An analytic method of describing R-related quantities in QCD

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Abstract

A model based on the analytic approach to QCD, involving a summation of threshold singularities and taking into account the nonperturbative character of the light quark masses, is applied to find hadronic contributions to different physical quantities. It is shown that the suggested model allows us to describe well such objects as the hadronic contribution to the anomalous magnetic moment of the muon, the ratio of hadronic to leptonic τ-decay widths in the vector channel, the Adler D-function, the smeared R_Δ-function, and the hadronic contribution to the evolution of the fine structure constant.

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I. INTRODUCTION

A comparison of QCD theoretical results with experimental data is often based on the concept of quark-hadron duality, which establishes a bridge between quarks and gluons, a language of theoreticians, and real measurements with hadrons performed by experimentalists. The idea of quark-hadron duality was formulated in the paper by Poggio, Quinn, and Weinberg [1] as follows: Inclusive hadronic cross sections, once they are appropriately averaged over an energy interval, must approximately coincide with the corresponding quantities derived from the quark-gluon picture. For many physical quantities and functions the corresponding interval of integration involves an infrared region and in this case nonperturbative effects may play an important role in their description.

In this paper we consider the following quantities and functions.

- The ratio of hadronic to leptonic τ-decay widths in the vector channel:

$$R_\tau^V = R^{(0)} \int\limits_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) R(s);$$

(1)

- the so-called "light" Adler function, which is constructed from τ-decay data [2]:

$$D(Q^2) = -Q^2 \frac{d\,\Pi(-Q^2)}{d\,Q^2} = Q^2 \int\limits_0^\infty ds\, \frac{R(s)}{(s + Q^2)^2};$$

(2)

- the smeared R_Δ function [1]:

$$R_\Delta(s) = \frac{\Delta}{\pi} \int\limits_0^\infty ds' \frac{R(s')}{(s - s')^2 + \Delta^2};$$

(3)

- the hadronic contribution to the anomalous magnetic moment of the muon (in the leading order in electromagnetic coupling constant):

$$a_\mu^{\text{had}} = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int\limits_0^\infty \frac{ds}{s} K(s)\, R(s),$$

(4)

where $K(s)$ is the vacuum polarization factor given by (18) below;

- and the strong interaction contribution to the running of the fine structure constant:

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = -\frac{\alpha(0)}{3\pi} M_Z^2\, \mathcal{P} \int\limits_0^\infty \frac{ds}{s} \frac{R(s)}{s - M_Z^2}.$$

(5)

A common feature of all these quantities and functions is that they are defined through the function $R(s)$, the normalized hadronic cross-section, integrated with some other functions. By definition, all these quantities and functions include an infrared region as a part of the interval of integration and, therefore, they cannot be directly calculated within perturbative quantum chromodynamics (pQCD).

The method that we use here to describe the quantities and functions mentioned above is based on the analytic approach to QCD suggested in [3, 4]. The analytic approach allows one to describe self-consistently the timelike region [5, 6], which is represented in the integration in Eqs. (1)–(5). It incorporates the required analytic properties and leads to an integral representation for $R(s)$. We formulate a model that also incorporates a summation of threshold singularities [7] and takes into account the nonperturbative character of the light quark masses.

II. METHOD AND BASIC RELATIONS

A. Analytic perturbation theory

Analytic perturbation theory (APT) [8] is based on the analytic approach to QCD. In this approach, in contrast to the behavior of the perturbative running coupling, the Euclidean analytic coupling has no unphysical singularities. The ghost pole and corresponding branch points, which appear in higher orders, are absent. APT preserves the correct analytic properties of such important objects as the two-point correlation function and also provides a well-defined algorithm for calculating higher-loop corrections [9]. In APT, processes with typical spacelike and timelike momenta are described self-consistently [5, 6, 10] and, for example, inclusive τ-decay can be described equivalently either in terms of Minkowskian or Euclidean variables [8]. In the framework of APT, the theoretical ambiguity associated with the choice of renormalization scheme is dramatically reduced.

