Finite-element quantum field theory

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An alternative approach to lattice gauge theory has been under development for the past decade. It is based on discretizing the operator Heisenberg equations of motion in such a way as to preserve the canonical commutation relations at each lattice site. It is now known how to formulate a non-Abelian gauge theory within this framework. The formulation appears to be free of fermion doubling. Since the theory is unitary, a time-development operator (Hamiltonian) can be constructed.

1. QUANTUM MECHANICS

Let us consider a one-dimensional field theory, namely quantum mechanics, governed by a continuum Hamiltonian $H = p^2/2 + V(q)$. The corresponding Heisenberg equations are

$$\dot{p} = -V'(q), \quad \dot{q} = p. \tag{1}$$

These equations are to be solved subject to the initial condition $[q(0), p(0)] = i$. The equations (1) preserve unitarity in the sense that at any subsequent time t the canonical commutation relations continue to hold, $[q(t), p(t)] = i$.

1.1. Finite-element discretization

We discretize the above equations of motion by dividing the time interval $[0, T]$ into N equal intervals of length h. On each interval we define a local time variable t, $0 \le t \le h$, and write the dynamical variables as rth degree polynomials in t,

$$p(t) \approx \sum_{k=0}^{r} a_k (t/h)^k, \quad q(t) \approx \sum_{k=0}^{r} b_k (t/h)^k. \tag{2}$$

We determine the $2r+2$ operator coefficients, a_k, b_k, by imposing continuity between the intervals, that is, at the lattice sites, and by imposing the equations (1) at r different points in the interval, at $t_i = \alpha_i h$. Unitarity, that is, that $[q_n, p_n] = i$ *exactly* at each lattice site, then uniquely determines that the points at which the equations of motion are to be imposed are the Gaussian knots, the solutions of $P_r(2\alpha - 1) = 0$. Remarkably, these are precisely the points at which classically the numerical error is minimized, the relative error after N steps then going like N^{-2r}.

1.2. Time-evolution operator

Because the lattice theory is unitary, there must exist a unitary time-evolution operator which advances operators through the lattice,

$$q_{n+1} = U q_n U^{-1}, \quad p_{n+1} = U p_n U^{-1}. \tag{3}$$

For linear finite elements, $r = 1$, U is given by

$$U = e^{ihp_n^2/4} e^{ihA(q_n)} e^{ihp_n^2/4}, \tag{4}$$

$$A(q) = \frac{2}{h^2}[q - g^{-1}(4q/h^2)]^2 + V[g^{-1}(4q/h^2)], \tag{5}$$

$$g(x) = \frac{4}{h^2}x + V'(x). \tag{6}$$

It is apparent that this is an implicit scheme, as the equations of motion must be solved to construct the operator A. Of course, we can use this evolution operator to define a self-adjoint lattice Hamiltonian, $\mathcal{H} = -\frac{i}{h}\ln U$, which differs from the continuum Hamiltonian by $O(h^2)$. [The order r finite element prescription would give a Hamiltonian differing from the continuum one by $O(h^{2r})$]. The details are given in [1].

1.3. Harmonic-Oscillator Matrix Elements of the Time-Evolution Operator

Remarkably, in view of the above, a closed expression, involving g not g^{-1}, can be derived for

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matrix elements of the transfer operator in a harmonic oscillator basis [2,3]. Although this formula is fairly complex, it can be easily used to extract expansions in powers of the lattice spacing h. For the anharmonic oscillator with potential $V = \lambda q^{2k}/2k$, we find in the harmonic oscillator groundstate

$$\langle U\rangle = 1 + ih\lambda^{1/(k+1)}f(\alpha) - \frac{h^2}{2}\lambda^{2/(k+1)}s(\alpha), \quad (7)$$

where $\alpha = \lambda\gamma^{2k+2}$, γ is the width of the harmonic oscillator state, and

$$f(\alpha) = \frac{1}{4\alpha^{1/(k+1)}}\left(1 + \frac{2\alpha}{k}\frac{\Gamma(k+1/2)}{\Gamma(1/2)}\right) \quad (8)$$

$$s(\alpha) = \frac{1}{16\alpha^{2/(k+1)}}\left(3 - 4\alpha\frac{2k-1}{k}\frac{\Gamma(k+1/2)}{\Gamma(1/2)} + \frac{4\alpha^2}{k^2}\frac{\Gamma(2k+1/2)}{\Gamma(1/2)}\right). \quad (9)$$

