OKHEP-95-07

Quasi-Local Formulation of Non-Abelian Finite-Element Gauge Theory

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(January 4, 1996)

Abstract

Recently it was shown how to formulate the finite-element equations of motion of a non-Abelian gauge theory, by gauging the free lattice difference equations, and simultaneously determining the form of the gauge transformations. In particular, the gauge-covariant field strength was explicitly constructed, locally, in terms of a path ordered product of exponentials (link operators). On the other hand, the Dirac and Yang-Mills equations were nonlocal, involving sums over the entire prior lattice. Earlier, Matsuyama had proposed a local Dirac equation constructed from just the above-mentioned link operators. Here, we show how his scheme, which is closely related to our earlier one, can be implemented for a non-Abelian gauge theory. Although both Dirac and Yang-Mills equations are now local, the field strength is not. The technique is illustrated with a direct calculation of the current anomalies in two and four space-time dimensions. Unfortunately, unlike the original finite-element proposal, this scheme is in general nonunitary.

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I. INTRODUCTION

An alternative approach to lattice field theories, based on the finite-element equations of motion, has been under development for over a decade. (For a recent review see [1].) Shortly after the introduction of this method, it was seen how a Abelian gauge field could be coupled to a fermion in this way [2]. The resulting Dirac equation was nonlocal: In Minkowski space-time, the term proportional to γ^j involved a sum over all values of the corresponding lattice coordinate, m_j, $1 \leq m_j \leq M$, where M is the number of lattice sites in the j direction, while the term proportional to γ^0 involved a sum over all previous times, $0 \leq n' \leq n$, where n is the current lattice time. Shortly after our paper appeared, Matsuyama [3] proposed a local finite-element Dirac equation for QED, based on the immediate introduction of link operators into the free Dirac equation. Although it could be argued that this latter approach was somewhat unnatural because it introduced interactions into the mass term, the primary reason this idea was not pursued was that it was quite unclear what the form of the non-Abelian gauge transformations should be on the finite-element lattice.

Instead, the first foray into non-Abelian finite element gauge theory [4] was based on straightforward gauging of the global phase symmetry of the free finite-element Dirac equation. The form of the interacting Dirac equation was determined, nonlocally as in the Abelian case, and at the same time, the form of the gauge transformations of the vector and scalar potentials was determined, in terms of an infinite sequence of nested commutators. The Yang-Mills equations were determined analogously. The only thing not explicitly determined at the time was the form of the construction of the field strength tensor in terms of the potentials; although it was perfectly clear that the process could be continued indefinitely, only the first four terms in the sequence in powers of potentials were given. Although [4] had been restricted to (1+1) dimensions for simplicity, that restriction was easily removed [5].

The completion of this construction was only given this spring [6]. The essential element was the recognition that under the previously-determined gauge transformations, a suitable link operator transformed appropriately. These operators then can be used to transform the field strength $F_{\mu\nu}$ averaged over the finite-element hypercube to the μ, ν plane where it may be expressed as a path-ordered product of link operators around a plaquette.

This development makes it possible to revisit Matsuyama's scheme [3]. We will see that it is not only possible to formulate a local Dirac equation in the non-Abelian regime, but local gauge-covariant Yang-Mills equations as well. The resulting equations are inequivalent to those given previously [6], but not so different either, for the previous equations were quasilocal, as seen in the simple difference equation given for the interaction terms. But the new formulation is still *nonlocal* in that the field strength that appears in the Yang-Mills equation involves the vector potential over the entire previous lattice.

In the next section we restate the gauge transformation properties of the link operators, and give the corresponding construction of the field strength. Then, in Sec. 3 we restate Matsuyama's prescription for the Dirac equation, followed by the corresponding local Yang-Mills equation. The resulting nonlocality in the construction of the field strength is shown. A simple calculation of the axial-vector anomaly in two dimensions is given in Sec. 4. However, it is not clear how to extend such calculations to four dimensions because, unlike the original formulation [7], in general interactions here break unitarity. In a particular gauge in which the transfer matrix is unitary, the current anomalies are computed in the smallest nontrivial four-dimensional lattice in Sec. 5. The corresponding calculation in the standard finite-element formulation is given in Sec. 6. A discussion of how symmetry breaking occurs here is given in the Conclusion and the Appendix.

II. GAUGE-COVARIANT LINK OPERATORS AND CONSTRUCTION OF THE FIELD STRENGTH

It is convenient to define the link operator using coordinates referring to a given finite element:

$$(L_\mu)_{ijkl} = e^{-igh(\mathcal{A}_\mu)_{m_1+i,m_2+j,m_3+k,m_4+l}}, \tag{2.1}$$

where h is the lattice constant, and the vector potential is an appropriately averaged one:

$$\mathcal{A}^{\mu}_{m_\mu,\mathrm{m}_\perp} = \frac{1}{2}\left(A^{\mu}_{m_\mu,\mathrm{m}_\perp} + A^{\mu}_{m_\mu-1,\mathrm{m}_\perp}\right). \tag{2.2}$$