In the APT scheme the QCD contributions $d(z)$ and $r(s)$ to the functions $D \propto 1 + d$ and $R \propto 1 + r$, respectively, are expressed in terms of the effective spectral function $\rho(\sigma)$ as [3–5]

$$d(z) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma - z}\, \rho(\sigma)\,, \qquad r(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho(\sigma)\,. \tag{6}$$

The APT spectral function $\rho(\sigma)$ is defined as the imaginary part of the perturbative function $d_{\mathrm{pt}}(z)$ and in the third order can be written in the form $\rho(\sigma) = \varrho_0(\sigma) + d_1 \varrho_1(\sigma) + d_2 \varrho_2(\sigma)$,

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where $\varrho_k(\sigma) = \mathrm{Im}[a_{\mathrm{pt}}^{k+1}(\sigma + i\epsilon)]$, $a = \alpha_s/\pi$, and d_k are the coefficients of the perturbative expansion of the D-function.

The function $\varrho_0(\sigma)$ defines the analytic spacelike, $\mathcal{A}(z)$, and timelike, $\mathfrak{A}(s)$, running couplings as follows

$$\mathcal{A}(z) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma - z}\, \varrho_0(\sigma)\,, \qquad \mathfrak{A}(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma}\, \varrho_0(\sigma)\,. \tag{7}$$

As has been argued from general principles in [11], the behavior of these couplings cannot be the same, i.e., they cannot be symmetrical in the spacelike and timelike domains, $\mathcal{A}(z) \neq \mathfrak{A}(z)$. The analytic and perturbative couplings have been compared in [6]. In analyzing hadronic processes with characteristic spacelike and timelike momenta, it is necessary to take into account this lack of symmetry between the behavior of the running coupling in the Euclidean and Minkowskian regions.

The analytic running coupling has no unphysical singularities and possesses the correct analytic properties, arising from Källén-Lehmann analyticity reflecting the general principles of the theory. The one-loop APT result is [3–5]

$$\mathcal{A}^{(1)}(z) = a_{\mathrm{pt}}^{(1)}(z) + \frac{4}{\beta_0} \frac{\Lambda^2}{\Lambda^2 + z}\,, \qquad \mathfrak{A}^{(1)}(s) = \frac{4}{\beta_0} \left[\frac{1}{2} - \frac{1}{\pi} \arctan \frac{\ln(s/\Lambda^2)}{\pi} \right]\,, \tag{8}$$

where $a_{\mathrm{pt}}^{(1)}(z) = \bar{\alpha}_s(z)/\pi = 4/\left[\beta_0 \ln(-z/\Lambda^2)\right]$ and $\beta_0 = 11 - 2f/3$ is the first coefficient of the renormalization group β-function.

Both the couplings (7) have the same infrared fixed point $\mathcal{A}(0) = \mathfrak{A}(0) = 4/\beta_0$. This value is defined by the leading contribution (8) and is not altered by higher-order corrections. The regular behavior in the infrared region of $\mathcal{A}^{(1)}(z)$ is provided by the power term in (8) which is invisible in the perturbative expansion. The reason for the regularity of the coupling $\mathfrak{A}^{(1)}(s)$ is connected with the summation of the so-called π^2-terms that play an important role in analyzing various hadronic processes [10, 12–15].

B. Resummation of threshold singularities

In describing a charged particle-antiparticle system near threshold, it is well known from QED that the so-called Coulomb resummation factor plays an important role. This resummation, performed on the basis of the nonrelativistic Schrödinger equation with the Coulomb potential $V(r) = -\alpha/r$, leads to the Sommerfeld-Sakharov S-factor [16, 17]. In the threshold region one cannot truncate the perturbative series and the S-factor should be taken into

account in its entirety. The S-factor appears in the parametrization of the imaginary part of the quark current correlator, which can be approximated by the Bethe-Salpeter amplitude of the two charged particles, $\chi_{\mathrm{BS}}(x = 0)$ [18]. The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential, leads to the appearance of the resummation factor in the parametrization of the $R(s)$-function discussed above.

