OKHEP-00-03

Comment on Casimir energy for spherical boundaries

I. Brevik*
Division of Applied Mechanics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

B. Jensen†
Faculty of Science, Sogn and Fjordane College, N-6851 Sogndal, Norway

K. A. Milton‡
Department of Physics and Astronomy, The University of Oklahoma, Norman, OK 73019, USA

(April 6, 2000)

It is shown that recent criticism by C. R. Hagen questioning the validity of stress tensor treatments of the Casimir energy for space divided into two parts by a spherical boundary is without foundation.

42.50.Lc,03.50.De,12.20.-m,03.70.+h

I. INTRODUCTION

In a recent paper, Hagen criticizes conventional, well-established techniques for computing the Casimir stress on spherical, or in general, curved boundaries [1]. In our view this paper is full of misconceptions and incorrect assertions, and sheds no light on the subject. The point seems to be that the Green's function method for calculating the Casimir stress on a spherical shell, as originally proposed in Ref. [2], is incorrect, yet, mysteriously, manages to arrive at the correct answer [3]. The simpler, and correct conclusion, would be, on the contrary, that that method is valid. As we will demonstrate, such is the case.

Three errors are claimed to have been made in such calculations: (i) the discontinuity of the stress tensor T_{ij} has been used to calculate the force per unit area on a spherical surface; (ii) poles are neglected in the rotation to Euclidean space; and (iii) the incorrect outgoing wave boundary conditions have been imposed on the Green's functions. On the contrary, all of the procedures used in the criticized papers are, in fact, correct.

Let us first address the point (iii). The causal or Feynman propagator is used both in the Milton et al. papers [2,4–6] and in Hagen's paper. Hagen seems not to appreciate the simple identity obeyed by the free photon propagator:

$$D_+(\mathbf{r},t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2-\omega^2-i\epsilon}=\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty}dk\,\sin kr\,e^{-ik|t|}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{1}{4\pi r}e^{i|\omega|r}, \tag{1}$$

where the second form is obtained by first integrating over ω, the third by integrating over $\mathbf{k}$. The last form explicitly shows that the casual propagator corresponds to outgoing-wave boundary conditions. [1]

As to point (i): There can be no doubt that the discontinuity of T_{rr} across the spherical surface represents the stress per unit area exerted on that surface. As shown in the Appendix of Milton and Ng [14], this is equivalent to the form given by the stationary principle of electrostatics, the force density being $\mathbf{f}=-\frac{1}{2}E^2\nabla\epsilon$ for a dielectric body. (See for example, Ref. [15].) Moreover, in most of Milton's papers, consistency was checked explicitly by rederiving the formulae by at least two different methods, either from the stress on the surface, or from the energy obtained by integrating T_{00} over all space. In fact in the original paper [2] a third independent variational method was used. The stress tensor formulation of the problem is further validated in Sec. 3. The problem that Hagen addresses seems to be the result of using undefined formal expressions [for example, the sum in his Eq. (8) does not exist] so statements about coefficients vanishing in the limit of the enclosing sphere becoming infinite are not meaningful, for one must regulate the expressions before a limit can be taken.

---

*E-mail address: iver.h.brevik@mtf.ntnu.no
†E-mail address: bjorn.jensen@hisf.no
‡E-mail address: milton@mail.nhn.ou.edu

[1] It is also possible to use the retarded Green's function, which follows from a statistical mechanics argument [7–13]. In general the connection between the causal and the retarded Green's dyadics is $\Im\boldsymbol{\Gamma}_+(\mathbf{r},\mathbf{r}',\omega)=\mathrm{sgn}(\omega)\Im\mathbf{G}^R(\mathbf{r},\mathbf{r}',\omega)$. The same results are obtained.

The final objection (ii) concerns the existence of poles in the complex frequency plane. Indeed such poles occur in the lower half frequency plane for the exterior contribution, when the exterior volume is infinite (no large exterior sphere). But exactly this issue was dealt with in the original paper [2], on p. 395, and will be explained in detail in the next section.

So none of Hagen's objections appear to be valid. Of course, it is not incorrect to enclose the sphere in question by a much larger sphere to keep the eigenvalues real, but it is not necessary, and the simplified procedure is correct.