The notation here is that $\mathrm{m}_\perp$ refers to space-time lattice coordinates other that the one singled out, m_μ. Each index in (2.1) takes on two values, 0 or 1. The result of a detailed, constructive calculation [6] is the following simple transformation law for the link operator

$$\delta(L_\mu)_{1000} = ig[\delta\omega_{0000}(L_\mu)_{1000} - (L_\mu)_{1000}\delta\omega_{1000}], \tag{2.3}$$

where we have assumed that the first coordinate index refers to the μ direction. Then, it is easy to see that the "transversely-local" field strength $\mathcal{F}_{\mu\nu}$ is given by the following path-ordered product of link operators around the μ, ν "plaquette":

$$e^{-igh^2(\mathcal{F}_{\mu\nu})_\mathrm{m}} = Pe^{-ig\oint A\cdot dl} = (L_\mu)_{1000}(L_\nu)_{1100}(L^\dagger_\mu)_{1100}(L^\dagger_\nu)_{0100}, \tag{2.4}$$

where the first index is the μ coordinate and the second index the ν coordinate. To construct the full field strength, which is forward-averaged over a hypercube with lower left-hand corner at m we use the gauge covariant average operators constructed from the links:

$$(\tilde{D}_\lambda \mathcal{F}^{\mu\nu})_{0000} = \frac{1}{2}\left[(L_\lambda)_1(\mathcal{F}^{\mu\nu})_1(L^\dagger_\lambda)_1 + (\mathcal{F}^{\mu\nu})_0\right], \tag{2.5}$$

where on the right side we have only displayed the λ coordinate (the rest are 0). That is, with an overbar representing forward averaging,

$$x_{\overline{m}} = \frac{1}{2}(x_m + x_{m+1}), \tag{2.6}$$

the field strength is given by[1]

$$(F_{\mu\nu})_{\overline{\mathrm{m}}} = \left(\prod_{\lambda\neq\mu,\nu} \tilde{D}_\lambda \mathcal{F}_{\mu\nu}\right)_\mathrm{m}, \tag{2.7}$$

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[1]The averaging over the finite element on the left side of (2.7), required by the finite-element prescription, is necessary to ensure unitarity. If $(F_{\mu\nu})_{\overline{\mathrm{m}}}$ were replaced by $(F_{\mu\nu})_\mathrm{m}$, even the free theory would not be unitary.

where symmetrical averaging is to be understood. It is immediately obvious that $F_{\mu\nu}$ given by (2.7) transforms covariantly,

$$(\delta F^{\mu\nu})_{\overline{\mathrm{m}}} = ig[\delta\omega_{\mathrm{m}}, (F^{\mu\nu})_{\overline{\mathrm{m}}}]. \tag{2.8}$$

III. LOCAL FORMULATION OF YANG-MILLS EQUATIONS

Matsuyama had proposed a local finite-element formulation of a fermion interacting with an Abelian gauge field [3]. He began by adopting a local form for the fermionic gauge transformation,

$$\delta\psi_{\mathrm{m}} = ig\delta\omega_{\mathrm{m}}\psi_{\mathrm{m}}, \tag{3.1}$$

defined on fields at the lattice sites, rather than in the middle of the finite element as in [2,4,6]. Then covariant derivative and averaging operators can be defined in terms of the link operators defined in (2.1):

$$(D_\mu\psi)_{0000} = \frac{1}{h}\left[(L_\mu)_1\psi_1 - \psi_0\right], \tag{3.2a}$$
$$(\tilde{D}_\mu\psi)_{0000} = \frac{1}{2}\left[(L_\mu)_1\psi_1 + \psi_0\right]. \tag{3.2b}$$

Up to an ordering ambiguity then, the gauge covariant Dirac equation is

$$i\gamma^\mu \prod_{\nu\neq\mu} \tilde{D}_\nu D_\mu\psi + \mu\prod_\nu \tilde{D}_\nu\psi = 0. \tag{3.3}$$

By virtue of (2.3) this equation (3.3) is covariant not only under Abelian gauge transformations, but under non-Abelian ones as well. Finally, we can transform to a gauge-covariant average Dirac field by averaging operators,

$$\Psi_{\overline{\mathrm{m}}} = \left(\prod_\lambda \tilde{D}_\lambda\psi\right)_{\mathrm{m}}, \tag{3.4}$$

[$\neq \psi_{\overline{m}}$ using the notation of (2.6)] which transforms covariantly in the sense of [6],

$$\delta\Psi_{\overline{\mathrm{m}}} = ig\delta\omega_{\mathrm{m}}\Psi_{\overline{\mathrm{m}}}. \tag{3.5}$$

However, in [6] it the "free" average of the Dirac field, $\psi_{\overline{m}}$ that transforms covariantly.