For a systematic relativistic analysis of quark-antiquark systems, it is essential from the very beginning to have a relativistic generalization of the S-factor. A new form for this relativistic factor in the case of QCD has been proposed in [7]

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}, \qquad X(\chi) = \frac{\pi\,\alpha}{\sinh\chi}, \tag{9}$$

where χ is the rapidity which related to s by $2m\cosh\chi = \sqrt{s}$, $\alpha \to 4\alpha_s/3$ in QCD. The function $X(\chi)$ can be expressed in terms of $v = \sqrt{1 - 4m^2/s}$: $X(\chi) = \pi\alpha\sqrt{1 - v^2}/v$. The relativistic resummation factor (9) reproduces both the expected nonrelativistic and ultrarelativistic limits and corresponds to a QCD-like Coulomb potential. Here we consider the vector channel for which a threshold resummation S-factor for the s-wave states is used. For the axial-vector channel the P-factor is required. The corresponding relativistic factor has recently been found in [19].

To incorporate the quark mass effects one usually uses the approximate expression proposed in [1, 20] above the quark-antiquark threshold

$$\mathcal{R}(s) = T(v)\left[1 + g(v)r(s)\right], \tag{10}$$

where

$$T(v) = v\frac{3 - v^2}{2}, \quad g(v) = \frac{4\pi}{3}\left[\frac{\pi}{2v} - \frac{3 + v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)\right], \quad v_f = \sqrt{1 - \frac{4m_f^2}{s}}. \tag{11}$$

The function $g(v)$ is taken in the Schwinger approximation [21].

One cannot directly use the perturbative expression for $r(s)$ in Eq. (10), which contains unphysical singularities, to calculate, for example, the Adler D-function. Instead, one can use the APT representation for $r(s)$. The explicit three-loop form for $r_{\mathrm{APT}}(s)$ can be found in [2]. Besides this replacement, one has to modify the expression (10) in such a way as to take into account summation of an arbitrary number of threshold singularities. Including

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the threshold resummation factor (9) leads to the following modification of the expression (10) [2, 22] for a particular quark flavour f

$$\mathcal{R}_f(s) = [R_{0,f}(s) + R_{1,f}(s)] \, \Theta(s - 4m_f^2), \tag{12}$$

$$R_0(s) = T(v) \, S(\chi), \qquad R_1(s) = T(v) \left[r_{\mathrm{APT}}(s) \, g(v) - \frac{1}{2}X(\chi) \right].$$

The usage of the resummation factor (9) reflects the assumption that the coupling is taken in the V renormalization scheme. To avoid double counting, the function R_1 contains the subtraction of $X(\chi)$. The potential term corresponding to the R_0 function gives the principal contribution to $\mathcal{R}(s)$, the correction R_1 amounting to less than twenty percent for the whole energy interval [19].

C.   Quark masses

The following considerations suggest the behavior of the mass function of the light quarks in the infrared region. A solution of the Schwinger-Dyson equations [23–25] demonstrates a fixed infrared behavior of the invariant charge and the quark mass function. The mass function of the light quarks at small momentum looks like a plateau with a height approximately equal to the constituent mass, then with increasing momentum the mass function rapidly decreases and approaches the small current mass.

This behavior can be understood by using the concept of the dynamical quark mass. This mass has an essentially nonperturbative nature. Its connection with the quark condensate has been established in [26]. By using an analysis based on the Schwinger-Dyson equations a similar relation has been found in [27]. It has been demonstrated in [28] that on the mass-shall one has a gauge-independent result for the dynamical mass