There are many other minor misstatements and incomplete remarks that render Hagen's paper objectionable. The recent experiments [16], while admittedly based on a somewhat different geometry than parallel plates, indeed confirm the theory at the 5–1% level because the correction to a spherical lens is easily done. Moreover, there can be little room for scepticism about the reality of Casimir forces, in view of the confirmation [17] of the closely related Lifshitz theory [18,9,10], and the demonstration of the equivalence of van der Waals forces with the Casimir force between and within dilute bodies [8,19]. The Casimir force for cylindrical and spherical geometries has recently been confirmed by several authors using completely different zeta function techniques [20]. These, and other methods, also give finite results for other geometries besides the "electromagnetic sphere," namely for cylinders, for scalar and fermionic modes, for dielectric bodies when $\epsilon\mu = \text{const}$, etc. We could go on, but our point is taken.

II. ON THE LEGITIMACY OF ROTATING THE INTEGRATION CONTOUR

Here we readdress in more detail the issue of "rotating the contour of frequency integration." As we will see, what is involved is a bit more sophisticated: It is a Euclidean transformation. The argument as given in [2] applied to the Green's functions, either inside or outside the sphere of radius a, for which either $r, r' < a$, or $r, r' > a$. In fact, as employed in that reference and elsewhere, the "rotation" is actually applied to the expression for the Casimir energy, or the stress at the surface. To perform the transformation, it is essential that both inside and outside contributions be included. We can, however, consider the scalar (or TE) modes separately. Those give for the force per unit area on the sphere (where we have not included a large external sphere—its inclusion does not modify the argument)

$$f = \frac{i}{8\pi^2 a^3} \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) \int_C d\omega\, e^{-i\omega\tau} \left(ka \left[\frac{H^{(1)\prime}_{l+1/2}(ka)}{H^{(1)}_{l+1/2}(ka)} + \frac{J'_{l+1/2}(ka)}{J_{l+1/2}(ka)} \right] + 1 \right), \tag{2}$$

where $k = |\omega|$. (For the derivation of Eq. (2) see [6].) Here the sign of the additive constant has been reversed, in effect, by adding a contact term, a term proportional to $\delta(\tau)$, so that the resulting frequency integral be convergent. Now it may be verified that the integrand in Eq. (2) has the following analytic properties in the complex variable $\zeta = ka$:

- The singularities lie in the lower half plane or on the real axis. Consequently, the integration contour C in ω lies just above the real axis for $\omega > 0$, and just below the real axis for $\omega < 0$.
- For $\Im\zeta > 0$, the integrand goes to zero as $1/|\zeta|^2$. (This is a weaker condition than specified in [2].) This convergent behavior is the result of including both interior and exterior contributions.

Then we may write the stress on the sphere as

$$f = \int_C \frac{d\omega}{2\pi} e^{-i\omega\tau} g(|\omega|), \tag{3}$$

where the integrand satisfies the dispersion relation

$$g(|\omega|) = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\zeta \frac{\zeta}{\zeta^2 - \omega^2 - i\epsilon} g(\zeta), \tag{4}$$

because the singularities of $g(\zeta)$ occur only for $\Im\zeta \le 0$. Now we can carry out the ω integral in Eq. (3) to obtain

$$f = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta \frac{\zeta}{|\zeta|} e^{-i|\zeta||\tau|} g(\zeta). \tag{5}$$

Finally, we re-write the result in Euclidean space by making the Euclidean transformation $i|\tau| \to |\tau_4| > 0$, so that we have the representation

$$\frac{1}{2|\zeta|}e^{-|\zeta||\tau_4|}=\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\frac{e^{ik_4\tau_4}}{k_4^2+\zeta^2}. \tag{6}$$

Thus the Euclidean transform of the stress is

$$f\to f_E=i\int\frac{dk_4}{2\pi}e^{ik_4\tau_4}g(i|k_4|). \tag{7}$$

In effect, then, the Euclidean transformation is given by the recipe $\omega\to ik_4$, $|\omega|\to i|k_4|$, $\tau\to i\tau_4$. In particular, the force per unit area given for a massless scalar field (2) is transformed into the expression

$$f_E=-\frac{1}{8\pi^2a^4}\sum_{l=0}^{\infty}(2l+1)\frac{1}{2}\int_{-\infty}^{\infty}dy\,e^{i\delta y}\left(x\frac{K'_{l+1/2}(x)}{K_{l+1/2}(x)}+x\frac{I'_{l+1/2}(x)}{I_{l+1/2}(x)}+1\right), \tag{8}$$

where we have adopted dimensionless variables, $y=\omega a$, $\delta=\tau_4/a\to 0$, $x=|y|$. It is now straightforward to evaluate this expression. The result is as (surprisingly) first given in 1994 [6]: $f=0.0028168/(4\pi a^4)$. Thus a careful analysis shows that the methodology in Ref. [2] is correct, and that, in fact, as long recognized, a finite result is only achievable if the difference between $T_{rr}(r=a-\epsilon)$ and $T_{rr}(r=a+\epsilon)$ is taken.

