left(z_1-z_2\right) - \mathsf{q}_1\times\mathsf{q}_2\tilde{D}_{\mu\nu}\left(z_1-z_2\right)\right) \\ &\times \int (dx_1)\mathrm{e}^{-ip_1x_1}\bar{u}(p_1)\left(m_1+v_1\cdot p_1\right)G_1(x_1,z_1|\bar{A}_1)\gamma^\mu \int (dy_1)\mathrm{e}^{ip_1'y_1}G_1(z_1,y_1|\bar{A}_1)\left(m_1+v_1'\cdot p_1'\right)u(p_1') \\ &\times \int (dx_2)\mathrm{e}^{-ix_2p_2}\bar{u}(p_2)\left(m_2+v_2\cdot p_2\right)G_2(x_2,z_2|\bar{A}_2)\gamma^\nu \int (dy_2)\mathrm{e}^{ip_2'y_2}G_2(z_2,y_2|\bar{A}_2)\left(m_2+v_2'\cdot p_2'\right)u(p_2'). \end{aligned} \tag{4.10}$$

Substituting Eq. (4.9) into Eq. (4.10), we simplify this to

$$\begin{aligned} T(p_1,p_1';p_2,p_2') &= -i\int_0^1 d\,a \int (dz_1)(dz_2)\mathrm{e}^{-iz_1\left(p_1-p_1'\right)}\mathrm{e}^{-iz_2\left(p_2-p_2'\right)}\bar{u}(p_1')\gamma^\mu u(p_1)\bar{u}(p_2')\gamma^\nu u(p_2) \\ &\times\left(\mathsf{q}_1\cdot\mathsf{q}_2\, D_{\mu\nu}(z_1-z_2) - \mathsf{q}_1\times\mathsf{q}_2\tilde{D}_{\mu\nu}(z_1-z_2)\right)\mathrm{e}^{a\mathbf{L}_{12}} \\ &\times\exp\left[i\int_0^\infty d\alpha_1\left\{p_1\cdot\left(\bar{A}_1\left(z_1+\alpha_1p_1\right)\right) + p_1'\cdot\left(\bar{A}_1\left(z_1-\alpha_1p_1'\right)\right)\right\}\right] \\ &\times\exp\left[i\int_0^\infty d\alpha_2\left\{p_2\cdot\left(\bar{A}_2\left(z_2+\alpha_2p_2\right)\right) + p_2'\cdot\left(\bar{A}_2\left(z_2-\alpha_2p_2'\right)\right)\right\}\right]. \end{aligned} \tag{4.11}$$

Choosing the incoming momenta to be in the z direction, in the center of momentum frame, $p_1^\mu = (E_1,0,0,p)$, $p_2^\mu = (E_2,0,0,-p)$, invoking the approximation of small recoil and passing the linkage operator through the exponentials containing the photon field, we find from Eq. (4.11)

$$\begin{aligned} T(p_1,p_1';p_2,p_2') &= -i\int_0^1 da\int (dz_1)(dz_2)\mathrm{e}^{-iz_1\left(p_1-p_1'\right)}\mathrm{e}^{-iz_2\left(p_2-p_2'\right)}\bar{u}(p_1')\gamma_\mu u(p_1)\bar{\upsilon}(p_2')\gamma_\nu\upsilon(p_2) \\ &\times\left(\mathsf{q}_1\cdot\mathsf{q}_2\, D^{\mu\nu}\left(z_1-z_2\right) - \mathsf{q}_1\times\mathsf{q}_2\tilde{D}^{\mu\nu}\left(z_1-z_2\right)\right)\mathrm{e}^{ia\Phi(p_1,p_2;z_1-z_2)}\,, \end{aligned} \tag{4.12}$$

where the "eikonal phase" integral is,

$$\Phi_n\left(p_1, p_2; z_1 - z_2\right) = p_1^\kappa p_2^\lambda \int_{-\infty}^{\infty} d\alpha_1\, d\alpha_2 \left(\mathsf{q}_1 \cdot \mathsf{q}_2\, D_{\kappa\lambda} - \mathsf{q}_1 \times \mathsf{q}_2 \tilde{D}_{\kappa\lambda}\right)\left(z_1 - z_2 + \alpha_1 p_1 - \alpha_2 p_2\right) . \tag{4.13}$$