$$m^3 = -\frac{4}{3}\pi\alpha_s \langle 0|\bar{q}\,q|0\rangle. \tag{13}$$

A result obtained in [29] demonstrates the step-like behaviour of the mass function. The height m of the plateau is given by the quark condensate (13). According to these results it is reasonable to assume that at small p^2 the function $m(p^2)$ is rather smooth (nearly constant). In the region $p^2 > 1$–2 GeV the principal behavior of the function $m(p^2)$ is defined by perturbation theory with the renormalization group improvement.
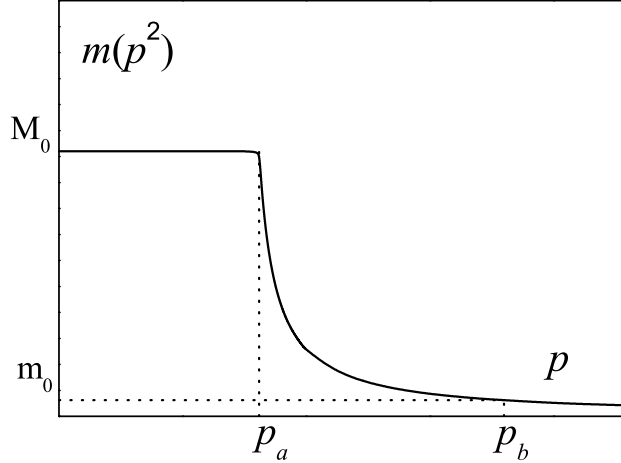
FIG. 1: Effective quark mass.

TABLE I: Typical values of m_0^f and M_0^f.

f	u	d	s	c	b	t
m_0^f (GeV)	0.004	0.007	0.130	1.35	4.4	174
M_0^f (GeV)	0.260	0.260	0.450	1.35	4.4	174.0

The following analysis was performed by using the model mass function $m(p^2)$ that is shown in Fig. 1. We take the curve that connects the points p_a and p_b to have the form $A^3/(p^2 - B^2)$. The parameters m_0 are taken from the known values of the running (current) masses at $p_b = 2$ GeV. The quantities considered here are not too sensitive to the parameters of the heavy quarks and we take for c, b and t quarks $m^f(p^2) = m_0^f = M_0^f = $ const. The values of m_0^f at 2 GeV [30] and typical values of M_0^f are shown in Table I.

III.   PHYSICAL QUANTITIES AND FUNCTIONS GENERATED BY $R(s)$

In this section we apply the model we have formulated to describe the physical quantities and functions connected with $R(s)$, described in the Introduction.

A.   Inclusive decay of the τ-lepton

The ratio of hadronic to leptonic τ-decay widths in the vector channel is expressed by Eq. (1), where $R^{(0)} = 3\,|V_{ud}|^2\,S_{EW}/2$, $|V_{ud}| = 0.9752 \pm 0.0007$ is the CKM matrix element, $S_{EW} = 1.0194 \pm 0.0040$ is the electroweak factor, and $M_\tau = 1776.99^{+0.29}_{-0.26}$ MeV is the mass of

the τ-lepton [30]. The experimental data obtained by the ALEPH and OPAL collaborations for this ratio is [31–33]: $R_{\tau,V}^{\text{ALEPH}} = 1.775 \pm 0.017$, $R_{\tau,V}^{\text{OPAL}} = 1.764 \pm 0.016$.

In our analysis we use the nonstrange vector channel spectral function obtained by the ALEPH collaboration [31] and keep in all further calculations the value $R_{\tau,V}^{\text{ALEPH}}$ as the normalization point. The range of estimates are obtained by varying the quark masses in the interval $M_0^{u,d} = 260 \pm 10$ MeV (this band is fixed rather definitely by the D-function considered below) and $M_0^c = 450 \pm 100$ MeV. The results for R_τ^V are given below.

B. D_V-function

The experimental information obtained by the ALEPH and OPAL collaborations allows us to construct the nonstrange vector channel "experimental" D-function. Within the analytic approach this function has been analysed in [2]. Here we improve our method of constructing the "light" D-function by taking into account the global duality relation. We demonstrate that this Euclidean object is useful from the point of view of defining the effective masses of the light quarks.

