Quantum mechanics using Fradkin's representation

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Fradkin's representation is a general method of attacking problems in quantum field theory, having as its basis the functional approach of Schwinger. As a pedagogical illustration of that method, we explicitly formulate it for quantum mechanics (field theory in one dimension) and apply it to the solution of Schrödinger's equation for the quantum harmonic oscillator.

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I. INTRODUCTION

In one of his papers[1] on functional methods E. S. Fradkin derives a formal solution (Fradkin's representation[2]) to a class of differential equations which occur in quantum field theory. The solution of the problem reduces to operations involving functional derivatives alone. As such, it is complementary to the far more familiar solution in terms of Feynman functional integrals. In a different context, Fradkin mentions that for the case of a harmonic oscillator "all operations are quite simple and brought to completion." In this pedagogical note we carry out this "simple" task (which is perhaps not quite so straightforward as it might appear), as an example of the use of this method, which is not widely discussed. Elsewhere, we will apply this method to nontrivial physical examples, particularly to describe the physics of magnetic monopoles.

Fradkin's representation was developed in the context of quantum field theories and in this regard it is an extension to Schwinger's functional representation[3] for the vacuum to vacuum persistence probability amplitude $Z \equiv \langle 0_+ | 0_- \rangle$. The latter object, expressed in terms of external sources, generates all the Green's functions of the theory, and therefore defines the theory. Schwinger's expression of field theory in this way was the content of important papers published in the 1950s.[4]

II. STATEMENT OF THE PROBLEM

Schrödinger's equation for a quantum mechanical oscillator in one dimension is

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2 \right]\psi(x,t). \tag{1}$$

In terms of the parameters ω and α, defined as $\frac{k}{\hbar} = \alpha^2\omega$ and $\frac{m}{\hbar} = \frac{\alpha^2}{\omega}$, we can construct the dimensionless pairs of variables, ωt and αx. Without any loss of generality, we can choose $\omega = 1$ and $\alpha = 1$, which renders t and x dimensionless. In terms of dimensionless t and x Schrödinger's equation takes the form

$$i \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 \right]\psi(x,t). \tag{2}$$

The solution to this differential equation for a particular initial condition

$$\psi(x,0) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(x-a)^2} \tag{3}$$

is[5]

$$\psi(x,t) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(x-a\cos t)^2} e^{-\frac{i}{2}[t+2ax\sin t-\frac{1}{2}a^2\sin 2t]}, \tag{4}$$

where a is interpreted as the amplitude of the corresponding classical oscillator. The goal here is to reproduce this result using Fradkin's representation for Schrödinger's equation.

III. FRADKIN'S REPRESENTATION FOR SCHRÖDINGER'S EQUATION

In this section we shall keep our treatment general for a time-dependent potential of the form $U(x,t)$. Consider the following differential equation, constructed by introducing a suitable source term $v(t)$ for the derivative operator in Schrödinger's equation,

$$\frac{\partial}{\partial t}W[x,t;v] = \left[\frac{i}{2}\frac{\partial^2}{\partial x^2} + v(t)\frac{\partial}{\partial x} - iU(x,t)\right] W[x,t;v], \tag{5}$$

where $W[x,t;v]$ is a functional of the source v and a function of x and t. The solution $\psi(x,t)$ to the original Schrödinger equation is related to $W[x,t;v]$ by the prescription

$$\psi(x,t) = \{W[x,t;v]\}_{v=0}\,. \tag{6}$$

The initial condition to Eq. (5) is prescribed, in terms of the initial condition to the original Schrödinger equation, to be

$$W[x,0;v] = \psi(x,0). \tag{7}$$

Notice that we have chosen our initial condition to be independent of v, thus we have

$$\frac{\delta}{\delta v(\tau)}W[x,0;v] = 0. \tag{8}$$

Before going ahead, it is good to appreciate the unitary property of the differential equation (5). We know that Schrödinger's equation preserves the norm of $\psi(x,t)$,

$$\frac{d}{dt}(\psi,\psi) = \frac{d}{dt}\int dx\, |\psi(x,t)|^2 = 0, \tag{9}$$

which is the statement of probability conservation. In other words, the time evolution of $\psi(x,t)$ is generated by a unitary transformation. Using the traditional technique, which involves multiplying Eq. (5) by the complex conjugate of $W[x,t;v]$, and then adding the modified equation to its complex conjugated version, we can see that

$$\frac{d}{dt}\int dx\, \Big|W[x,t;v]\Big|^2 = 0, \tag{10}$$

if $v(t) = v(t)^*$. (We are also assuming that the potential U is real.) This exposes a very welcome feature of Eq. (5), the time evolution involved in $W[x,t;v]$ is unitary if $v(t)$ is real.

