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Casimir Energy for a Spherical Cavity in a Dielectric: Toward a Model for Sonoluminescence?[1]

Kimball A. Milton[2]
Department of Physics and Astronomy
The University of Oklahoma, Norman OK 73019, USA

Abstract

In the final few years of his life, Julian Schwinger proposed that the "dynamical Casimir effect" might provide the driving force behind the puzzling phenomenon of sonoluminescence. Motivated by that exciting suggestion, I have computed the static Casimir energy of a spherical cavity in an otherwise uniform material with dielectric constant ϵ and permeability μ. As expected the result is divergent; yet a plausible finite answer is extracted, in the leading uniform asymptotic approximation. That result gives far too small an energy to account for the large burst of photons seen in sonoluminescence. If the divergent result is retained (which is different from that guessed by Schwinger), it is of the wrong sign to drive the effect. Dispersion does not resolve this contradiction. However, dynamical effects are not yet included.

1 Introduction

In a series of papers in the last three years of his life, Julian Schwinger proposed [1] that the dynamical Casimir effect could provide the energy that drives the copious production of photons in the puzzling phenomenon of sonoluminescence [2, 3]. In fact, however, he guessed an approximate (static) formula for the Casimir energy of a spherical bubble in water, based on a general, but incomplete, analysis [4]. He apparently was unaware that I had, at the time I left UCLA, completed the analysis of the Casimir force for a dielectric ball [5]. It is my purpose here to carry out the very straightforward calculation for the complementary situation, for a cavity in an infinite dielectric medium. In fact, I will consider the general case of spherical region, of radius a, having permittivity ϵ' and permeability μ', surrounded by an infinite medium of permittivity ϵ and permeability μ.

Of course, this calculation is not directly relevant to sonoluminescence, which is anything but static. It is offered as only a first step, but it should give an idea of the orders of magnitude of the energies involved. It is an improvement over the crude estimation used in [1]. Attempts at dynamical calculations exist [6, 7]; but they are subject to possibly serious methodological objections. Sonoluminescence aside, this calculation is of interest for its own sake, as one of a relatively few nontrivial Casimir calculations with nonplanar boundaries [8, 9, 10, 11, 12, 13, 14, 15]. It represents a significant generalization on the calculation of Brevik and Kolbenstvedt [16],

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[2]e-mail: milton@phyast.nhn.uoknor.edu

who consider the same geometry with $\mu\epsilon = \mu'\epsilon' = 1$, a special case, possibly relevant to hadronic physics, in which the result is unambiguously finite.

In the next section we review the Green's dyadic formalism we shall employ, and compute the Green's functions in this case for the TE and TM modes. Then, in Section 3, we compute the force on the shell from the discontinuity of the stress tensor. The energy is computed similarly in Section 4, and the expected relation between stress and energy is found. Estimates in Section 5 show that the result so constructed, even with physically required subtractions, and including both interior and exterior contributions, is divergent, but that if one supplies a plausible contact term, a finite result (at least in leading approximation) follows. (Physically, we expect that the divergence is regulated by including dispersion.) Numerical estimates of both the divergent and finite terms are given in the conclusion, and comparison is made with the calculations of Schwinger.

2 Green's Dyadic Formulation

I follow closely the formulation given in [10, 5]. We start with Maxwell's equations in rationalized units, with a polarization source $\mathbf{P}$: (in the following we set $c = \hbar = 1$)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{H} &= \frac{\partial}{\partial t}\mathbf{D} + \frac{\partial}{\partial t}\mathbf{P}\ , \qquad \boldsymbol{\nabla} \cdot \mathbf{D} = -\boldsymbol{\nabla} \cdot \mathbf{P}, \\ -\boldsymbol{\nabla} \times \mathbf{E} &= \frac{\partial}{\partial t}\mathbf{B}\ , \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \end{aligned} \tag{1}$$

where, for an homogeneous, isotropic, nondispersive medium

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}. \tag{2}$$

We define a Green's dyadic $\boldsymbol{\Gamma}$ by

$$\mathbf{E}(\mathbf{r}, t) = \int (d\mathbf{r}')\, dt'\, \boldsymbol{\Gamma}(\mathbf{r}, t; \mathbf{r}', t') \cdot \mathbf{P}(\mathbf{r}', t') \tag{3}$$

and introduce a Fourier transform in time

$$\boldsymbol{\Gamma}(\mathbf{r}, t; \mathbf{r}', t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega), \tag{4}$$

where in the following the ω argument will be suppressed. Maxwell's equations then become (which define $\boldsymbol{\Phi}$)