We transform to the center of momentum coordinates, by decomposing the relative coordinate accordingly,

$$(z_1 - z_2)^\mu = x_\perp^\mu - \tau_1 p_1^\mu + \tau_2 p_2^\mu, \tag{4.14}$$

where the Jacobian of the transformation is

$$J = p\sqrt{s} \tag{4.15}$$

and $s = -(p_1 + p_2)^2$ is the square of the center of mass energy. Here we use the *symmetric* (see [22,23] for details) infinite string function, which has the momentum-space form,

$$f^\mu(k) = \frac{n^\mu}{2i}\left(\frac{1}{n \cdot k - i\delta} + \frac{1}{n \cdot k + i\delta}\right). \tag{4.16}$$

Inserting the momentum-space representation of the propagator, we cast Eq. (4.13) into the form

$$\begin{aligned}\Phi_n\left(p_1, p_2; x\right) &= p_1^\kappa p_2^\lambda \int_{-\infty}^{\infty} d\alpha_1\, d\alpha_2 \int \frac{(dk)}{(2\pi)^4} \frac{\mathrm{e}^{i\,k\cdot(x+\alpha_1 p_1 - \alpha_2 p_2)}}{k^2 + \mu^2} \\ &\times \left[\mathsf{q}_1 \cdot \mathsf{q}_2\, g_{\kappa\lambda} - \mathsf{q}_1 \times \mathsf{q}_2 \epsilon_{\kappa\lambda\sigma\tau} k^\sigma \frac{n^\tau}{2}\left(\frac{1}{n \cdot k - i\delta} + \frac{1}{n \cdot k + i\delta}\right)\right],\end{aligned} \tag{4.17}$$

where we have introduced the standard infrared photon-mass regulator, μ^2. The delta functions that result from performing the integrations over parameters α_1 and α_2 in Eq. (4.17) in the eikonal phase suggests the momentum decomposition

$$k^\mu = k_\perp^\mu + \lambda_1 e_1^\mu + \lambda_2 e_2^\mu\,,\ \text{where } \lambda_1 = p_2 \cdot k\,, \text{ and } \lambda_2 = p_1 \cdot k, \tag{4.18}$$

and the four-vector basis is given by

$$e_1^\mu = \frac{-1}{\sqrt{s}}\left(1, 0, 0, \frac{p_1^0}{p}\right) \quad \text{and} \quad e_2^\mu = \frac{-1}{\sqrt{s}}\left(1, 0, 0, -\frac{p_2^0}{p}\right), \tag{4.19}$$

which have the following properties,

$$e_1 \cdot e_1 = \frac{1}{s}\frac{M_1^2}{p^2}, \quad e_2 \cdot e_2 = \frac{1}{s}\frac{M_2^2}{p^2}, \quad \text{and} \quad e_1 \cdot e_2 = \frac{1}{s}\frac{p_1 \cdot p_2}{p^2}. \tag{4.20}$$

The corresponding measure and Jacobian are, respectively,

$$(dk) = J d^2\boldsymbol{k}_\perp d\lambda_1 d\lambda_2 \quad \text{and} \quad J = \left(p\sqrt{s}\right)^{-1}. \tag{4.21}$$

Using the definition of the Møller amplitude, $M(s,t)$, given by removing the momentum-conserving delta function,

$$T(p_1, p_1'; p_2, p_2') = (2\pi)^4 \delta^{(4)}\left(P - P'\right) M(s,t)\,, \tag{4.22}$$

we put Eq. (4.12) into the form

$$M(s,t) = -i\int_0^1 da \int d^2\boldsymbol{x}_\perp \mathrm{e}^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{x}_\perp} \bar{u}(p_1')\gamma^\mu u(p_1)\bar{u}(p_2')\gamma^\nu u(p_2) I_{\mu\nu} \mathrm{e}^{ia\Phi_n(p_1,p_2;x)}\,, \tag{4.23}$$