To determine Fradkin's representation for Schrödinger's equation we start by integrating Eq. (5) with respect to time from 0 to t and obtain the following integral equation

$$W[x,t;v] = W[x,0;v] + \int_0^t dt' \left[\frac{i}{2}\frac{\partial^2}{\partial x^2} + v(t')\frac{\partial}{\partial x} - iU(x,t')\right] W[x,t';v]. \tag{11}$$

Using the above integral equation in conjunction with Eq. (8) it is easy to derive a very useful identity

$$\lim_{\epsilon\to 0}\frac{\delta}{\delta v(t-\epsilon)}W[x,t;v] = \frac{\partial}{\partial x}W[x,t;v], \tag{12}$$

which involves the functional derivative of $W[x,t;v]$, and ϵ is a positive infinitesimal quantity. This, of course, is the reason the source $v(t)$ was introduced. The above identity will play a crucial role in the following development. For the sake of dimensional analysis, it might be helpful to remind ourselves that the dimension of a functional derivative is not equal to the inverse of the dimension of the function, because the functional derivative is defined as

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y), \tag{13}$$

where the delta function has the dimension of the inverse of the variable x.

A. Proof of Eq. (12)

We begin the proof of Eq. (12) by introducing a variable τ which is defined in the domain $t-\epsilon<\tau<t$, where ϵ is a positive infinitesimal quantity. Starting from Eq. (11) and using Eq. (8) we can write

$$\frac{\delta}{\delta v(\tau)}W[x,t;v] \;=\; \int_0^t dt'\delta(t'-\tau)\frac{\partial}{\partial x}W[x,t';v]+\int_0^t dt'\left[\frac{i}{2}\frac{\partial^2}{\partial x^2}+v(t')\frac{\partial}{\partial x}-iU(x,t')\right]\frac{\delta}{\delta v(\tau)}W[x,t';v], \tag{14}$$

where we have used the definition of functional derivative in Eq. (13). The last expression when iterated gives us

$$\begin{aligned}\frac{\delta}{\delta v(\tau)}W[x,t;v] \;&=\; \int_0^t dt'\delta(t'-\tau)\frac{\partial}{\partial x}W[x,t';v]\\ &+\int_0^t dt'\left[\frac{i}{2}\frac{\partial^2}{\partial x^2}+v(t')\frac{\partial}{\partial x}-iU(x,t')\right]\int_0^{t'} dt''\delta(t''-\tau)\frac{\partial}{\partial x}W[x,t'';v]+\cdots,\end{aligned} \tag{15}$$

which is a series expression for the functional derivative of W with respect to v. The first term in the series trivially evaluates to our desired result because $0<\{t',\tau\}<t$. In evaluating the second term we should be careful because the contribution from the delta function depends on whether or not τ is greater than t'. We evaluate the second term by splitting the integral over t' at $t'=t-\epsilon$. It is clear that only the integral over t' from $t-\epsilon$ to t can contribute, and, by the mean value theorem, this integral is of order ϵ. Higher terms in the series are even smaller, so we conclude that in the limit $\epsilon\to 0$ the result (12) holds true.

B. Expression for wavefunction

To proceed let us write the solution to $W[x,t;v]$ in terms of an auxiliary functional W_1:

$$W[x,t;v]=\lim_{\epsilon\to 0}e^{\frac{i}{2}\int_0^t d\tau\frac{\delta^2}{\delta v(\tau-\epsilon)^2}}W_1[x,t;v], \tag{16}$$

which when substituted into Eq. (5) and after the use of Eq. (12) gives

$$\frac{\partial}{\partial t}W_1[x,t;v] \;=\; \lim_{\epsilon\to 0}e^{-\frac{i}{2}\int_0^t d\tau\frac{\delta^2}{\delta v(\tau-\epsilon)^2}}\left[v(t)\frac{\partial}{\partial x}-iU(x,t)\right]e^{\frac{i}{2}\int_0^t d\tau\frac{\delta^2}{\delta v(\tau-\epsilon)^2}}\;W_1[x,t;v], \tag{17}$$