We proceed similarly in the gauge sector. If the field strength like the Dirac field transformed locally at the lattice sites,

$$(\delta F^{\mu\nu})_{\mathrm{m}} = ig[\delta\omega_{\mathrm{m}}, (F^{\mu\nu})_{\mathrm{m}}] \tag{3.6}$$

rather than as in [6], gauge covariant derivative and averaging operators could be defined by (2.5) and

$$(D_\lambda F^{\mu\nu})_{0000} = \frac{1}{h}\left[(L_\lambda)_1 (F^{\mu\nu})_1 (L_\lambda^\dagger)_1 - (F^{\mu\nu})_0\right], \tag{3.7}$$

which yields the local covariant Yang-Mills equation:

$$\prod_{\lambda\neq\nu} \tilde{D}_\lambda D_\nu F^{\mu\nu} = j^\mu, \tag{3.8}$$

where we can adopt the following as a gauge-covariant current (see Sec. 6):

$$(j^\mu)_{\mathrm{m}} = g\overline{\Psi}_{\mathrm{m}} T \gamma^\mu \Psi_{\mathrm{m}}. \tag{3.9}$$

However, (3.6) *does not hold!* As shown in Sec. 2 the field strength constructed locally in terms of the link operators transforms according to (2.8), defined at the center of the finite element. However, given $F_{\mu\nu}$ constructed in Sec. 2, we can construct a locally covariant field strength $f_{\mu\nu}$ according to

$$(F_{\mu\nu})_{\overline{\mathrm{m}}} = \left(\prod_\lambda \tilde{D}_\lambda f_{\mu\nu}\right)_{\mathrm{m}}, \tag{3.10}$$

which does transform according to (3.6), and which then satisfies (3.8). Note that, in part, (3.10) undoes the transformation (2.7), so that if we choose an appropriate ordering

$$(\mathcal{F}_{\mu\nu}) = \tilde{D}_\mu \tilde{D}_\nu f_{\mu\nu}. \tag{3.11}$$

However, as a result of inverting the averaging operators $\tilde{D}_\mu$, that is solving the difference equation (3.11), $f_{\mu\nu}$ is not local, but depends on vector potentials over the entire prior lattice. It appears impossible to have a completely local formalism.

IV. AXIAL-VECTOR CURRENT ANOMALY IN (1+1) DIMENSIONS

Let us illustrate the calculational aspects of this scheme in the simplest context, that of an Abelian theory in which the only nontrivial element is the Dirac equation (3.3). (We will henceforward replace g by e.) Although the result was stated in [3], it is useful to first revisit the two-dimensional case of the Schwinger model, with the fermion mass $\mu = 0$. In terms of chiral components, that is, eigenvectors of $i\gamma_5$ with eigenvalue equal to ± 1, the solution of that equation is (for a particular ordering)

$$\psi^{(+)}_{m+1,n+1} = e^{ieh(\mathcal{A}_0)_{m,n+1}} e^{ieh(\mathcal{A}_1)_{m+1,n+1}} \psi^{(+)}_{m,n}, \tag{4.1a}$$

$$\psi^{(-)}_{m,n+1} = e^{ieh(\mathcal{A}_0)_{m,n+1}} e^{-ieh(\mathcal{A}_1)_{m+1,n}} \psi^{(-)}_{m+1,n}. \tag{4.1b}$$

The current is given by (3.9) with $T = 1$; in terms of chiral components what we wish to compute are the following combinations:

$$\langle \text{“}\partial_\mu j^\mu\text{”} \rangle = \frac{1}{h} \langle j^{(+)}_{m+1,n+1} - j^{(+)}_{m,n} + j^{(-)}_{m,n+1} - j^{(-)}_{m+1,n} \rangle, \tag{4.2a}$$

$$\langle \text{“}\partial_\mu j_5^\mu\text{”} \rangle = \frac{1}{h} \langle j^{(+)}_{m+1,n+1} - j^{(+)}_{m,n} - j^{(-)}_{m,n+1} + j^{(-)}_{m+1,n} \rangle, \tag{4.2b}$$

where the quotation marks signify finite-element lattice derivatives. We use the solution (4.1) to refer all Dirac fields to the intermediate time $n+1$ and we evaluate the fermion matrix elements according to the Fock space rule [8]

$$\langle \psi^{(\pm)\dagger}_{m,n} \psi^{(\pm)}_{m+q,n} \rangle = \pm q \frac{i}{L} \frac{1}{\sin \pi/M}, \qquad \text{for } M \text{ even}, \tag{4.3a}$$

$$\langle \psi^{(\pm)\dagger}_{m,n} \psi^{(\pm)}_{m+q,n} \rangle = \pm q \frac{i}{L} \frac{\cos^2 \pi/2M}{\sin \pi/M}, \qquad \text{for } M \text{ odd}, \tag{4.3b}$$

where $q = \pm 1$. In both cases, the vacuum expectation value is taken to zero if $m = m'$. (As noted in [8] the actual value in that case is irrelevant.) Then using (4.1) in, for example,

$$\begin{aligned}\left(\prod_\nu \tilde{D}_\nu \psi^{(+)} \right)_{m,n} &= \frac{1}{4} \Big(\psi^{(+)}_{m,n} + e^{-ieh(\mathcal{A}_1)_{m+1,n}} \psi^{(+)}_{m+1,n} \\ &\quad + e^{-ieh(\mathcal{A}_0)_{m,n+1}} \psi^{(+)}_{m,n+1} + e^{-ieh(\mathcal{A}_1)_{m+1,n}} e^{-ieh(\mathcal{A}_0)_{m+1,n+1}} \psi^{(+)}_{m+1,n+1} \Big),\end{aligned} \tag{4.4}$$