where

$$\begin{aligned}I_{\mu\nu} = \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2}\frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi} &\frac{\mathrm{e}^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} 2\pi\delta(\lambda_1) 2\pi\delta(\lambda_2)}{\left(\boldsymbol{k}_\perp^2 + \mu^2 + \frac{1}{s\,p^2}\left(\lambda_1^2 M_1^2 + \lambda_2^2 M_2^2 + 2\lambda_1\lambda_2 p_1 \cdot p_2\right)\right)} \\ &\times \left[\mathsf{q}_1 \cdot \mathsf{q}_2\, g_{\mu\nu} - \mathsf{q}_1 \times \mathsf{q}_2 \epsilon_{\mu\nu\sigma\tau} k^\sigma \frac{n^\tau}{2}\left(\frac{1}{n \cdot k - i\delta} + \frac{1}{n \cdot k + i\delta}\right)\right]\end{aligned} \tag{4.24}$$

Here $P = p_1 + p_2$ and $P' = p_1' + p_2'$, and $q = p_1 - p_1'$ is the momentum transfer. The factor

$$\exp(i\tau_1 p_1 \cdot q - i\tau_2 p_2 \cdot q) = \exp\left[i\frac{1}{2}q^2(\tau_1 + \tau_2)\right] \tag{4.25}$$

has been omitted because it is unity in the eikonal limit, and correspondingly, we have carried out the integrals on τ_1 and τ_2. The eikonal phase Eq. (4.17) now takes the very similar form

$$\Phi_n(p_1, p_2; x) = \frac{p_1^\kappa p_2^\lambda}{p\sqrt{s}} I_{\kappa\lambda}. \tag{4.26}$$

Choosing a spacelike string[14], $n^\mu = (0, \boldsymbol{n})$, integrating over the coordinates λ_1, λ_2, and introducing "proper-time" parameter representations of the propagators, we reduce Eq. (4.26) to

$$\begin{aligned}
\Phi_n(p_1, p_2; x) &= \frac{1}{p\sqrt{s}} \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int_0^\infty d\mathfrak{s}\, \mathrm{e}^{-\mathfrak{s}(\boldsymbol{k}^2+\mu^2)} \\
&\quad \times \left\{ \mathsf{q}_1 \cdot \mathsf{q}_2\, p_1 \cdot p_2 - \mathsf{q}_1 \times \mathsf{q}_2 p_1^\mu p_2^\nu \epsilon_{\mu\nu\sigma\tau} \frac{n^\sigma}{2i} \frac{\partial}{\partial n_\tau} \left(\int_0^\infty \frac{dt}{it} \mathrm{e}^{i\mathfrak{t}(\boldsymbol{n}\cdot\boldsymbol{k}+i\delta)} - \int_{-\infty}^0 \frac{dt}{it} \mathrm{e}^{i\mathfrak{t}(\boldsymbol{n}\cdot\boldsymbol{k}-i\delta)} \right) \right\} \\
&= \frac{1}{2\pi} \left\{ \mathsf{q}_1 \cdot \mathsf{q}_2 \frac{p_1 \cdot p_2}{p\sqrt{s}} K_0(\mu|\boldsymbol{x}|) - \mathsf{q}_1 \times \mathsf{q}_2 \epsilon_{3jk} n^j \frac{\partial}{\partial n^k} \frac{1}{2} \int \frac{d\mathfrak{t}}{\mathfrak{t}} K_0(\mu|(\boldsymbol{x} + \mathfrak{t}\boldsymbol{n})|) \right\},
\end{aligned} \tag{4.27}$$

in terms of modified Bessel functions, where we have dropped the subscript $\perp$.

We perform the parameter integral over $\mathfrak{t}$ in the limit of small μ^2:

$$-\frac{1}{2}\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{x}) \left[\int_0^\infty - \int_{-\infty}^0\right] \frac{d\mathfrak{t}\, e^{-\delta|\mathfrak{t}|}}{(\mathfrak{t} + \hat{\boldsymbol{n}} \cdot \boldsymbol{x})^2 + x^2 - (\hat{\boldsymbol{n}} \cdot \boldsymbol{x})^2} = \arctan\left[\frac{\boldsymbol{n} \cdot \boldsymbol{x}}{\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{x})}\right], \tag{4.28}$$

so the phase is

$$\Phi_n(p_1, p_2; x) = \frac{1}{2\pi} \left\{ \mathsf{q}_1 \cdot \mathsf{q}_2 \ln(\tilde{\mu}|\boldsymbol{x}|) - \mathsf{q}_1 \times \mathsf{q}_2 \arctan\left[\frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{x}}{\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{x})}\right] \right\}. \tag{4.29}$$