the functional operator having removed the second derivative term in Schrödinger's equation. The exponential of the functional operator in the above equation contributes nothing when it acts on the quantity in the brackets due to the fact that the sources are defined at different times, that is, because $\tau-\epsilon<t$. As a consequence, the exponential terms in Eq. (17) nullify each other by passing through the bracketed quantity. The differential equation for W_1 thus takes the form

$$\frac{\partial}{\partial t}W_1[x,t;v]=\left[v(t)\frac{\partial}{\partial x}-iU(x,t)\right]W_1[x,t;v]. \tag{18}$$

Let us eliminate the $\frac{\partial}{\partial x}$ term by writing

$$W_1[x,t;v]=\;e^{\int_0^t d\tau v(\tau)\frac{\partial}{\partial x}}\;W_2[x,t;v], \tag{19}$$

which when substituted in Eq. (18) gives

$$\begin{aligned}i\frac{\partial}{\partial t}W_2[x,t;v] \;&=\; e^{-\int_0^t d\tau v(\tau)\frac{\partial}{\partial x}}U(x,t)e^{\int_0^t d\tau v(\tau)\frac{\partial}{\partial x}}\;W_2[x,t;v]\\ &=\; U\left(x-\textstyle\int_0^t d\tau\, v(\tau),t\right)W_2[x,t;v].\end{aligned} \tag{20}$$

The last equation can be immediately exponentiated,

$$W_2[x,t;v]=W_2[x,0;v]\;e^{-i\int_0^t dt' U\left(x-\int_0^{t'}d\tau\, v(\tau),t'\right)}. \tag{21}$$

Thus, following the sequential substitutions in Eqs. (16), (19), and (21), we have a formal solution to the differential equation in Eq. (5):

$$W[x,t;v] = \lim_{\epsilon\to 0} e^{\frac{i}{2}\int_0^t d\tau \frac{\delta^2}{\delta v(\tau-\epsilon)^2}} e^{\int_0^t d\tau v(\tau)\frac{\partial}{\partial x}} W_2[x,0;v] e^{-i\int_0^t dt' U\left(x-\int_0^{t'} d\tau v(\tau),t'\right)}. \tag{22}$$

Using the above formal representation for $W[x,t;v]$ we can write Fradkin's representation for Schrödinger's equation for the time-evolved wavefunction using the prescription in Eq. (6) as [because $W(x,0,v) = W_1(x,0,v) = W_2(x,0,v) = \psi(x,0)$]

$$\begin{aligned}
\psi(x,t) &= \lim_{\epsilon\to 0} e^{\frac{i}{2}\int_0^t d\tau \frac{\delta^2}{\delta v(\tau-\epsilon)^2}} e^{\int_0^t d\tau v(\tau)\frac{\partial}{\partial x}} \psi(x,0) e^{-i\int_0^t dt' U\left(x-\int_0^{t'} d\tau v(\tau),t'\right)}\bigg|_{v=0} \\
&= \lim_{\epsilon\to 0} e^{\frac{i}{2}\int_0^t d\tau \frac{\delta^2}{\delta v(\tau-\epsilon)^2}} \psi\left(x+\int_0^t d\tau\ v(\tau),0\right) e^{-i\int_0^t dt' U\left(x+\int_{t'}^t d\tau v(\tau),t'\right)}\bigg|_{v=0}.
\end{aligned} \tag{23}$$

IV. QUANTUM HARMONIC OSCILLATOR

For the case of the quantum harmonic oscillator we have $U(x,t) = \frac{1}{2}x^2$, for which Schrödinger's equation takes the form in Eq. (2), where we impose the initial condition $\psi(x,0)$ given in Eq. (3). Fradkin's representation for the quantum harmonic oscillator is obtained by substituting $U(x,t)$ and $\psi(x,0)$ into Eq. (23), which gives

$$\psi(x,t) = \lim_{\epsilon\to 0} \pi^{-\frac{1}{4}} e^{\frac{i}{2}\int_0^t d\tau \frac{\delta^2}{\delta v(\tau-\epsilon)^2}} e^{-\frac{1}{2}\left(x-a+\int_0^t d\tau v(\tau)\right)^2} e^{-\frac{i}{2}\int_0^t dt'\left(x+\int_{t'}^t d\tau v(\tau)\right)^2}\bigg|_{v=0}. \tag{24}$$