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{\Gamma} &= i\omega\boldsymbol{\Phi}\ , \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Phi} = 0, \\ \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{\Phi} &= -i\omega\epsilon\boldsymbol{\Gamma}'\ , \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}' = 0, \end{aligned} \tag{5}$$

in which $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma} + \mathbf{1}/\epsilon$, where $\mathbf{1}$ includes a spatial delta function. The two solenoidal Green's dyadics given here satisfy the following second-order equations:

$$(\nabla^2 + \omega^2\epsilon\mu)\boldsymbol{\Gamma}' = -\frac{1}{\epsilon}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}), \tag{6}$$

$$(\nabla^2 + \omega^2\epsilon\mu)\boldsymbol{\Phi} = i\omega\mu\boldsymbol{\nabla} \times \mathbf{1}. \tag{7}$$

They can be expanded in vector spherical harmonics [17, 18] defined by

$$\mathbf{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}, \tag{8}$$

as follows:

$$\mathbf{\Gamma}'(\mathbf{r},\mathbf{r}') = \sum_{lm}\left(f_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega) + \frac{i}{\omega\epsilon\mu}\boldsymbol{\nabla}\times g_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega)\right), \tag{9}$$

$$\mathbf{\Phi}(\mathbf{r},\mathbf{r}') = \sum_{lm}\left(\tilde{g}_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega) - \frac{i}{\omega}\boldsymbol{\nabla}\times \tilde{f}_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega)\right). \tag{10}$$

When these are substituted in Maxwell's equations (5) we obtain, first,

$$g_l = \tilde{g}_l, \qquad f_l = \tilde{f}_l + \frac{1}{\epsilon}\frac{1}{r^2}\delta(r-r')\mathbf{X}_{lm}(\Omega'), \tag{11}$$

and then the second-order equations

$$(D_l + \omega^2\mu\epsilon)g_l(r,\mathbf{r}') = i\omega\mu\int d\Omega''\,\mathbf{X}^*_{lm}(\Omega'')\cdot\boldsymbol{\nabla}''\times\mathbf{1}, \tag{12}$$

$$\begin{aligned}(D_l + \omega^2\mu\epsilon)f_l(r,\mathbf{r}') &= -\frac{1}{\epsilon}\int d\Omega''\,\mathbf{X}^*_{lm}(\Omega'')\cdot\boldsymbol{\nabla}''\times(\boldsymbol{\nabla}''\times\mathbf{1}),\\ &= \frac{1}{\epsilon}D_l\frac{1}{r^2}\delta(r-r')\mathbf{X}^*_{lm}(\Omega'),\end{aligned} \tag{13}$$

where

$$D_l = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}. \tag{14}$$

These equations can be solved in terms of Green's functions satisfying

$$(D_l + \omega^2\epsilon\mu)F_l(r,r') = -\frac{1}{r^2}\delta(r-r'), \tag{15}$$

which have the form

$$F_l(r,r') = \begin{cases} ik'j_l(k'r_<)[h_l(k'r_>) - Aj_l(k'r_>)], & r,r' < a,\\ ikh_l(kr_>)[j_l(kr_<) - Bh_l(kr_<)], & r,r' > a,\end{cases} \tag{16}$$

where

$$k = |\omega|\sqrt{\mu\epsilon}, \qquad k' = |\omega|\sqrt{\mu'\epsilon'}, \tag{17}$$

and $h_l = h_l^{(1)}$ is the spherical Hankel function of the first kind. Specifically, we have

$$\tilde{f}_l(r,\mathbf{r}') = \omega^2\mu F_l(r,r')\mathbf{X}^*_{lm}(\Omega'), \tag{18}$$

$$g_l(r,\mathbf{r}') = -i\omega\mu\boldsymbol{\nabla}'\times G_l(r,r')\mathbf{X}^*_{lm}(\Omega'), \tag{19}$$

where F_l and G_l are Green's functions of the form (16) with the constants A and B determined by the boundary conditions given below. Given F_l, G_l, the fundamental Green's dyadic is given by

$$\begin{aligned}\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}') = \sum_{lm}\Big\{&\omega^2\mu F_l(r,r')\mathbf{X}_{lm}(\Omega)\mathbf{X}^*_{lm}(\Omega') \\ &-\frac{1}{\epsilon}\boldsymbol{\nabla}\times G_l(r,r')\mathbf{X}_{lm}(\Omega)\mathbf{X}^*_{lm}(\Omega')\times\overleftarrow{\boldsymbol{\nabla}}' \\ &+\frac{1}{\epsilon}\frac{1}{r^2}\delta(r-r')\sum_{lm}\mathbf{X}_{lm}(\Omega)\mathbf{X}^*_{lm}(\Omega')\Big\}.\end{aligned} \tag{20}$$