In this limit we have used the asymptotic limit of the modfied Bessel function

$$K_0(x) \sim -\ln\left(\frac{\mathrm{e}^\gamma x}{2}\right), \tag{4.30}$$

where $\gamma = 0.577\ldots$ is Euler's constant and we have defined $\tilde{\mu} = \mathrm{e}^\gamma \mu/2$. Similarly, Eq. (4.23) becomes

$$\begin{aligned}
M(s,t) &= -\frac{i}{2\pi} \int_0^1 da \int d^2\boldsymbol{x}\, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} \bar{u}(p_1')\gamma^\mu u(p_1) \bar{u}(p_2')\gamma^\nu u(p_2) \\
&\quad \times \left\{ g_{\mu\nu} \mathsf{q}_1 \cdot \mathsf{q}_2 K_0(\mu|\boldsymbol{x}|) - \epsilon_{\mu\nu\sigma\tau} \mathsf{q}_1 \times \mathsf{q}_2 n^\tau \frac{\partial}{\partial n_\sigma} \frac{1}{2} \int \frac{d\mathfrak{t}}{\mathfrak{t}} K_0(\mu|(\boldsymbol{x} + \mathfrak{t}\boldsymbol{n})|) \right\} \mathrm{e}^{ia\Phi_n(p_1, p_2; x)}.
\end{aligned} \tag{4.31}$$

Although in the eikonal limit, no spin-flip processes occur, it is, as always, easier to calculate the helicity amplitudes, of which there is only one in this case. In the high-energy limit, $p^0 \gg m$, the Dirac spinor in the helicity basis is

$$u^\sigma(p) = \sqrt{\frac{p^0}{2m}}(1 + i\gamma_5\sigma)v_\sigma, \tag{4.32}$$

where the v_σ may be thought of as two-component spinors satisfying $\gamma^0 v_\sigma = v_\sigma$. They are further eigenstates of the helicity operator $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}$ with eigenvalue σ:

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[14]We choose a spacelike string in order that we formally have a local interaction in momentum space.

$$v_+^\dagger(\hat{\boldsymbol{p}}') = \left(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2} \right) \quad v_-^\dagger(\hat{\boldsymbol{p}}') = \left(-\sin\tfrac{\theta}{2}, \cos\tfrac{\theta}{2} \right) \quad v_+(\hat{\boldsymbol{p}}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_-(\hat{\boldsymbol{p}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.33}$$

We employ the definition

$$\gamma_5 = \gamma_0 \gamma^1 \gamma^2 \gamma^3 \tag{4.34}$$

and consequently $\gamma^0 \boldsymbol{\gamma} = i\gamma^5 \boldsymbol{\sigma}$, where $\sigma_{ij} = \epsilon_{ijk}\sigma^k$. We then easily find upon integrating over the parameter a that the spin non-flip part of Eq. (4.31) becomes $(\theta \to 0)$

$$M(s,t) = \frac{s}{2M_1 M_2} \left\{ \int d^2\boldsymbol{x}\, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} \mathrm{e}^{i\Phi_n(p_1,p_2;x)} - (2\pi)^2\, \delta^2\left(\boldsymbol{q}\right) \right\}. \tag{4.35}$$

Now notice that the arctangent function is discontinuous when the xy component of $\hat{\boldsymbol{n}}$ and $\boldsymbol{x}$ lie in the same direction. We require that the eikonal phase factor $e^{i\Phi_n}$ be continuous, which leads to the Schwinger quantization condition (1.4):

$$\mathsf{q}_1 \times \mathsf{q}_2 = 4N\pi\,. \tag{4.36}$$

Now using the integral form for the Bessel function of order ν

$$i^\nu J_\nu(t) = \int_0^{2\pi} \frac{d\phi}{2\pi} \mathrm{e}^{i(t\cos\phi - \nu\phi)}\,, \tag{4.37}$$