In order to construct the Euclidean D-function (2) we use for $R(s)$ the following expression

$$R(s) = R^{\text{expt}}(s)\,\theta(s_0 - s) + R^{\text{theor}}(s)\,\theta(s - s_0)\,. \tag{14}$$

The continuum threshold s_0 we find from the global duality relation [34]

$$\int_0^{s_0} ds\, R^{\text{expt}}(s) = \int_0^{s_0} ds\, R^{\text{theor}}(s). \tag{15}$$

This gives $s_0 \simeq 1.5$ GeV^2. The value of s_0 agrees with the results of papers [35–37]. A similar value of the continuum parameter is used in the QCD sum rules [38, 39]. Note, for some parameters there are two possible solutions of the duality condition (15). We exclude the second solution, $s_0 \simeq 2.5$ GeV^2, at this stage of the analysis due to the requirement of describing, in a self-consistent manner, different experimental data.

The low energy τ-data in the nonstrange vector channel results in the curve for $D(Q^2)$ in Fig. 2. In this figure we also plot three theoretical curves corresponding to masses of the light quarks of 150, 260 and 350 MeV. Fig. 2 demonstrates that the shape of the infrared tail of the D-function is quite sensitive to the value of the light quark masses.
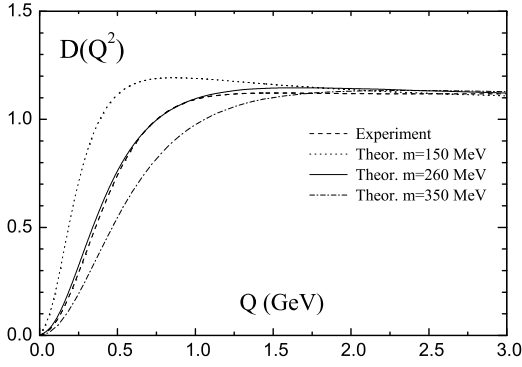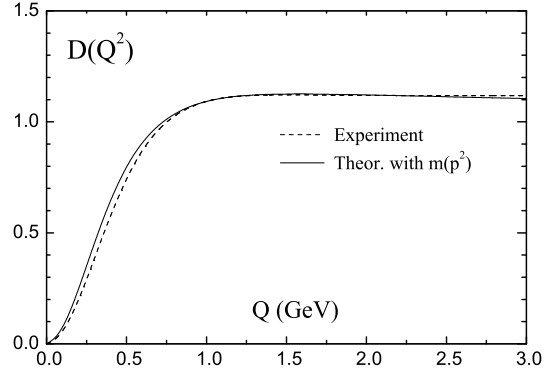
FIG. 2: D-function for $m = \text{const.}$

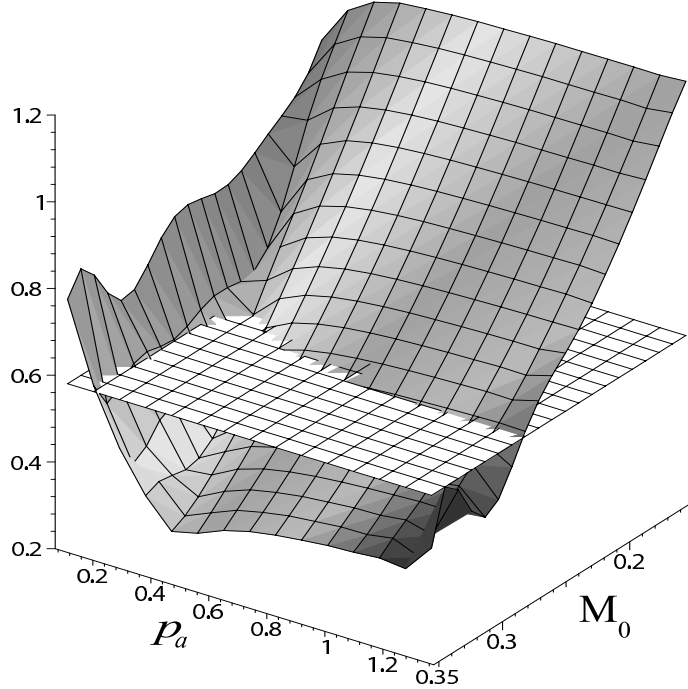FIG. 3: D-function for $m = m(p^2)$.