Now we consider a sphere of radius a centered at the origin, with properties ϵ', μ' in the interior and ϵ, μ outside. Because of the boundary conditions that

$$\mathbf{E}_\perp, \quad \epsilon E_r, \quad B_r, \quad \frac{1}{\mu}\mathbf{B}_\perp \tag{21}$$

be continuous at $r = a$, we find for the constants A and B in the two Green's functions in (20)

$$A_F = \frac{\sqrt{\epsilon\mu'}\tilde{e}_l(x')\tilde{e}_l'(x) - \sqrt{\epsilon'\mu}\tilde{e}_l(x)\tilde{e}_l'(x')}{\Delta_l}, \tag{22}$$

$$B_F = \frac{\sqrt{\epsilon\mu'}\tilde{s}_l(x')\tilde{s}_l'(x) - \sqrt{\epsilon'\mu}\tilde{s}_l(x)\tilde{s}_l'(x')}{\Delta_l}, \tag{23}$$

$$A_G = \frac{\sqrt{\epsilon'\mu}\tilde{e}_l(x')\tilde{e}_l'(x) - \sqrt{\epsilon\mu'}\tilde{e}_l(x)\tilde{e}_l'(x')}{\tilde{\Delta}_l}, \tag{24}$$

$$B_G = \frac{\sqrt{\epsilon'\mu}\tilde{s}_l(x')\tilde{s}_l'(x) - \sqrt{\epsilon\mu'}\tilde{s}_l(x)\tilde{s}_l'(x')}{\tilde{\Delta}_l}. \tag{25}$$

Here we have introduced $x = ka$, $x' = k'a$, the Riccati-Bessel functions

$$\tilde{e}_l(x) = xh_l(x), \qquad \tilde{s}_l(x) = xj_l(x), \tag{26}$$

and the denominators

$$\begin{aligned}\Delta_l &= \sqrt{\epsilon\mu'}\tilde{s}_l(x')\tilde{e}_l'(x) - \sqrt{\epsilon'\mu}\tilde{s}_l'(x')\tilde{e}_l(x), \\ \tilde{\Delta}_l &= \sqrt{\epsilon'\mu}\tilde{s}_l(x')\tilde{e}_l'(x) - \sqrt{\epsilon\mu'}\tilde{s}_l'(x')\tilde{e}_l(x),\end{aligned} \tag{27}$$

and have denoted differentiation with respect to the argument by a prime.

3 Stress on the sphere

We can calculate the stress (force per unit area) on the sphere by computing the discontinuity of the (radial-radial component) of the stress tensor:

$$\mathcal{F} = T_{rr}(a-) - T_{rr}(a+), \tag{28}$$

where

$$T_{rr} = \frac{1}{2}\langle[\epsilon(E_\perp^2 - E_r^2) + \mu(H_\perp^2 - H_r^2)]\rangle. \tag{29}$$

The vacuum expectation values of the product of field strengths are given directly by the Green's dyadics computed in Section 2:

$$i\langle\mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \mathbf{\Gamma}(\mathbf{r},\mathbf{r}'), \tag{30}$$

$$i\langle\mathbf{B}(\mathbf{r})\mathbf{B}(\mathbf{r}')\rangle = -\frac{1}{\omega^2}\boldsymbol{\nabla}\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\times\overleftarrow{\boldsymbol{\nabla}}', \tag{31}$$

where here and in the following we ignore δ functions because we are interested in the *limit* as $\mathbf{r}'\to\mathbf{r}$. It is then rather immediate to find for the stress on the sphere (the *limit* $t'\to t$ is assumed)