we easily find that the vector current is conserved exactly[2],

$$\langle \text{“}\partial_\mu j^\mu\text{”} \rangle = 0 \tag{4.5}$$

while the axial-vector current is anomalous,

$$\begin{aligned}\langle \text{“}\partial_\mu j_5^\mu\text{”} \rangle &= \frac{e}{2Mh^2}\frac{1}{\sin \pi/M}[\sin eh(\mathcal{A}_1)_{m+2,n+1} + \sin eh(\mathcal{A}_1)_{m+1,n+1} \\ &\quad + \sin eh[(\mathcal{A}_0)_{m+1,n+1} - (\mathcal{A}_0)_{m+2,n+1} - (\mathcal{A}_1)_{m+2,n}] \\ &\quad - \sin eh[(\mathcal{A}_0)_{m+1,n+1} + (\mathcal{A}_1)_{m+1,n} - (\mathcal{A}_0)_{m,n+1}] \\ &\approx \frac{e}{M \sin \pi/M} E, \quad (h \to 0),\end{aligned} \tag{4.6}$$

where $E =$ “$\partial_0 A_1 - \partial_1 A_0$” is the lattice electric field, the familiar finite-element lattice result [1].

The above calculation seems quite similar to that given for the nonlocal finite-element formulation in [8], and is certainly no simpler. In fact, the calculations are identical if, as in [8], we choose the gauge $A_0 = 0$. For then the massless Dirac equation can be written in terms of the transfer matrix, defined by

$$\psi_{n+1} = T\psi_n, \tag{4.7}$$

which is to be understood as a matrix equation in the spatial coordinates. Here the transfer matrix is

$$T = \frac{1 + i\gamma_5 \mathcal{D}}{1 - i\gamma_5 \mathcal{D}}, \tag{4.8}$$

where the covariant derivative, $\mathcal{D} = -(h/2)(\tilde{D}_1)^{-1} D_1$, is defined in terms of (the time coordinate n is suppressed)

$$(\tilde{D}_1)_{m,m'} = \frac{1}{2}\left(\delta_{m,m'} + \delta_{m+1,m'} e^{-ieh(\mathcal{A}_1)_{m+1}}\right), \tag{4.9a}$$

$$(D_1)_{m,m'} = \frac{1}{h}\left(-\delta_{m,m'} + \delta_{m+1,m'} e^{-ieh(\mathcal{A}_1)_{m+1}}\right). \tag{4.9b}$$

[2]In [3] results (4.5) and (4.6) were established only to $O(h)$.

Equation (4.9a) is inverted as in (17) of [8] with the result that the covariant derivative operator coincides exactly with that given in (10) of that reference, and hence the same conclusions follow.

V. CURRENT ANOMALIES IN (3+1) DIMENSIONS

We wish to repeat the above calculation in four dimensions. Again we will set the mass $\mu = 0$, so we have to solve the following symbolic Dirac equation involving the link operators (2.1) for the simplest possible ordering:

$$\gamma^0(L_3+1)(L_2+1)(L_1+1)(L_0-1)\psi + \gamma^1(L_3+1)(L_2+1)(L_1-1)(L_0+1)\psi$$
$$+\gamma^2(L_3+1)(L_2-1)(L_1+1)(L_0+1)\psi + \gamma^3(L_3-1)(L_2+1)(L_1+1)(L_0+1)\psi = 0, \quad (5.1)$$

where, for example, $\psi = \psi_{0000}$, $L_0\psi = (L_0)_{0001}\psi_{0001}$, and $L_2 L_0\psi = (L_2)_{0100}(L_0)_{0101}\psi_{0101}$. We must solve this system of equations at each lattice site at a given time. For simplicity, let us consider the simplest nontrivial rectangular spatial lattice, with the number of sites in the 1 direction being $M_1 = 2$, while in the other two directions there is but a single site, $M_2 = M_3 = 1$. We anticipate the periodic/antiperiodic boundary condition, [2],

$$\psi_{m+M} = (-1)^{M+1}\psi_m, \tag{5.2}$$

so we expect that the fields should be antiperiodic in the 1 direction. The system of Dirac equations reduces to the following simple matrix problem

$$R\psi_1 = S\psi_0, \tag{5.3}$$

where the Dirac fields at time 1 and time 0 are

$$\psi_1 = \begin{pmatrix} \psi_{01} \\ \psi_{11} \end{pmatrix}, \quad \psi_0 = \begin{pmatrix} \psi_{00} \\ \psi_{10} \end{pmatrix}, \tag{5.4}$$

and the matrices are given by

$$\gamma^0 R^+ = \begin{pmatrix} 0 & -l_0 c & 2\tilde{l}_0\tilde{l}_1 & -\tilde{l}_0\tilde{l}_1 c \\ l_0 c^* & 2l_0 & \tilde{l}_0\tilde{l}_1 c^* & 0 \\ -2l_0 l_1 & l_0 l_1 d & 0 & -\tilde{l}_0 d \\ -l_0 l_1 d^* & 0 & \tilde{l}_0 d^* & 2\tilde{l}_0 \end{pmatrix}, \quad \gamma^0 R^- = \begin{pmatrix} 2l_0 & l_0 c & 0 & \tilde{l}_0\tilde{l}_1 c \\ -l_0 c^* & 0 & -\tilde{l}_0\tilde{l}_1 c^* & 2\tilde{l}_0\tilde{l}_1 \\ 0 & -l_0 l_1 d & 2\tilde{l}_0 & \tilde{l}_0 d \\ l_0 l_1 d^* & -2l_0 l_1 & -\tilde{l}_0 d^* & 0 \end{pmatrix}, \tag{5.5}$$