FIG. 4: D-function surface at $Q_0 = 0.5$ GeV vs. parameters M_0 and p_a. The plane corresponds to $D_{\text{expt}}(Q_0^2)$.

The measured quantity R_τ^V defined in Eq. (1) is less sensitive to m_u and m_d values than the infrared tail of the D_V-function. Varying the light quark masses over a wide range one finds $R_\tau^V = 1.79$ for $m_u = m_d = 150$ MeV and $R_\tau^V = 1.66$ for 350 MeV. The values of masses $m_u = m_d \simeq 260$ MeV agree with the experimental value $R_\tau^V = 1.775 \pm 0.017$ [31]. The values of the light quark masses are close to the constituent quark masses and therefore incorporate

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nonperturbative effects. These values are consistent with other results [40–42] and with the analysis performed in [35].

A result for the D-function that is obtained by using the mass function $m(p^2)$ with parameters defined in Table I and $p_a = 0.8$ GeV is shown in Fig. 3. Thus we obtain results that are rather close to the results obtained for $m(p^2) = \text{const} = 260$ MeV. In Fig. 4 we plot a 3-dimensional graph of the function $D(Q_0^2)$ as function of the parameters M_0 and a. The plane corresponds to the experimental value $D_{\text{expt}}(Q_0^2) = 0.58$ at $Q_0 = 0.5$ GeV. Fig. 4 demonstrates that for $p_a > 0.4$–0.5 GeV the curve of intersection of the surface $D(Q_0^2; M_0, p_a)$ and the plane is approximately a straight line, corresponding to $M_0 = 260$ MeV. The large p_a-limit reproduces the results with $m(p^2) = \text{const}$.

C.   Smeared R_Δ-function

To compare experimental and theoretical results from the point of view of the quark-hadron duality, in [1] it was proposed to use the smeared function $R_\Delta(s)$. Instead of the Drell ratio $R(s)$ defined in terms of the discontinuity of the correlation function $\Pi(q^2)$ across the physical cut

$$R(s) = \frac{1}{2\pi i} \left[\Pi(s + i\epsilon) - \Pi(s - i\epsilon)\right], \tag{16}$$

the smeared function $R_\Delta(s)$ is defined as

$$R_\Delta(s) = \frac{1}{2\pi i} \left[\Pi(s + i\Delta) - \Pi(s - i\Delta)\right], \tag{17}$$

with a finite value of Δ to keep away from the cut. If Δ is sufficiently large and both the experimental data and the theory prediction are smeared, it is possible to compare theory with experiment.

Equation (17) and the dispersion relation for the correlator $\Pi(q^2)$ give the representation (3). Note that the smeared function $R_\Delta(s)$ is defined both in the Minkowskian region of positive s, where a trace of resonances still remains for not too large Δ, and in the Euclidean domain of negative argument s, where like the Adler function $D(Q^2)$ the function $R_\Delta(s)$ is smooth and monotone.

As with the Adler function we will construct the "light" experimental function $R_\Delta(s)$. For this purpose we match the experimental data taken with $s < s_0$ to the theoretical result
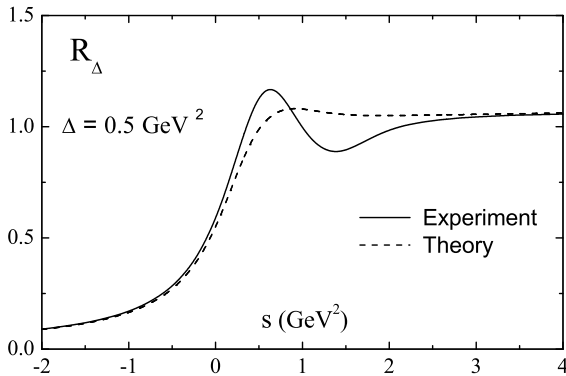
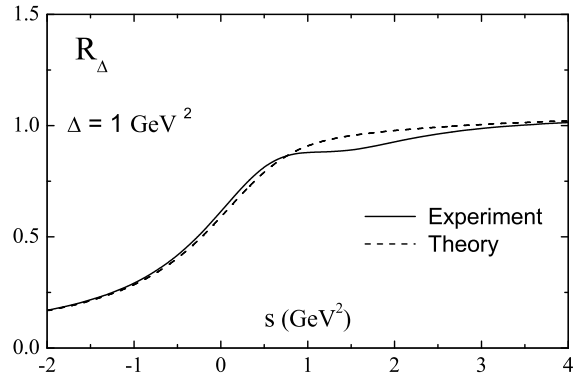FIG. 5: Smeared function for $\Delta = 0.5 \text{ GeV}^2$.   FIG. 6: Smeared function for $\Delta = 1.0 \text{ GeV}^2$.