$$\begin{aligned}\mathcal{F} = {} & \frac{1}{2ia^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi} \\ & \times\Bigg\{(\epsilon'-\epsilon)\left[\frac{k^2}{\epsilon}a^2F_l(a+,a+) + \left(\frac{l(l+1)}{\epsilon'} + \frac{1}{\epsilon}\frac{\partial}{\partial r}r\frac{\partial}{\partial r'}r'\right)G_l(r,r')\Big|_{r=r'=a+}\right] \\ & +(\mu'-\mu)\left[\frac{k^2}{\mu}a^2G_l(a+,a+) + \left(\frac{l(l+1)}{\mu'} + \frac{1}{\mu}\frac{\partial}{\partial r}r\frac{\partial}{\partial r'}r'\right)F_l(r,r')\Big|_{r=r'=a+}\right]\Bigg\} \end{aligned}\tag{32}$$

$$= \frac{i}{2a^4}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{-iy\delta}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}x\frac{d}{dx}\ln\Delta_l\tilde{\Delta}_l, \tag{33}$$

where $y=\omega a$, $\delta=(t-t')/a$, and

$$\ln\Delta_l\tilde{\Delta}_l = \ln\left[(\tilde{s}_l(x')\tilde{e}_l'(x) - \tilde{s}_l'(x')\tilde{e}_l(x))^2 - \xi^2(\tilde{s}_l(x')\tilde{e}_l'(x) + \tilde{s}_l'(x')\tilde{e}_l(x))^2\right] + \text{constant}. \tag{34}$$

Here the parameter ξ is

$$\xi = \frac{\sqrt{\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}}-1}{\sqrt{\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}}+1}. \tag{35}$$

This is not yet the answer. We must remove the term which would be present if either medium filled all space (the same was done in the case of parallel dielectrics [19]). The corresponding Green's function is

$$F_l^{(0)} = \begin{cases} ik'j_l(k'r_<)h_l(k'r_>), & r,r'<a \\ ikj_l(kr_<)h_l(kr_>), & r,r'>a \end{cases} \tag{36}$$

The resulting stress is

$$\mathcal{F}^{(0)} = \frac{1}{a^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\left\{x'[\tilde{s}_l'(x')\tilde{e}_l'(x') - \tilde{e}_l(x')\tilde{s}_l''(x')] - x[\tilde{s}_l'(x)\tilde{e}_l'(x) - \tilde{e}_l(x)\tilde{s}_l''(x)]\right\}. \tag{37}$$

The final formula for the stress is obtained by subtracting (37) from (33):

$$\begin{aligned}\mathcal{F} = {} & -\frac{1}{2a^4}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{iy\delta}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}\{x\frac{d}{dx}\ln\Delta_l\tilde{\Delta}_l \\ & +2x'[s_l'(x')e_l'(x') - e_l(x')s_l''(x')] - 2x[s_l'(x)e_l'(x) - e_l(x)s_l''(x)]\}, \end{aligned}\tag{38}$$

where we have now performed a Euclidean rotation,

$$\begin{aligned} &y \to iy, \quad x \to ix, \quad \tau = t - t' \to i(x_4 - x_4') \quad [\delta = (x_4 - x_4')/a], \\ &\tilde{s}_l(x) \to s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x), \quad \tilde{e}_l(x) \to e_l(x) = \frac{2}{\pi}\sqrt{\frac{\pi x}{2}} K_{l+1/2}(x). \end{aligned} \tag{39}$$

4 Total energy

In a similar way we can directly calculate the Casimir energy of the configuration, starting from the energy density

$$U = \frac{\epsilon E^2 + \mu H^2}{2}. \tag{40}$$

In terms of the Green's dyadic, the total energy is

$$\begin{aligned} E &= \int (d\mathbf{r})\, U \\ &= \frac{1}{2i}\int r^2 dr\, d\Omega \left[\epsilon \mathrm{Tr}\,\mathbf{\Gamma}(\mathbf{r},\mathbf{r}) - \frac{1}{\omega^2 \mu}\mathrm{Tr}\,\mathbf{\nabla}\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r})\times \overleftarrow{\mathbf{\nabla}}\right] \qquad (41) \\ &= \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} e^{-i\omega(t-t')}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty r^2\, dr \\ &\quad \times \left\{ 2k^2[F_l(r,r) + G_l(r,r)] + \frac{1}{r^2}\frac{\partial}{\partial r} r \frac{\partial}{\partial r'} r' [F_l + G_l](r,r')\big|_{r'=r}\right\}, \qquad (42) \end{aligned}$$

where there is no explicit appearance of ϵ or μ. (However, the value of k depends on which medium we are in.) As in [10] we can easily show that the total derivative term integrates to zero. We are left with

$$E = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty r^2\, dr\, 2k^2[F_l(r,r) + G_l(r,r)]. \tag{43}$$