we find the dyon-dyon scattering amplitude (4.35) to be

$$M(s,t) = \frac{\pi s}{M_1 M_2} \mathrm{e}^{i2N\psi} \int_0^\infty dx\, x\, J_{2N}\left(qx\right) \mathrm{e}^{i2\tilde{\alpha}\ln(\tilde{\mu}x)}\,, \tag{4.38}$$

where $\tilde{\alpha} = \mathsf{q}_1 \cdot \mathsf{q}_2 / 4\pi$, and ψ is the angle between $\boldsymbol{q}$ and $\boldsymbol{n}$. The integral over x is just a ratio of gamma functions,

$$\int_0^\infty dx\, \left(\tilde{\mu}x\right)^{1+2i\tilde{\alpha}} J_{2N}\left(qx\right) = \frac{1}{2\tilde{\mu}} \left(\frac{4\tilde{\mu}^2}{q^2} \right)^{i\tilde{\alpha}+1} \frac{\Gamma\left(1+N+i\tilde{\alpha}\right)}{\Gamma\left(N-i\tilde{\alpha}\right)}. \tag{4.39}$$

Then Eq. (4.38) becomes

$$M(s,t) = \frac{s}{M_1 M_2} \frac{2\pi}{q^2} (N - i\tilde{\alpha}) \mathrm{e}^{i2N\psi} \left(\frac{4\tilde{\mu}^2}{q^2} \right)^{i\tilde{\alpha}} \frac{\Gamma\left(1+N+i\tilde{\alpha}\right)}{\Gamma\left(1+N-i\tilde{\alpha}\right)}. \tag{4.40}$$

This result is almost identical in structure to the nonrelativistic form of the scattering amplitude for the Coulomb potential, which result is recovered by setting $N = 0$. (See, for example, Ref. [67].) Following the standard convention [68] we calculate the spin-averaged cross section for dyon-dyon scattering in the high energy limit,

$$\frac{d\sigma}{dt} = \frac{\left(\mathsf{q}_1 \cdot \mathsf{q}_2\right)^2 + \left(\mathsf{q}_1 \times \mathsf{q}_2\right)^2}{4\pi t^2}\,. \tag{4.41}$$

While the Lagrangian is string-dependent, because of the charge quantization condition, the cross section, Eq (4.41), is string independent.

For the case of charge-monopole scattering $e_1 = g_2 = 0$, this result, of course, coincides with that found by Urrutia [45], which is also string independent as a consequence of (1.1). This is to be contrasted with *ad hoc* prescriptions that average over string directions or eliminate its dependence by simply dropping string-dependent terms because they cannot contribute to any gauge invariant quantities (*cf.* Ref. [28]).

V. CONCLUSION

In this paper we have responded to the challenge of Schwinger [27], to construct a realistic theory of relativistic magnetic charges. He sketched such a development in source theory language, but restricted his consideration to classical point particles, explicitly leaving the details to the reader. Urrutia applied this skeletal formulation in the eikonal limit [45], as already suggested by Schwinger.

We believe that we have given a complete formulation, in modern quantum field theoretic language, of an interacting electron-monopole or dyon-dyon system. The resulting Schwinger-Dyson equations, although to some extent implicit in the work of Schwinger and others, are given here for the first time.

The challenge is to apply these equations to the calculation of monopole and dyon processes. Perturbation theory is useless, not only because of the strength of the coupling, but more essentially because the graphs are fatally string- (or gauge-) dependent. The most obvious nonperturbative technique for transcending these limitations in scattering processes lies in the high energy regime where the eikonal approximation is applicable; in that limit, our formalism generalizes the lowest-order result of Urrutia and charts the way to include systematic corrections. More problematic is the treatment of monopole production processes—we defer that discussion to a subsequent publication.

In addition we have also detailed how the Dirac string dependence disappears from physical quantities. It is by no means a result of string averaging or a result of dropping string-dependent terms as in Ref. [28]. In fact, it is a result of summing the soft contributions to the dyon-dyon or charge-monopole process. There is good reason to believe that inclusion of hard scattering contributions will not spoil this consistency. At the level of the eikonal approximation and its corrections one might suspect the occurence of a factorization of hard string-independent and soft string-dependent contributions in a manner similar to that argued recently in strong-coupling QCD.