III. ON THE LOCAL INTERPRETATION OF THE STRESS TENSOR FOR A CURVED BOUNDARY

A. Theory

Let us readdress the issue about the relation between the change in the stress tensor $T^{\alpha\beta}$ across an *arbitrary* surface, and the resulting force density f^{α} on the surface. In Hagen's paper it is argued that the well-known relation between f^{α} and a change in T_{rr} across a planar surface in Cartesian coordinates is lost when one is dealing with curved surfaces, and apparently even when one is using non-Cartesian coordinates independent of the presence or non-presence of curved surfaces. Hagen bases his reasoning on the assumption that $\nabla_{\alpha}T^{\alpha\beta}=0$ holds everywhere. However, it is an almost trivial fact that this is not the case when sources for the electromagnetic field are present. It is in particular straightforward to show that one in general has

$$\nabla_{\alpha}T^{\alpha\beta}=-F^{\beta\lambda}J_{\lambda}=-f^{\beta} \tag{9}$$

where J^{λ} is the source current, $F^{\alpha\beta}$ is the electromagnetic field strength tensor, and f^{β} is the four-vector force density acting on the material sources in the system. Since this relation is covariant we conclude that the interpretation of f^{α} as a force density holds in any situation. This interpretation is in particular independent of the explicit system of coordinates employed or whether one is dealing with a curved spacetime. One can always locally recover the usual Minkowski spacetime results by transforming to a local vierbein.

The *total force* F^{β} on the source is simply the volume integral of the force density in some three-dimensional spacelike hyper-surface Σ. The total force vector field on a simply connected volume V in Σ with boundary ∂V is most simply computed using Stokes' theorem

$$F^{\beta}=\int_V f^{\beta}=-\int_V\nabla_{\alpha}T^{\alpha\beta}=-\oint_{\partial V}n_{\alpha}T^{\alpha\beta}\,. \tag{10}$$

In this expression n^{α} represents a unit spacelike outward normal to V. We will now specialize to the case when we are dealing with a thin conducting spherical shell, and when Maxwell's vacuum field equations hold everywhere away from the shell. Thus V represents the shell, and ∂V its inner and outer surfaces. Clearly, in general the total stress F in the direction of a unit vector field u^{α} is $F=\int_V f^{\beta}u_{\beta}$. Due to time-independence and the exact spherical symmetry of the shell, we choose u^{α} to point in the radial direction, and we find that the total outward stress on the shell *per unit solid angle* is

$$\mathcal{F}=r_-^2T^{rr}(r_-)-r_+^2T^{rr}(r_+)\,. \tag{11}$$

In this expression the components of the stress tensor are to be evaluated at the inner and the outer radii of the shell, which are denoted by r_- and r_+ respectively. In the limit when the shell is infinitely thin ($r_+\to r_-$) with radius a we thus find that the force per unit area is

$$f = \frac{\mathcal{F}}{a^2} = T^{rr}(a_-) - T^{rr}(a_+). \tag{12}$$

The same result can also be obtained by integrating the divergence of the stress tensor directly, using the fact that $\mathrm{Tr}(T_{\alpha\beta}) = 0$. This should prove once and for all that Hagen's critical discussion about the use of the change in T_{rr} as a direct measure of the stress on a spherical shell is, from the theoretical side, completely wrong. Let us now turn to the experimental side.

B. Experiments

We wish to present two counterexamples from optics which demonstrate that Hagen's claim about the use of T_{rr} in calculating the force on a spherical shell is incorrect in this regard: The predictions obtained from local stress tensor calculations are in accordance with experimental evidence.

1. The first example concerns the oscillations of a small water droplet illuminated by a laser pulse. In 1988 Zhang and Chang [21] took photographs of water droplets (50 μm radius) at various instants after illumination by a λ=0.60 μm dye laser pulse of 0.4 μs duration. In the case of a 100 mJ pulse, the droplet began oscillating with surface elevations of up to 30 per cent of the equilibrium radius. Since a spherical drop acts as a lens, the maximum elevations were observed, as expected, at the rear section.

From a theoretical point of view this problem is a hybrid problem: the natural way to proceed is to calculate the electromagnetic surface force density using Eq. (12) as if the surface of the sphere were at rest. Thereafter one can calculate the hydrodynamic response of the sphere by means of the Navier-Stokes equation. The problem was studied by Lai *et al.* [22], and by Brevik and Kluge [23], for various polarizations in the linear approximation for the surface elevation. Wave optics was required. Although the linear approximation may appear rough at the rear section, it is clear from the figures that this kind of theory is in reasonable agreement with the observations. This example, as far as we are aware, is one of the very few cases where the observations of the radiation pressure are *local.* The viability of the expression (12) is quite evident.