However, again we should subtract off that contribution which the formalism would give if either medium filled all space. That means we should replace F_l and G_l by

$$\tilde{F}_l, \tilde{G}_l = \begin{cases} -ik' A_{F,G} j_l(k'r) j_l(k'r'), \; r, r' < a \\ -ik B_{F,G} h_l(kr) h_l(kr'), \quad r, r' > a \end{cases} \tag{44}$$

so then (43) says

$$E = -\sum_{l=1}^{\infty}(2l+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left\{\int_0^a r^2 dr\, k'^3(A_F + A_G) j_l^2(k'r) + \int_a^\infty r^2 dr\, k^3 (B_F + B_G) h_l^2(kr)\right\}. \tag{45}$$

The radial integrals may be done by using the following indefinite integral for any spherical Bessel function j_l:

$$\int dx\, x^2 j_l^2(x) = \frac{x}{2}[((xj_l)')^2 - j_l(xj_l)' - xj_l(xj_l)'']. \tag{46}$$

But we must remember to add the contribution of the total derivative term in (42) which no longer vanishes when the replacement (44) is made. The result is precisely that expected from the stress (38),

$$E = 4\pi a^3 \mathcal{F}, \qquad \mathcal{F} = \frac{1}{4\pi a^2}\left(-\frac{\partial}{\partial a}\right) E, \tag{47}$$

where the derivative is the naive one, that is, the cutoff δ has no effect on the derivative.

5 Asymptotic analysis and numerical results

The result for the stress (38) is an immediate generalization of that given in [5], and therefore, the asymptotic analysis given there can be applied nearly unchanged. The result for the energy is new, and seems not to have been recognized earlier.

We first remark on the special case $\sqrt{\epsilon\mu} = \sqrt{\epsilon'\mu'}$. Then $x = x'$ and the energy reduces to

$$E = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\, e^{iy\delta} \sum_{l=1}^{\infty}(2l+1)x\frac{d}{dx}\ln[1-\xi^2((s_l e_l)')^2], \tag{48}$$

where

$$\xi = \frac{\mu - \mu'}{\mu + \mu'}. \tag{49}$$

If $\xi = 1$ we recover the case of a perfectly conducting spherical shell, treated in [10], for which E is finite. In fact (49) is finite for all ξ, and if we use the leading uniform asymptotic approximation for the Bessel functions we obtain

$$U \sim \frac{3}{64a}\xi^2. \tag{50}$$

Further analysis of this special case is given by Brevik and Kolbenstvedt [16].

In general, using the uniform asymptotic behavior, with $x = \nu z$, $\nu = l + 1/2$, and, for simplicity looking at the large z behavior, we have

$$E \sim -\frac{1}{2\pi a}\frac{1}{\sqrt{\epsilon\mu}}\sum_l \nu^2 \int_{-\infty}^{\infty} dz\, e^{iz\nu\delta/\sqrt{\epsilon\mu}} z\frac{d}{dz}\ln\left[1 + \frac{1}{16z^4}\left(\frac{\epsilon\mu}{\epsilon'\mu'} - 1\right)^2 (1-\xi^2)\right], \tag{51}$$

which exhibits a cubic divergence as $\delta \to 0$. To be more explicit, let us content ourselves with with the case when $\epsilon - 1$, $\epsilon' - 1$ are both small and $\mu = \mu' = 1$. Then, the leading ν term is

$$\begin{aligned} E &\sim -\frac{(\epsilon' - \epsilon)^2}{16\pi a}\sum_{l=1}^{\infty}\nu^2 \frac{1}{2}\int_{-\infty}^{\infty} dz\, e^{i\nu z\delta} z\frac{d}{dz}\frac{1}{(1+z^2)^2} \\ &= -\frac{(\epsilon'-\epsilon)^2}{64a}\left(\frac{16}{\delta^3} + \frac{1}{4}\right) \to -\frac{(\epsilon'-\epsilon)^2}{256a}. \end{aligned} \tag{52}$$

Here, the last arguable step is made plausible by noting that since $\delta = \tau/a$ the divergent term represents a contribution to the surface tension on the bubble, which should be cancelled by a suitably chosen counter term (contact term). This argument is given somewhat more weight by the discussion in [20]. In essence, justification is provided there for the use of zeta-function regularization, which directly gives the finite part here:

$$E \sim \frac{(\epsilon'-\epsilon)^2}{32\pi a}\sum_{l=1}^\infty \nu^2\frac{\pi}{2} = \frac{(\epsilon'-\epsilon)^2}{64a}\left(-\frac{1}{4}\right), \tag{53}$$

because $\sum_{l=0}^\infty \nu^s = (2^{-s}-1)\zeta(-s)$ vanishes at $s=2$.