For our purpose it will be convenient to rewrite the above equation in the general form

$$\psi(x,t) = \lim_{\epsilon\to 0} \pi^{-\frac{1}{4}} e^{\frac{1}{2}\int_0^t dt' \int_0^t dt'' \frac{\delta}{\delta v(t'-\epsilon)} A(t',t'') \frac{\delta}{\delta v(t''-\epsilon)}} e^{\frac{1}{2}\int_0^t dt' \int_0^t dt'' v(t') B(t',t'') v(t'') + \int_0^t dt' v(t') C(t') + R(t)}\bigg|_{v=0}, \tag{25}$$

where the various kernels are

$$A(t',t'') = i\delta(t'-t'') \qquad\qquad C(t') = -(x-a) - ixt' \tag{26a}$$

$$B(t',t'') = -\left\{1 + i\ t_<(t',t'')\right\} \qquad\qquad R(t) = -\frac{1}{2}(x-a)^2 - \frac{i}{2}x^2 t. \tag{26b}$$

Here $t_<(t',t'')$ stands for the function which picks the minimum among t' and t''. In this form, we notice that it is possible to carry out the operations of taking the functional derivatives in Eq. (25) because the exponents are quadratic in v.

To evaluate Eq. (25), consider the discrete analog, constructed from symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, column vectors $\mathbf{c}, \mathbf{x}$ and $\nabla = \frac{\partial}{\partial \mathbf{x}}$, and the number r, for which we have the identity

$$e^{\frac{1}{2}\nabla^{\mathrm{T}}\cdot\mathbf{A}\cdot\nabla} e^{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{x}+\mathbf{c}^{\mathrm{T}}\cdot\mathbf{x}+r} = e^{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\cdot\mathbf{B}\cdot\mathbf{K}\cdot\mathbf{x}+\mathbf{c}^{\mathrm{T}}\cdot\mathbf{K}\cdot\mathbf{x}+\frac{1}{2}\mathbf{c}^{\mathrm{T}}\cdot\mathbf{K}\cdot\mathbf{A}\cdot\mathbf{c}+r+\frac{1}{2}\mathrm{Tr}\ln\mathbf{K}}, \tag{27}$$

where $\mathbf{K}$ is the solution to the matrix equation

$$\Big[\mathbf{1} - \mathbf{A}\cdot\mathbf{B}\Big]\cdot\mathbf{K} = \mathbf{1}. \tag{28}$$

Generalizing the identity (27) to functions, and, using it to evaluate the functional derivatives in Eq. (25), and at the end setting $v = 0$, we get

$$\psi(x,t) = \pi^{-\frac{1}{4}}\ e^{R(t)+\frac{1}{2}\int_0^t dt' \int_0^t dt'' \int_0^t dt''' C(t') K(t',t'') A(t'',t''') C(t''') + \frac{1}{2}\mathrm{Tr}\ln K}, \tag{29}$$

where $K(t',t'')$ is the solution to the integral equation [which is the meaning of Eq. (28) for functions]

$$\int_0^t d\tau' \left[\delta(t'-\tau') - \int_0^t d\tau'' A(t',\tau'') B(\tau'',\tau')\right] K(\tau',t'') = \delta(t'-t''). \tag{30}$$

Using the expressions for $A(t',t'')$ and $B(t',t'')$ in Eqs. (26a) and (26b) the above equations simplify to

$$\psi(x,t) = \pi^{-\frac{1}{4}}\ e^{R(t)+\frac{i}{2}\int_0^t dt' \int_0^t dt'' C(t') K(t',t'') C(t'') + \frac{1}{2}\mathrm{Tr}\ln K}, \tag{31}$$

where $K(t',t'')$ and $\mathrm{Tr}\ln K$ are to be determined using the integral equation

$$K(t',t'') + i\int_0^t d\tau \Big[1 + it_<(t',\tau)\Big] K(\tau,t'') = \delta(t'-t''). \tag{32}$$