2. Our second example is about the optical forces observed on microparticles in an evanescent laser field. There is a striking experimental demonstration of this effect, by Kawata and Sugiura [24]: The particles are lifted up from the dielectric interface and moved parallel to the interface. Although this effect is not a local effect such as above, one has nevertheless, in order to describe it theoretically, to make use of the same formula (12) as before. The theory of the effect has been worked out by Almaas and Brevik [25], Lester and Nieto-Vesperinas [26], and others. The agreement between wave theory and observations is reasonable, again justifying the legitimacy of Eq. (12).

---

[1] C. R. Hagen, Phys. Rev. D **61**, 065005 (2000).
[2] K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (N.Y.) **115**, 388 (1978).
[3] T. H. Boyer, Phys. Rev. **174**, 1764 (1968); B. Davies, J. Math. Phys. **13**, 1324 (1972); R. Balian and B. Duplantier, Ann. Phys. (N.Y.) **112**, 165 (1978); S. Leseduarte and A. Romeo, Europhys. Lett. **34**, 79 (1996); Ann. Phys. (N.Y.) **250**, 448 (1996); V. V. Nesterenko and I. G. Pirozhenko, Phys. Rev. D **57**, 1284 (1997); M. E. Bowers and C. R. Hagen, Phys. Rev. D **59**, 0250007 (1999).
[4] L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (N.Y.) **136**, 229 (1981).
[5] K. A. Milton and Y. J. Ng, Phys. Rev. D **46**, 842 (1992).
[6] C. M. Bender and K. A. Milton, Phys. Rev. D **50**, 6547 (1994); K. A. Milton, Phys. Rev. D **55**, 4940 (1997).
[7] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 3rd ed. (Pergamon, Oxford, 1980), Part 1.
[8] I. Brevik and V. M. Marachevsky, Phys. Rev. D **60**, 085006 (1999).
[9] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Usp. Fiz. Nauk **73**, 381 (1961) [Sov. Phys.-Usp. **4**, 153 (1961)].
[10] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960), pp. 368–376.
[11] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover Publications, New York, 1975).
[12] Yu. S. Barash and V. L. Ginzburg, Usp. Fiz. Nauk **116**, 5 (1975) [Sov. Phys.-Usp. **18**, 306 (1975)].
[13] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part 2 (Pergamon, Oxford, 1981).
[14] K. A. Milton and Y. J. Ng, Phys. Rev. E **55**, 4207 (1997).
[15] J. Schwinger, L. L. DeRaad, Jr., K. A. Milton, and W.-y. Tsai, *Classical Electrodynamics* (Perseus Books, Reading MA, 1998).

[16] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 6 (1997); **81**, 5475(E) (1998); Phys. Rev. A **59**, R3149 (1999); U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998); A. Roy and U. Mohideen, Phys. Rev. Lett. **82**, 4380 (1999); A. Roy, C.-Y. Lin, and U. Mohideen, Phys. Rev. D **60**, 111101 (1999).

[17] E. S. Sabisky and C. H. Anderson, Phys. Rev. A **7**, 790 (1973).

[18] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 894 (1955) [Sov. Phys. JETP **2**, 73 (1956)].

[19] G. Barton, J. Phys. A **32**, 525 (1999); I. Brevik, V. N. Marachevsky, and K. A. Milton, Phys. Rev. Lett. **82**, 3948 (1999).

[20] S. Leseduarte and A. Romeo, Europhys. Lett. **34**, 79 (1996); Ann. Phys. (N.Y.) **250**, 448 (1996); V. V. Nesterenko and I. G. Pirozhenko, Phys. Rev. D **57**, 1284 (1997); P. Gosdzinsky and A. Romeo, Phys. Lett. B **441**, 265 (1998); K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, Phys. Rev. D **59**, 105009 (1999).

[21] J. Z. Zhang and R. K. Chang, Opt. Lett. **13**, 916 (1988).

[22] H. M. Lai, P. T. Leung, K. L. Poon, and K. Young, J. Opt. Soc. Am. B **6**, 2430 (1989).

[23] I. Brevik and R. Kluge, J. Opt. Soc. Am. B **16**, 976 (1999).

[24] S. Kawata and T. Sugiura, Opt. Lett. **17**, 772 (1992).

[25] E. Almaas and I. Brevik, J. Opt. Soc. Am. B **12**, 2429 (1995).

[26] M. Lester and M. Nieto-Vesperinas, Opt. Lett. **24**, 936 (1999).