and

$$\gamma^0 S^+ = \begin{pmatrix} 2 & c & 0 & \tilde{l}_1 c \\ -c^* & 0 & -\tilde{l}_1 c^* & 2\tilde{l}_1 \\ 0 & -l_1 d & 2 & d \\ l_1 d^* & -2l_1 & -d^* & 0 \end{pmatrix}, \quad \gamma^0 S^- = \begin{pmatrix} 0 & -c & 2\tilde{l}_1 & -\tilde{l}_1 c \\ c^* & 2 & \tilde{l}_1 c^* & 0 \\ -2l_1 & l_1 d & 0 & -d \\ -l_1 d^* & 0 & d^* & 2 \end{pmatrix}, \tag{5.6}$$

where

$$l_0 = (L_0)_{01}, \quad \tilde{l}_0 = (L_0)_{11}, \quad l_1 = (L_1)_{00}, \quad \tilde{l}_1 = (L_1)_{10}, \tag{5.7}$$

and

$$c = \tan\frac{eh}{2}(\mathcal{A}_3)_{00} + i\tan\frac{eh}{2}(\mathcal{A}_2)_{00}, \quad d = \tan\frac{eh}{2}(\mathcal{A}_3)_{10} + i\tan\frac{eh}{2}(\mathcal{A}_2)_{10}, \tag{5.8}$$

and we have denoted eigenvalues of $i\gamma_5$ by the $\pm$ superscripts, have used the following representation of Dirac matrices:

$$\gamma^0\boldsymbol{\gamma} = i\gamma_5\boldsymbol{\sigma}, \tag{5.9}$$

and have chosen σ_1 to be the Pauli σ_z matrix. The transfer matrix T is then given by

$$T = R^{-1}S. \tag{5.10}$$

Unfortunately, it turns out in general the theory is not unitary, that is $TT^\dagger \neq 1$. This is true even in the temporal gauge, $A_0 = 0$. This is traced back to the failure of the covariant derivative, given by (3.3),

$$\boldsymbol{\mathcal{D}} = \frac{1}{\tilde{\Delta}_0}\boldsymbol{\Delta} \tag{5.11}$$

where symbolically

$$\tilde{\Delta}_0 = (L_3+1)(L_2+1)(L_1+1), \quad \Delta_1 = (L_3+1)(L_2+1)(L_1-1),$$
$$\Delta_2 = (L_3+1)(L_2-1)(L_1+1), \quad \Delta_3 = (L_3-1)(L_2+1)(L_1+1), \tag{5.12}$$

to be skew Hermitian.[3] In fact, it is easily verified that although $\mathcal{D}_1$ is skew Hermitian, $\mathcal{D}_2$ is not unless $l_1\tilde{l}_1 = 1$. So, to proceed, we will simply choose the gauge with $A_0 = A_1 = 0$, which should suffice for the anomaly calculation. This simplifies the transfer matrix dramatically by replacing

$$l_0 \to 1, \quad \tilde{l}_0 \to 1, \quad l_1 \to 1, \quad \tilde{l}_1 \to 1. \tag{5.13}$$

Then it is easily seen that T is unitary. (If ψ were periodic rather than antiperiodic in the 1 direction, which would be accomplished by changing the sign of l_1 in (5.5) and (5.6), T would not be unitary.) In fact, it will suffice in the following to expand T to bilinears in c and d. It then turns out to be

$$T^+ \approx \begin{pmatrix} v & d & -1+\xi & 0 \\ -c^* & v & 0 & 1+\tilde{\xi} \\ 1+\tilde{\xi} & 0 & v^* & c \\ 0 & -1+\xi & -d^* & v^* \end{pmatrix}, \quad T^- \approx \begin{pmatrix} v^* & -c & 1+\tilde{\xi} & 0 \\ d^* & v^* & 0 & -1+\xi \\ -1+\xi & 0 & v & -d \\ 0 & 1+\tilde{\xi} & c^* & v \end{pmatrix}, \tag{5.14}$$

where we have abbreviated

$$\xi = \frac{1}{2}dd^*, \quad \tilde{\xi} = -\frac{1}{2}cc^*, \quad v = -\frac{1}{2}dc^*. \tag{5.15}$$

In the following we will use the block form of T, which refers to specific spatial coordinates:

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}. \tag{5.16}$$

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[3] It is easily seen that no other ordering will resolve this problem. For the given ordering, $\mathcal{D}_2 = (L_1+1)^{-1}(L_2-1)^{-1}(L_2+1)(L_1-1)$; the inner factor involving L_2 is clearly skew Hermitian, but the appearance of the L_1 terms destroys that property.