A. Solution of integral equation

We start by differentiating the above integral equation twice with respect to t' to obtain the differential equation satisfied by $K(t', t'')$:

$$\left[\frac{\partial^2}{\partial t'^2} + 1\right] K(t', t'') = \frac{\partial^2}{\partial t'^2}\delta(t' - t''). \tag{33}$$

The boundary conditions on $K(t', t'')$ are read off from Eq. (32) to be

$$K(0, t'') = -i \int_0^t d\tau K(\tau, t''), \tag{34a}$$

$$K(t, t'') = K(0, t'') + \int_0^t d\tau\, \tau K(\tau, t''). \tag{34b}$$

We have presumed that the delta function in Eq. (32) does not contribute at $t' = 0$ and $t' = t$. (But see below.) In terms of a Green's function $M(t', t'')$, which satisfies

$$\left[\frac{\partial^2}{\partial t'^2} + 1\right] M(t', t'') = \delta(t' - t''), \tag{35}$$

we can write

$$K(t', t'') = \frac{\partial^2}{\partial t'^2} M(t', t'') = \delta(t' - t'') - M(t', t''). \tag{36}$$

The continuity conditions for $M(t', t'')$ are dictated by the Green's function equation (35) to be

$$\{M(t', t'')\}_{t'=t''+\delta} - \{M(t', t'')\}_{t'=t''-\delta} = 0, \tag{37a}$$

$$\{M'(t', t'')\}_{t'=t''+\delta} - \{M'(t', t'')\}_{t'=t''-\delta} = 1, \tag{37b}$$

and the boundary conditions on $M(t', t'')$ are prescribed by the boundary conditions on $K(t', t'')$ in Eqs. (34a) and (34b). We start by writing the solution to $M(t', t'')$ in the form

$$M(t', t'') = \begin{cases} \alpha(t'') \sin t' + \beta(t'') \cos t', & 0 \leq t' < t'' \leq t, \\ \eta(t'') \sin t' + \xi(t'') \cos t', & 0 \leq t'' < t' \leq t, \end{cases} \tag{38}$$

in terms of four arbitrary constants. Using the continuity conditions (37) to determine two of the four constants gives us

$$K(t', t'') = \delta(t' - t'') - \alpha(t'') \sin t' - \xi(t'') \cos t' - \sin t_> \cos t_<, \tag{39}$$

where we have suppressed the t' and t'' dependence in $t_<(t', t'')$ and $t_>(t', t'')$. Using the above expression for $K(t', t'')$ in Eqs. (34a) and (34b) we get the equations determining $\alpha(t'')$ and $\xi(t'')$ to be

$$\alpha(t'') i[1 - \cos t] + \xi(t'')[1 + i \sin t] = i \cos t \cos t'' - \sin t'', \tag{40a}$$

$$\alpha(t'') \cos t - \xi(t'') \sin t = -\cos t \cos t'', \tag{40b}$$

which easily yields

$$\alpha(t'') = -e^{-it} \cos(t - t''), \tag{41a}$$

$$\xi(t'') = i e^{-i(t-t'')} \cos t. \tag{41b}$$

Using Eqs. (41a) and (41b) in eqn. (39) we can obtain the solution to $K(t', t'')$ in the form

$$K(t', t'') = \delta(t' - t'') - i \cos(t - t') \cos(t - t'') - \sin(t - t_<) \cos(t - t_>), \tag{42}$$

which can be verified to satisfy the original integral equation (32) by substitution.

B. Evaluation of $\mathbf{Tr}\ln K$

We start by observing that $K(t',t'')$ and $M(t',t'')$ are related by Eq. (36), and $M(t',t'')$ satisfies Eq. (35). These observations guide us to construct the eigenvalue equation

$$\left[\frac{\partial^2}{\partial t'^2}+1\right]\psi_n(t')=E_n\psi_n(t') \tag{43}$$

where we presume that there exists a Hilbert space where $\psi_n(t')$ are orthonormal eigenfunctions satisfying the orthonormality condition

$$\int_0^t dt'\,\psi_n(t')^*\psi_m(t')=\delta_{nm} \tag{44a}$$

and the completeness relation

$$\sum_n \psi_n(t')\psi_n(t'')^*=\delta(t'-t''). \tag{44b}$$

We shall defer the derivation of the boundary conditions on $\psi_n(t')$.