taken with $s > s_0$ as in (14). The value $s_0 \simeq 1.6 \text{ GeV}^2$ is found from the duality relation (15).

For the charm region the value of Δ is about 3 GeV^2. An adequate choice in the case of the light smeared function is $\Delta \simeq 0.5$–1.0 GeV^2. In Figs. 5 and 6 the experimental and theoretical curves for $\Delta = 0.5 \text{ GeV}^2$, $\Delta = 1.0 \text{ GeV}^2$ and $m = m(p^2)$ are shown. Let us emphasize that, for reasonable values of Δ, in the spacelike region ($s < 0$) there is a good agreement between data and theory starting from $s = 0$.

D.   Hadronic contribution to a_μ

The hadronic contribution to the anomalous magnetic moment of the muon in the leading order in the electromagnetic coupling constant is defined by (4), where $\alpha^{-1} = \alpha(0)^{-1} = 137.035\,999\,11(46)$ [30], and (see, for example, [21])

$$K(s) = \int\limits_0^1 dx \, \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2}. \tag{18}$$

The muon mass is $m_\mu = 105.7 \text{ MeV}$.

The expression (4) can be rewritten in terms of the D-function

$$a_\mu^{\text{had}} = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{2} \int\limits_0^1 \frac{dx}{x} (1-x)(2-x) D\left(\frac{x^2}{1-x} m_\mu^2\right). \tag{19}$$

It is should be emphasized that the expressions (4) and (19) are equivalent due to the analytic properties of the function $\Pi(q^2)$. If one uses a method that does not maintain the required properties of $\Pi(q^2)$, expressions (4) and (19) will no longer be equivalent and will

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imply different results (see [43] for details). This situation is similar to that which occurs in the analysis of inclusive τ-decay [8], where the initial integral, performed over an interval including a nonperturbative region, for which a perturbative QCD calculation is not valid, is transformed based on the analytic properties into a contour representation. Within APT one is justified in doing this, and can use equally well either the expression (4) or the expression (19).

TABLE II: Dependence of a_μ^{had} on light quark masses.

m_q (MeV)		$a_\mu^{\text{had}} \times 10^{10}$			
		LO		NNLO	
$q = u, d$	$q = s$	$m_q = $ const	$m_q \neq $ const	$m_q = $ const	$m_q \neq $ const
250	400	736	760	725	763
250	500	716	736	705	726
260	400	691	715	682	711
260	500	671	690	661	685

The value of a_μ^{had} is not very sensitive to the values of the heavy quark masses, which we take as given in Table I. The relative contributions of u and d quarks are about 72 and 19 %, respectively. The relative factor of 4 between u and d contributions is explained by the ratio of quark charges. The relative contribution of the s-quark to a_μ^{had} is about 5–9 % for $M_0^s = 400$–500 MeV. The contribution of the c-quark is about 2 %. Contributions of b and t quarks are very small.