We now take matrix elements of the current in Fock-space states defined in terms of the momentum-space expansion of the free Dirac field [8,7]

$$\psi_{m,1} = \sum_{s,p} \sqrt{\frac{\mu}{\omega}} \left(b_{ps} u_{ps} e^{i(p+1/2)m2\pi/M} + d^\dagger_{ps} v_{ps} e^{-i(p+1/2)m2\pi/M} \right). \tag{5.17}$$

Because of the particular choice of gauge (5.13) the current is proportional to the usual finite-element one, and the matrix element of j^0, for example, is

$$\langle j^0_{m,n} \rangle = C \frac{e}{16} \langle [(\psi^\dagger_1(1+T))_1 + (\psi^\dagger_1(1+T))_0][((1+T^\dagger)\psi_1)_1 + ((1+T^\dagger)\psi_1)_0] \rangle, \tag{5.18}$$

where the Dirac fields are those given in (5.4),

$$C = \cos^2 \frac{eh}{2} (\mathcal{A}_2)_{00} \cos^2 \frac{eh}{2} (\mathcal{A}_3)_{00}, \tag{5.19}$$

and the matrix subscripts refer to the spatial coordinate. The matrix elements are evaluated using the following easily derived formula for $M = 2$:

$$\langle \psi^\dagger_m \Gamma \psi_{m'} \rangle = \frac{1}{h^3} (\delta_{mm'} \mathrm{tr}\, \Gamma + 2i\epsilon_{mm'} \mathrm{tr}\, \gamma^0 \gamma^1 \Gamma). \tag{5.20}$$

Then a straightforward calculation gives the result

$$\langle j^0_{m,n} \rangle = \frac{e}{h^3} (1+\lambda) C, \tag{5.21}$$

where

$$\lambda = \frac{1}{4} (dd^* - cd^* - dc^* - cc^*). \tag{5.22}$$

A similar calculation reveals that $\langle j^1 \rangle = 0$. The axial-vector current matrix elements vanish. Consequently, for this tiny lattice, there is no axial-vector current anomaly, but there is a vector current anomaly,

$$\begin{aligned} \langle \text{“}\partial_\mu j^\mu\text{”} \rangle = -\frac{e^3}{8h^2} \{ & [(\mathcal{A}_2)^2_{11} + (\mathcal{A}_2)_{11}(\mathcal{A}_2)_{01} + (\mathcal{A}_2)^2_{01} - (\mathcal{A}_2)^2_{10} - (\mathcal{A}_2)_{10}(\mathcal{A}_2)_{00} - (\mathcal{A}_2)^2_{00}] \\ & + [2 \leftrightarrow 3] \}, \end{aligned} \tag{5.23}$$

a somewhat curious result which will be discussed below.

VI. FINITE-ELEMENT CURRENT ANOMALY CALCULATION

Because of the various difficulties seen in the calculation exhibited in the previous section, we return now to original formulation. There, the *unitary* transfer matrix in (3+1) dimensions is, in the gauge $A_0 = 0$, [7]

$$T = \frac{1+\gamma^0\boldsymbol{\gamma}\cdot\mathcal{D}}{1-\gamma^0\boldsymbol{\gamma}\cdot\mathcal{D}}, \tag{6.1}$$

where (the time coordinate n is suppressed)

$$\mathcal{D}^i_{m_i,\mathbf{m}_\perp;m_i',\mathbf{m}_\perp'} = -(-1)^{m_i+m_i'}(\epsilon_{m_i m_i'}\cos\hat{\zeta}_{m_i m_i'} - i\sin\hat{\zeta}_{m_i m_i'})\sec\zeta^{(i)}\delta_{\mathbf{m}_\perp\mathbf{m}_\perp'}, \tag{6.2}$$

where (and now all the local dependence on the other spatial coordinates is also suppressed)

$$\zeta_{m_i} = \frac{eh}{2}\mathcal{A}^i_{m_i}, \quad \zeta^{(i)} = \sum_{m_i=1}^{M}\zeta_{m_i}, \tag{6.3}$$

and

$$\hat{\zeta}_{m_i,m_i'} = \sum_{m_i''=1}^{M}\operatorname{sgn}(m_i''-m_i)\operatorname{sgn}(m_i''-m_i')\zeta_{m_i''}. \tag{6.4}$$

Here, the sign function is

$$\operatorname{sgn}(m) = \begin{cases} 1, & m>0, \\ -1, & m\le 0. \end{cases} \tag{6.5}$$

Again, let us consider the smallest possible nontrivial lattice, with the number of lattice points in the 1 direction being $M_1 = 2$, while the 2 and 3 directions have but one site, $M_2 = M_3 = 1$. This leads again to an 4×4 transfer matrix for each chirality, which for small ehA is, in the block form given in (5.16),

$$T^\pm_{00} = \begin{pmatrix} \lambda & \pm\frac{c+d}{2} \\ \mp\frac{c^*+d^*}{2} & \lambda \end{pmatrix}, \quad T^+_{01} = T^{-\dagger}_{10} = \frac{s}{2}\begin{pmatrix} \alpha & \beta \\ \beta^* & -\alpha^* \end{pmatrix}, \tag{6.6a}$$

$$T^+_{10} = T^{-\dagger}_{01} = \frac{s^*}{2}\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\beta}^* & -\tilde{\alpha}^* \end{pmatrix}, \quad T^\pm_{11} = \begin{pmatrix} -\tilde{\lambda} & \pm\frac{c+d}{2} \\ \mp\frac{c^*+d^*}{2} & -\tilde{\lambda} \end{pmatrix}. \tag{6.6b}$$