We match this expansion to that of $\exp(i\omega_0 h)$ to determine approximations to the ground state energy. If we use the order h data only, we must determine the variational parameter α by, say, the principle of minimum sensitivity. The energy so determined is accurate to a few percent. If the order h^2 data is used α is determined, and the results are considerably more accurate. If, instead, we use a two-state approximation to the anharmonic oscillator ground state, improvement by more than an order of magnitude results. See Table 1. Wavefunctions are equally impressive.

2. NON-ABELIAN GAUGE THEORY

Recently it was discovered how to implement non-Abelian gauge invariance on a linear finite-element lattice [4]. This was done constructively, by gauging the free lattice Dirac and Yang-Mills equations (which, by the linear finite-element prescription, are forward difference equations, forward-averaged in the non-differentiated directions), and simultaneously determining the form of the gauge transformation. The result is most conveniently expressed in terms of the following *link operator* defined on a particular finite-element (hypercube):

$$(L_\mu)_{\mathbf{m}} = \exp(-igh(\mathcal{A}_\mu)_{\mathbf{m}}), \quad (10)$$

where $\mathbf{m}$ is a local four-vector coordinate, having components 0 or 1, indicating the 16 corners of the hypercube, and, in terms of absolute coordinates

$$(\mathcal{A}_\mu)_{m_\mu, m_\perp} = (A_\mu)_{\overline{m_\mu - 1}, m_\perp}, \quad (11)$$

with the overbar indicating forward averaging in the indicated coordinate, and $\perp$ indicating coordinates other than the one specified. The result of the above construction is that the link operator transforms very simply,

$$\delta(L_\mu)_1 = ig[\delta\omega_0(L_\mu)_1 - (L_\mu)_1\delta\omega_1], \quad (12)$$

where only the local value of the m_μ coordinate is shown (the other coordinates are the same throughout). The covariant, "transversely local" field strength is a path ordered product of link operators around the μ-ν plaquette:

$$\exp(-igh^2(\mathcal{F}_{\mu\nu})_m) = P\exp(-ig\oint A\cdot dl) = (L_\mu)_{10}(L_\nu)_{11}(L_\mu^\dagger)_{11}(L_\nu^\dagger)_{01}, \quad (13)$$

where now the local coordinates refer to the values of m_μ and m_ν. This, however, does not reduce to the free finite element form if $g = 0$, and therefore would violate unitarity. So we introduce a covariant averaging operator,

$$(\tilde{\mathcal{D}}_\lambda X)_0 = \frac{1}{2}[(L_\lambda)_1 X_1 (L_\lambda^\dagger)_1 + X_0], \quad (14)$$

where now the coordinate refers to m_λ. The field strength is then (symmetric averaging)

$$(F_{\mu\nu})_{\overline{m}} = \left(\prod_{\lambda\neq\mu,\nu}\tilde{\mathcal{D}}_\lambda\mathcal{F}_{\mu\nu}\right)_m. \quad (15)$$

The field strength transforms covariantly in the sense

$$\delta(F_{\mu\nu})_{\overline{m}} = ig[\delta\omega_m, (F_{\mu\nu})_{\overline{m}}]. \quad (16)$$