It is convenient to evaluate the trace of an operator in terms of its representation as a kernel. Let us denote $\mathbf{L}\equiv\frac{\partial^2}{\partial t'^2}+1$ and introduce the matrix representation

$$\mathbf{L}\equiv L(t',t'')=\sum_n E_n\psi_n(t')\psi_n(t'')^*, \tag{45}$$

in terms of which our original Eq. (43) takes the form

$$\int_0^t dt''L(t',t'')\psi_n(t'')=E_n\psi_n(t'). \tag{46}$$

In general, we can write the kernel associated with an arbitrary operator F constructed out of $\mathbf{L}$ to be

$$F(\mathbf{L})\equiv F(t',t'')=\sum_n F(E_n)\psi_n(t')\psi_n(t'')^*. \tag{47}$$

As a particular example we can write the kernel introduced in Eq. (35) to be

$$\mathbf{L}^{-1}=\mathbf{M}\equiv M(t',t'')=\sum_n\frac{1}{E_n}\;\psi_n(t')\psi_n(t'')^*. \tag{48}$$

The trace of a kernel in this representation is well defined by

$$\mathrm{Tr}\,F(\mathbf{L})\equiv\int_0^t dt'F(t',t')=\int_0^t dt'\sum_n F(E_n)\psi_n(t')\psi_n(t')^*=\sum_n F(E_n), \tag{49}$$

where in the last step we have used the orthonormality relation (44a).

Using the above prescription and using Eqs. (36) and (48) we can write the kernel representation of the operator $K(t',t'')$ to be

$$K(t',t'')=\delta(t'-t'')-\sum_n\frac{1}{E_n}\psi_n(t')\psi_n(t'')^*=\sum_n\left(1-\frac{1}{E_n}\right)\psi_n(t')\psi_n(t'')^*, \tag{50}$$

where in the last step we have used the completeness relation (44b). This implies that the eigenvalues of $\mathbf{K}$ are $(1-1/E_n)$. Therefore, the trace of the logarithm of the kernel $K(t',t'')$ is

$$\mathrm{Tr}\ln\mathbf{K}\;=\;\sum_n\ln\left[1-\frac{1}{E_n}\right]. \tag{51}$$

Now we return to the derivation of the boundary conditions on $\psi_n(t')$. We multiply $\psi_n(t'')$ by the integral equation for the kernel $K(t',t'')$ in Eq. (32), integrate with respect to t'', and recognize that acting on ψ_n, K may be replaced by its eigenvalue:

$$\left(1-\frac{1}{E_n}\right)\psi_n(t')+i\int_0^t d\tau\Big[1+it_<(t',\tau)\Big]\left(1-\frac{1}{E_n}\right)\psi_n(\tau)=\psi_n(t'), \tag{52}$$

which further simplifies into

$$\psi_n(t')=-i\,(1-E_n)\int_0^t d\tau\Big[1+it_<(t',\tau)\Big]\psi_n(\tau). \tag{53}$$

From the last equation we read off the boundary conditions on $\psi_n(t')$ to be[6]

$$\psi_n(0) \;=\; -i\,(1-E_n)\int_0^t d\tau\,\psi_n(\tau), \tag{54a}$$

$$\psi_n(t) \;=\; \psi_n(0)+(1-E_n)\int_0^t d\tau\,\tau\,\psi_n(\tau). \tag{54b}$$

Now in terms of $z_n=1-E_n$, the desired trace takes the form

$$\mathrm{Tr}\,\ln\mathbf{K}=\sum_n \ln\left(1-\frac{1}{z_n}\right)^{-1}=\ln\det\mathbf{K}\quad\text{where}\quad\det\mathbf{K}=\prod_n\left(1-\frac{1}{z_n}\right)^{-1}. \tag{55}$$