There is a significant dependence on the mass parameters of the light quarks. This dependence we illustrate in Table II. In our calculations we take into account the matching conditions at quark thresholds according to the procedure described in [6]. The mass parameters of u and d quarks are fixed rather well by the infrared tail of the light D-function and the value of R_τ^V. If we take for the parameter $M_0^{u,d}$ in the function $m = m(p^2)$ the best fit value 260 MeV and vary $M_0^s = 400$–500 MeV, we get

$$a_\mu^{\text{had}} = (698 \pm 13) \times 10^{-10}. \tag{20}$$

Alternative "theoretical" values of a_μ^{had} are extracted from e^+e^- annihilation and τ decay data: $(696.3 \pm 6.2_{\text{exp}} \pm 3.6_{\text{rad}}) \times 10^{-10}$ (e^+e^--based) [44], which is 1.9σ below the BNL

experiment [45]; $(711.0 \pm 5.0_{\text{exp}} \pm 0.8_{\text{rad}} \pm 2.8_{SU(2)}) \times 10^{-10}$ (τ-based) [44], which is within 0.7σ of experiment; and $(693.4 \pm 5.3_{\text{exp}} \pm 3.5_{\text{rad}}) \times 10^{-10}$ (e^+e^--based) [46], 2.7σ below experiment. An even lower value $(692.4 \pm 5.9_{\text{exp}} \pm 2.4_{\text{rad}}) \times 10^{-10}$ is given by [47]. The quantity a_μ^{had} is rather sensitive to the light quark mass parameters, which are known only with large uncertainties. For this reason our estimations at this stage cannot give a preference to one or another of the above-mentioned fits to experimental data.

E. Hadronic contributions to $\Delta\alpha$

Consider the hadronic correction to the electromagnetic fine structure constant α at the Z-boson scale. The evolution of the running electromagnetic coupling is described by

$$\alpha(s) = \frac{\alpha(0)}{1 - \Delta\alpha_{\text{lept}}(s) - \Delta\alpha_{\text{had}}^{(5)}(s) - \Delta\alpha_{\text{had}}^{\text{top}}(s)}. \tag{21}$$

The leptonic part $\Delta\alpha_{\text{lept}}(s)$ is known to the three loop level, $\Delta\alpha_{\text{lept}}(M_Z^2) = 0.03149769$ [48]. It is conventional to separate the contribution $\Delta\alpha_{\text{had}}^{(5)}(s)$ coming from the first five quark flavors. The contribution of the t-quark is estimated as $\Delta\alpha_{\text{had}}^{\text{top}}(M_Z^2) = -0.000070(05)$ [49].

The quantity $\Delta\alpha_{\text{had}}^{(5)}(s)$ at the Z-boson scale can be represented in the form of the dispersion integral (5). The total function $R(s)$ is

$$R(s) = 3 \sum_f Q_f^2 \mathcal{R}_f(s), \tag{22}$$

where Q_f is the quark electric charge of flavour f. For the calculation of $R(s)$ we use (22) with five quark flavors $f = u, d, s, c, b$. Varying the parameters as has been described above and using $m_c = 1.3$–1.5 GeV, we get

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (278.2 \pm 3.5) \times 10^{-4}. \tag{23}$$

This value is to be compared with predictions extracted from a wide range of data describing $e^+e^- \to$ hadrons [47]:

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (275.5 \pm 1.9_{\text{expt}} \pm 1.3_{\text{rad}}) \times 10^{-4}. \tag{24}$$

We see that our result (23) is consistent with previous theoretical/experimental evaluations, with comparable uncertainties.

The relative error in (23) is substantially less than the error that appears in the quantity a_μ^{had} and therefore one can obtain a more exact result. In comparison with the a_μ^{had} result, where the contribution of the c-quark was about $2\,\%$, now it is about $30\,\%$. The contribution of the b-quark is about $5\,\%$ and the relative contribution of the t-quark is a fraction of a percent.

IV. CONCLUSIONS

A method of performing QCD calculations in the nonperturbative domain has been developed. This method is based on the analytic approach to QCD, in which there are no unphysical singularities, and takes into account the summation of threshold singularities and the involvement of nonperturbative light quark masses.

The following quantities have been analysed: the inclusive τ-decay characteristic in the vector channel, R_τ^V; the light-quark Adler function, $D(Q^2)$; the smeared R_Δ-function; the hadronic contribution to the anomalous magnetic moment of the muon, a_μ^{had}; and the hadronic contribution to the fine structure constant, $\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2)$. We have demonstrated that the proposed method allows us to describe these quantities rather well.

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