It is also of interest to investigate other nonperturbative methods of calculation in order to demonstrate gauge invariance of Green's functions and scattering amplitudes in both electron-monopole and dyon-dyon scattering and in Drell-Yan production processes.[15] In a subsequent paper we will apply the techniques and results found here to the Drell-Yan production of monopole-antimonopole processes, and obtain phenomenologically relevant estimates for the laboratory production of monopole-antimonopole pairs.

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APPENDIX A: PATH INTEGRAL

In this appendix we summarize the main steps to obtain the covariant path integral for the string dependent action corresponding to the generating functional, Eq. (3.44).

This path integral is obtained by calculating the functional Fourier transform of Z_0 which in the photonic sector amounts to transforming Eq. (3.40), according to Eq. (3.46), the functional transform of which is given by

$$\tilde{Z}_0\,(A,B) = \int [d\,J]\,[d\,{}^*\!J]\,Z_0(J,\,{}^*\!J)\exp\left[-i\int (J\cdot A + {}^*\!J\cdot B)\right]\,. \tag{A1}$$

After performing the Gaussian functional integration over J in Eq. (A1), we obtain

$$\begin{aligned}\tilde{Z}_0(A,B) = \int [d^*J]\exp\Big\{ &- i\int (dx)\,{}^*\!J_\mu(x)B^\mu(x) \\ &+ \frac{i}{2}\int (dx)(dx')\,{}^*\!J_\mu(x)D^{\mu\nu}(x-x')\,{}^*\!J_\nu(x') \\ &- \frac{i}{2}\int (dx)(dx')A'_\mu(x)K^{\mu\nu}(x-x')A'_\nu(x')\Big\}\,,\end{aligned} \tag{A2}$$

where $K_{\mu\nu}$ is the kernel given by Eq. (3.16), the inverse to $D^{\mu\nu}$, and

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[15] In addition there is a formalism recently employed in Ref. [69] based on Fradkin's [70] Green's function representation, which includes approximate vertex and self-energy polarization corrections using nonperturbative techniques.

$$A'_\mu(x) = A_\mu(x) + \epsilon_{\mu\nu\sigma\tau}\int(dx)(dx')(dx'')D(x-x')\partial'^\nu f^\sigma(x'-x'')\,{}^*J^\tau(x''). \tag{A3}$$

We now use the following identity involving the contraction of ϵ symbols:

$$\begin{aligned}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\nu\sigma\tau} = &-g^\alpha_\nu g^\beta_\sigma g^\gamma_\tau + g^\alpha_\nu g^\beta_\tau g^\gamma_\sigma + g^\alpha_\sigma g^\beta_\nu g^\gamma_\tau - g^\alpha_\sigma g^\beta_\tau g^\gamma_\nu \\ &- g^\alpha_\tau g^\beta_\nu g^\gamma_\sigma + g^\alpha_\tau g^\beta_\sigma g^\gamma_\nu,\end{aligned} \tag{A4}$$

to simplify the final term in the exponential in Eq. (A2):

$$\begin{aligned}-\frac{i}{2}\int(dx)(dx')A'_\mu(x)D^{-1}(x-x')A'^\mu(x') &= -\frac{i}{2}\int(dx)(dx')A_\mu(x)D^{-1}(x-x')A^\mu(x') \\ +\, i\int(dx)(dx')\,{}^*J^\mu(x)\epsilon_{\mu\nu\sigma\tau}\partial^\nu f^\sigma(x-x')A^\tau(x') &- \frac{i}{2}\int(dx)(dx')\,{}^*J_\mu(x)\Delta^{\mu\nu}(x-x')\,{}^*J_\nu(x') \\ -\,\frac{i}{2}\int(dx)(dx')\,{}^*J^\mu(x)D_+(x-x')\,{}^*J_\mu(x'),&\end{aligned} \tag{A5}$$

the last term in which cancels the second term in the exponential in Eq. (A2). Here we see the "string propagator," Eq. (3.50). Now we carry out the $^*J$ functional integration, noticing that the second term on the right side of Eq. (A5) converts B^μ to B'^μ, Eq. (3.49):

$$\begin{aligned}Z_0(A,B) = \int[dA]\,[dB]\exp\Big\{&-\frac{i}{2}\int(dx)(dx')A^\mu(x)K_{\mu\nu}(x-x')A^\nu(x') \\ &+\frac{i}{2}\int(dx)(dx')B'^\mu(x)\tilde\Delta^{-1}_{\mu\nu}(x-x')B'^\nu(x')\Big\}\end{aligned} \tag{A6}$$

which implies the effective action (3.48).