The solution to Eq. (43) is

$$\psi_n(t')=A_n\cos\sqrt{z_n}t'+B_n\sin\sqrt{z_n}t', \tag{56}$$

where the coefficients A_n and B_n are in principle determined by the boundary conditions. For the special case of $z_n=0$, the solution is

$$\psi_n(t')=c_1t'+c_2 \tag{57}$$

which leads to the trivial solution $\psi_n(t')=0$, because the only solution to c_1 and c_2 allowed by the boundary conditions are $c_1=0$ and $c_2=0$. Thus we conclude that $z_n=0$ is not an eigenvalue. Then we substitute the solutions (56) into Eqs. (54a) and (54b) to get the equations relating A_n and B_n:

$$A_n\left[1+i\sqrt{z_n}\sin\sqrt{z_n}t\right]+B_ni\sqrt{z_n}\,(1-\cos\sqrt{z_n}t) \;=\; 0, \tag{58a}$$

$$A_n\sin\sqrt{z_n}t-B_n\cos\sqrt{z_n}t \;=\; 0, \tag{58b}$$

which leads to nontrivial solutions to $\psi_n(t')$ only when the eigenvalues satisfy the characteristic equation $P(z)=0$, where the determinant of the coefficient matrix in Eq. (58) is

$$P(z)\equiv\cos\sqrt{z}t+i\sqrt{z}\sin\sqrt{z}t. \tag{59}$$

As an illustrative example consider a polynomial $Q^{(N)}(z)$ of order N, which can be written in terms of the roots $z_1,z_2,\ldots,z_N$ of the equation $Q^{(N)}(z)=0$ as

$$Q^{(N)}(z)=d_N(z-z_1)(z-z_2)\ldots(z-z_N). \tag{60}$$

In this form we can deduce

$$\frac{Q^{(N)}(1)}{Q^{(N)}(0)}=\left(1-\frac{1}{z_1}\right)\left(1-\frac{1}{z_2}\right)\ldots\left(1-\frac{1}{z_N}\right). \tag{61}$$

Presuming that the nonpolynomial $P(z)$ in Eq. (59) has a similar infinite product representation in terms of its roots, we can write

$$\det\,\mathbf{K}=\prod_n\left(1-\frac{1}{z_n}\right)^{-1}=\frac{P(0)}{P(1)}=e^{-it}. \tag{62}$$

Thus we have established the simple result

$$\mathrm{Tr}\,\ln\mathbf{K}=-it. \tag{63}$$

V. RESULTS AND CONCLUSIONS

Using the solution for the kernel (42) we can evaluate the first two terms in the exponent in Eq. (31) to be

$$R(t) + \frac{i}{2}\int_0^t dt' \int_0^t dt'' C(t')K(t',t'')C(t'') = -\frac{1}{2}(x - a\cos t)^2 - \frac{i}{2}\left\{2ax\sin t - \tfrac{1}{2}a^2\sin 2t\right\}. \tag{64}$$

Then when we add $\frac{1}{2}\mathrm{Tr}\,\ln\mathbf{K}$ in Eq. (63) we obtain

$$\psi(x,t) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(x-a\cos t)^2} e^{-\frac{i}{2}[t+2ax\sin t - \frac{1}{2}a^2 \sin 2t]} \tag{65}$$

which is the required solution (4).

The reader might now rightfully object that we have honed a mighty machine to crack open a peanut. However, it seems to us that we have performed a useful pedagogical purpose here in exposing explicitly functional techniques which should have much broader applicability in nontrivial contexts. We will present such applications in a subsequent publication.

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[1] E. S. Fradkin, "Application of functional methods in quantum field theory and quantum statistics (II)," Nucl. Phys. **76**, 588-624 (1966), section 5.

[2] To our knowledge, the term Fradkin's representation was first used by H. M. Fried in his book entitled *Basics of Functional Methods and Eikonal Models*, (Editions Frontières, France, 1990).

[3] J. Schwinger, "On the Green's functions of quantized fields. I - II," Proc. Natl. Acad. Sci. USA **37** (7), 452-455, 455-459 (1951).

[4] For a recent historical introduction to Schwinger's ideas on Green's functions, and the relation to Feynman's path integrals, see S. S. Schweber, "The sources of Schwinger's Green's functions," Proc. Natl. Acad. Sci. USA **102** (22), 7783-7788 (2005).

[5] See, for example, L. I. Schiff, *Quantum Mechanics*, 3rd ed., (McGraw-Hill Book Company, New York, 1955), 3rd ed., section 13.

[6] To derive these results from the boundary conditions on the kernels (34), we must include the δ-function terms there: in effect

$$K(0,t'') \to K(0,t'') - \delta(0-t'') = -M(0,t''), \quad K(t,t'') \to K(t,t'') - \delta(t-t'') = -M(t,t'').$$

Then the eigenfunction construction (50) correctly implies the boundary condition (54).