Alternatively, one could argue that dispersion should be included [21, 22, 23], crudely modelled by

$$\epsilon(\omega)-1 = \frac{\epsilon_0-1}{1-\omega^2/\omega_0^2}. \tag{54}$$

If this rendered the expression for the stress finite [we consider the stress, not the energy, for it is not necessary to consider the dispersive factor $d(\omega\epsilon(\omega))/d\omega$ there], we could drop the cutoff δ and the sign of the force would be positive: (at last, we set $\epsilon'=1$)

$$\mathcal{F} \sim +\frac{(\epsilon_0-1)^2}{128\pi^2a^4}\sum_{l=1}^\infty \nu^2\int_{-\infty}^\infty dz\,\frac{1}{(1+z^2)^2}\frac{1}{(1+z^2/z_0^2)^2}, \tag{55}$$

where $z_0=\omega_0 a/\nu$. As $\nu\to\infty$, $z_0\to 0$, and the integral here approaches $\pi z_0/2$, and so

$$\mathcal{F} \sim \frac{(\epsilon_0-1)^2}{256\pi a^3}\omega_0\sum_{l=1}^{\nu_c}\nu \sim \frac{(\epsilon_0-1)^2}{512\pi a}\omega_0^3, \tag{56}$$

if we take as the cutoff[3] of the angular momentum sum $\nu_c\sim\omega_0 a$. The corresponding energy is obtained by integrating $-4\pi a^2\mathcal{F}$,

$$E \sim -\frac{(\epsilon_0-1)^2}{256}\omega_0^3a^2, \tag{57}$$

which is of the form of (52) with $1/\delta\to\omega_0/4$.

6 Conclusions

So finally, what can we say about sonoluminescence? To calibrate our remarks, let us recall (a simplified version of) the argument of Schwinger [1]. On the basis of a provocative but incomplete analysis he argued that a bubble ($\epsilon'=1$) in water ($\epsilon\approx(4/3)^2$) possessed a positive Casimir energy[4]

$$E_c \sim \frac{4\pi a^3}{3}\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2}k\left(1-\frac{1}{\sqrt{\epsilon}}\right) \sim \frac{a^3K^4}{12\pi}\left(1-\frac{1}{\sqrt{\epsilon}}\right), \tag{58}$$

where K is a wavenumber cutoff. Putting in his estimate, $a\sim 4\times10^{-3}$cm, $K\sim 2\times10^5\text{cm}^{-1}$ (in the UV), we find a large Casimir energy, $E_c\sim 13$MeV, and something like 3 million photons would be liberated if the bubble collapsed.

[3] Inconsistently, for then $z_0\sim1$. If $z_0=1$ in (55), however, the same angular momentum cutoff gives 5/12 of the value in (56).

[4] Note, for small $\epsilon-1$, Schwinger's result goes like $(\epsilon-1)$, indicating that he had not removed the "vacuum" contribution corresponding to (37). This is the essential physical reason for the discrepancy between his results and mine.

What does our full (albeit static) calculation say? If we believe the subtracted result, the last form in (52), and say that the bubble collapses from an initial radius $a_i = 4 \times 10^{-3}$cm to a final radius $a_f = 4 \times 10^{-4}$cm, we find that the change in the Casimir energy is $\Delta U \sim +10^{-4}$eV. This is far to small to account for the observed emission.

On the other hand, perhaps we should retain the divergent result, and put in reasonable cutoffs. If we do so, we have

$$E = -\frac{(\epsilon - 1)^2}{4} a^2 K^3 \sim -4 \times 10^5 \text{eV}, \tag{59}$$

perhaps of acceptable magnitude, but of the *wrong* sign. The same conclusion follows if one uses dispersion, as (57) shows.

So we are unable to see how the Casimir effect could possibly supply energy relevant to the copious emission of light seen in sonoluminescence. Of course, dynamical effects could change this conclusion, but elementary arguments suggest that this is impossible unless ultrarelativistic velocities are achieved. (See also [7].) Yet the subject of vacuum energy is sufficiently subtle that surprises could be in store. A more complete analysis will be provided elsewhere.

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