Here λ is defined above in (5.22),

$$\tilde{\lambda} = \frac{1}{4}(dd^* + cd^* + dc^* - cc^*), \tag{6.7}$$

c and d are as in (5.8),

$$\alpha = \frac{1}{2}(4 + 4it_1 - 4t_1 - dd^* + d^*c - c^*d - cc^*), \tag{6.8a}$$

$$\tilde{\alpha} = \frac{1}{2}(-4 - 4it_1 + 4t_1 + dd^* + d^*c - c^*d + cc^*), \tag{6.8b}$$

$$\beta = = d(1 + it_1) - c(1 - it_1), \tag{6.8c}$$

$$\tilde{\beta} = = d(1 - it_1) - c(1 + it_1), \tag{6.8d}$$

and

$$t_i = \tan\zeta^{(i)}, \quad s = -e^{i(\zeta_1^1 - \zeta_2^1)} \sec\zeta^{(1)}. \tag{6.9}$$

Even though the transfer matrices are quite different (the form in (6.6) is more complicated because $A_1 \neq 0$), the results of the calculation are very similar: The axial-vector anomaly is zero, while the vector anomaly is

$$\langle \text{“}\partial_\mu j^\mu\text{”} \rangle = -\frac{e}{2h^4}[(t_2)_{21}(t_2)_{11} + (t_3)_{21}(t_3)_{11} - (t_2)_{20}(t_2)_{10} - (t_3)_{20}(t_3)_{10}]. \tag{6.10}$$

Here, the current is not (3.9) but

$$j^\mu_{\mathbf{m},n} = e\overline{\psi}_{\overline{\mathbf{m}},\overline{n}}\gamma^\mu\psi_{\overline{\mathbf{m}},\overline{n}}, \tag{6.11}$$

the averaging being over the finite element without the link operators. The result (6.10), which is manifestly gauge invariant, appears to be, like (5.23), a lattice artifact.[4] It is not a lattice version of F^2, as one might anticipate. However, we should not be discouraged, since the calculation given is for a truly tiny, unrealistic lattice. The extension of this calculation to larger lattices will be presented elsewhere.

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[4]It is probable that this result reflects the rectangular nature of the lattice considered here. A similar “anomalous” anomaly was found when the space-time lattice was not chosen to be square: That is, when the lattice spacing in the time direction was not equal to that in the space directions. See [8].

VII. HERMITICITY, UNITARITY, AND THE EXISTENCE OF A SYMMETRY CURRENT

In this paper we pursued a variation on the finite-element formulation of lattice gauge theory which seemed at first sight very promising. The idea was to use the link variables, in term of which the local field strength was constructed, to express the Dirac and Yang-Mills equations in local form, rather than the form involving the entire prior lattice given previously. A locally gauge-invariant construction can indeed be done, one which is inequivalent to the earlier formulation. However, the new scheme is less advantageous than it first seemed, and ultimately fails to be consistent:

- The field strength which appears in the Yang-Mills equation is not locally constructed in terms of local link operators. It appears impossible to have a local formulation consistent with local non-Abelian gauge symmetry.
- Moreover, detailed calculations, even in the Abelian theory where the equations are local, turn out to be no simpler, and perhaps more complicated than those in the original formulation.
- Disastrously, unitarity is violated. Explicitly, we have seen in the Abelian case that the transfer matrix is not unitary, even in a temporal gauge. This is traced to the fact that the covariant derivative operators are not skew Hermitian. In contrast, the original formulation is manifestly unitary (canonical). It should be recalled that preservation of the canonical commutation relations at the lattice sites was the original motivation for adopting the finite-element prescription for field theory on a Minkowski lattice.

We can interpret these results in a positive light. It was always apparent that, although some arbitrary choices had to be made to implement local gauge invariance, the requirement of that invariance, that is, that the transformation equations could be "integrated," was rather rigid. The findings presented here strengthen that conclusion. The fact that we have now shown that a rather natural and attractive alternative gauging process is ultimately inconsistent makes the pursuit of extracting physical information from the orginal, consistent, approach that much more compelling.

Finally, we should add some remarks about the current employed in both formulations studied here. It should be noted that the choice of current was essentially arbitrary, subject only to the requirement that it be locally gauge covariant. It is essential to note that *the current cannot be derived from the Dirac equation.* In fact, because our equations of motion cannot be derived from an action, there is no connection between symmetry (say chiral symmetry) and conservation laws (say axial-vector current conservation). In our consistent formulation this is because the Dirac equation is asymmetric between past and future. If one were interested in a Euclidean formulation, one would choose a Dirac equation symmetric in the fourth coordinate, and an action, and corresponding current could be constructed. That current would, however, also involve all lattice sites in the fourth coordinate, and therefore would be be unusable in the Minkowski context, where we wish to solve the operator equations of motion by time-stepping through the lattice. This Euclidean construction and its Minkowski failure is sketched in the Appendix.