2.1. Yang-Mills equations

Although the above field strength construction is *local*, it is not possible to obtain local Dirac and Yang-Mills equations without sacrificing unitarity [5]. The inductive approach in [4] gives the following covariant Yang-Mills equations

$$\sum_{\nu\neq\mu}\left\{\frac{1}{h}[(F^{\mu\nu})_1 - (F^{\mu\nu})_0] + \mathcal{K}_0^\nu\right\} = j_0^\mu, \quad (17)$$

Table 1
Ground-state energies for a $\lambda q^{2}k/2k$ potential. (Figures in parentheses are percentage relative errors.)

k	$O(h)$		$O(h^2)$		Exact
	One state	Two state	One state	Two state	
2	0.4293(+2)	0.4212(+0.1)	0.4178(−0.7)	0.4205(−0.06)	0.42081
3	0.4639(+7)	0.4391(+0.9)	0.4453(+2)	0.4328(−0.5)	0.43493
4	0.5230(+13)	0.4772(+3)	0.5171(+11)	0.4647(−0.2)	0.46450

where the value of the local ν coordinate is shown (the curl involves averaging in $m_\perp$), and the interaction term $\mathcal{K}_n^\nu = \mathcal{K}_n$ can be easily constructed from the difference equation

$$\mathcal{K}_{n-1}L_n + L_n\mathcal{K}_n = \frac{2}{h}[L_n, (F^{\mu\nu})_{n,\overline{m}_\perp}], \tag{18}$$

where $L_n = L_n^\nu$. The current in (17) may be taken to be the local form,

$$j_m^\mu = g\overline{\psi}_{\overline{m}}T\gamma^\mu\psi_{\overline{m}}, \tag{19}$$

which transforms like (16).

3. AXIAL-VECTOR ANOMALIES AND FERMION DOUBLING

For an Abelian theory it is easy to obtain an explicit form for the transfer matrix, which carries the Dirac operator from one time to the next, $\psi_{n+1} = T_n\psi_n$. In the temporal gauge, $A_0 = 0$:

$$T = \frac{1 - ih\mu\gamma^0/2 + \gamma^0\vec{\gamma}\cdot\vec{\mathcal{D}}}{1 + ih\mu\gamma^0/2 - \gamma^0\vec{\gamma}\cdot\vec{\mathcal{D}}}, \tag{20}$$

where $\vec{\mathcal{D}}$ is a covariant derivative operator [6]. Using this, nonzero vector and axial-vector anomalies have been computed in 2 and 4 space-time dimensions [6,5]. For example, in 2 dimensions

$$\langle \text{``}\partial_\mu J_5^\mu\text{''}\rangle = \frac{e^2E}{M\sin\pi/M}\left\{\begin{array}{c}\cos^2\pi/2M\\ 1\end{array}\right\}, \tag{21}$$

for M odd or even, respectively, where E is the electric field and M is the number of spatial lattice sites. Note this reduces to continuum form, with error, as expected, of $O(M^{-2})$, as $M\to\infty$. At the same time we can compute the lattice vacuum polarization (one fermion loop) and obtain results very close to the expected value of e^2/π. So there is no evidence of species doubling, which freedom is already signaled by the form of the free dispersion relation [7],

$$\omega^2 = \sum_{i=1}^{3}\frac{4}{h^2}\tan^2\frac{p_i\pi}{M} + \mu^2. \tag{22}$$

Note that the Minkowski-space formulation is crucial to this conclusion. If we were working in Euclidean space, it is possible to define an action from which the equations of motion could be derived. Since that action possess chiral symmetry, the currents derived from it (which involve all times, "past" and "future") will be anomaly free. Explicitly, fermion doublers then appear and contribute to the vacuum polarization [5].

But in Minkowski space, it is not possible to derive an action. Therefore, there is no connection between symmetry and conserved currents. In fact, the current (19) must be freely invented (consistent with gauge invariance), and can, and does, possess anomalies. Explicitly, the vacuum polarization gets its sole contribution from the neighborhood of zero-momentum fermions.

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