APPENDIX B: FUNCTIONAL REARRANGEMENT

We consider the expansion about the minima of the effective action $\Gamma_0[A,B,J,{}^*J]$, in particular, the impact on Eqs. (3.52) of the transformation (3.53). Using the properties of Volterra expansions for functionals, the shift in variables results in the translation of the loop functionals

$$F_1\left(\bar A_\mu + \phi_\mu\right) = \exp\left\{i\int(dx)\phi_\alpha(x)\frac{\delta}{i\delta\bar A_\alpha(x)}\right\}F_1\left(\bar A_\mu\right)\,, \tag{B1a}$$

$$F_2\left(\hat B'_\mu + \phi'_\mu\right) = \exp\left\{i\int(dx)\left(\phi'_\alpha(x) - \epsilon_{\alpha\beta\gamma\delta}\int(dx')\partial^\beta f^\gamma(x-x')\phi^\delta(x')\right)\frac{\delta}{i\delta\bar B_\alpha(x)}\right\}F_2\left(\bar B_\mu\right). \tag{B1b}$$

where

$$\hat B'_\mu(x) = \bar B_\mu(x) - \epsilon_{\mu\nu\sigma\tau}\int(dx')\partial^\nu f^\sigma\left(x-x'\right)\phi^\tau(x') \tag{B2}$$

Substituting Eqs. (B1a), (B1b) back into (3.52)

$$\begin{aligned}Z(\mathcal J) = \exp\Big\{&\frac{i}{2}\int(dx)(dx')\Big(J^\mu(x)D_{\mu\nu}\left(x-x'\right)J^\nu\left(x'\right) + {}^*J^\mu\left(x\right)D_{\mu\nu}\left(x-x'\right){}^*J^\nu\left(x'\right)\Big) \\ &- i\epsilon_{\mu\nu\sigma\tau}\int(dx)(dx')(dx'')J_\kappa(x)D^{\kappa\mu}\left(x-x'\right)\partial'^\nu f^\sigma\left(x'-x''\right){}^*J^\tau\left(x''\right)\Big\} \\ \times\int[\mathrm d\phi]\,[\mathrm d\phi']\exp\Big\{&i\int(dx)\Big(\phi_\mu(x)\Big[\frac{\delta}{i\delta\bar A_\mu(x)} + \epsilon^{\mu\nu\sigma\tau}\int(dx')\partial_\nu f_\sigma\left(x-x'\right)\frac{\delta}{i\delta\bar B^\tau(x')}\Big] + \phi'_\mu(x)\frac{\delta}{i\delta\bar B_\mu(x)}\Big) \\ &-\frac{i}{2}\int(dx)(dx')\Big(\phi^\mu(x)K_{\mu\nu}(x-x')\phi^\nu(x') - \phi'^\mu(x)\tilde\Delta^{-1}_{\mu\nu}(x-x')\phi'^\nu(x')\Big)\Big\}F_1(\bar A)F_2(\bar B)\,,\end{aligned} \tag{B3}$$

and performing the resulting quadratic integration over $\phi(x)$ and $\phi'(x)$, we obtain the results in Eq. (3.56).

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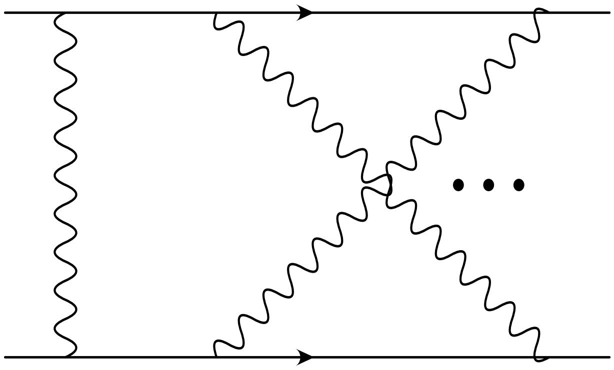

FIG. 1. Dyon-dyon scattering amplitudes in the quenched approximation.