ACKNOWLEDGMENTS

I thank the UK PPARC for the award of a Senior Visiting Fellowship and Imperial College for its hospitality during the initial stages of writing this paper. I further thank the US Department of Energy and the University of Oklahoma College of Arts and Sciences for partial financial support of this research. I am grateful to Maarten Golterman, Simon Hands, and Tai Wu for useful conversations.

APPENDIX: CURRENT CONSTRUCTED FROM EUCLIDEAN LAGRANGIAN

In the text we simply assumed a form of the current (3.9) and (6.11) which was manifestly gauge invariant. We are at liberty to do so, because the Minkowski finite-element equations of motion are not derivable from a Lagrangian. The current cannot be derived from the Dirac equation, but is an independent source for the Maxwell equations, see Ref. [2,4]. However, if one were to work in Euclidean space-time (periodic or antiperiodic in all four directions), it is possible to construct an action from which the equations of motion are derivable, and which therefore supplies a lattice current. The fermion part of that action is (a factor of i is absorbed in going to Euclidean space)

$$W_f = h^4 \sum_{\mathbf{m},\mathbf{m}'} \psi^\dagger_{\overline{\mathbf{m}}} \left(\frac{2}{h} \boldsymbol{\Gamma} \cdot \boldsymbol{\mathcal{D}} + i\mu \right)_{\mathbf{m},\mathbf{m}'} \psi_{\overline{\mathbf{m}}'}, \quad \boldsymbol{\Gamma} = (\gamma^0 \gamma_k, \gamma^0). \tag{A1}$$

If we vary (A1) with respect to $\psi^\dagger$ we obtain the Euclidean Dirac equation

$$\left(\frac{2}{h} \boldsymbol{\Gamma} \cdot \boldsymbol{\mathcal{D}} + i\mu \right) \psi = 0. \tag{A2}$$

where, as in (A1), a four-dimensional scalar product is implied.

Given an action, we can construct a conserved vector current by making a local gauge transformation

$$\delta\psi_{\overline{\mathbf{m}}} = ie\,\delta\Omega_{\mathbf{m}} \psi_{\overline{\mathbf{m}}}. \tag{A3}$$

Because the Dirac equation, and hence the action, is invariant under the global version of (A3), $\delta\Omega_{\mathbf{m}} = \delta\Omega =$ constant, we must have by the action principle

$$\delta W_f = 0 = -h^4 \sum_{\mathbf{m}} J^i_{\mathbf{m}} \frac{1}{h} (\delta\Omega_{m_i,\mathbf{m}_\perp} - \delta\Omega_{m_i-1,\mathbf{m}_\perp}), \tag{A4}$$

from which we read off the conserved current

$$\begin{aligned} J^i_{\mathbf{m}} = -e \sum_{m_i', m_i''} & \psi^\dagger_{\overline{m}_i', \overline{\mathbf{m}}_\perp} \Gamma^i \psi_{\overline{m}_i'', \overline{\mathbf{m}}_\perp} \operatorname{sgn}(m_i - m_i') \operatorname{sgn}(m_i - m_i'') \\ & \times (-1)^{m_i' + m_i''} \sec\zeta^{(i)} \exp(-i\epsilon_{m_i', m_i''} \hat{\zeta}_{m_i', m_i''}). \end{aligned} \tag{A5}$$

(The same result, of course, can be obtained by varying $\mathcal{D}$ (6.2) with respect to $\mathcal{A}^i_{m_i,\mathbf{m}_\perp}$.) The expression for this Euclidean current has been simplified by deleting constant terms. It is easy to verify explicitly that this current is both conserved and gauge invariant. Similarly, by making a chiral transformation,

$$\delta\psi_{\overline{\mathbf{m}}} = \gamma^0\gamma_5\delta\Omega_{\mathbf{m}}\psi_{\overline{\mathbf{m}}}, \tag{A6}$$

we can construct the axial-vector current $J^i_{5\mathbf{m}}$, which has the form of (A5) with the replacement

$$e\Gamma^i \to \gamma^0 i\gamma_5\Gamma^i \equiv \Gamma^i_5, \quad \Gamma^i_5 = (-i\gamma_5\gamma^i, -i\gamma_5). \tag{A7}$$

By construction, these currents possess no anomalies. However, they appear to be completely unacceptable, because they are horribly nonlocal. In particular, they possess no Minkowski analogues, in the sense that it is not possible to analytically continue back to real unbounded times. Crucial to our formulation is the propagation of the operators from past times to the present time, so that we can solve for the field operators by time-stepping through the lattice. The Euclidean current (A5) involves fermion field operators at all Euclidean times, which would make it impossible to solve for the operators at time n in terms of operators at earlier times. Therefore, for the considerations of the text we use the gauge-invariant current (3.9) and (6.11) and their axial analogues, currents which can and do possess anomalies.

It is further illuminating to note that if we were to use the current (A5) in a one-loop lattice calculation of the vacuum polarization in two dimensions, we would find a vanishing anomaly, rather than the value e^2/π reported in [8]. This is because species doublers occur in the action defined by (A1). Such doublers are absent in the finite-element scheme based on equations of motion and the current (6.11).

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