How Does Casimir Energy Fall?

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Doubt continues to linger over the reality of quantum vacuum energy. It has been suggested that fluctuating fields may not gravitate, or may do so anomalously. Here we show that for the simple case of parallel conducting plates, the associated Casimir energy gravitates just as required by the equivalence principle, and that therefore the inertial and gravitational masses of a system possessing Casimir energy E_c are both E_c/c^2. This simple result disproves recent claims in the literature. We clarify some pitfalls in the calculation that can lead to spurious dependences on coordinate system.

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The subject of quantum vacuum energy (the Casimir effect) dates from the same year as the discovery of renormalized quantum electrodynamics, 1948 [1]. It puts the lie to the naive presumption that zero-point energy is not observable. On the other hand, it continues to be surrounded by controversy, in large part because sharp boundaries give rise to divergences in the local energy density near the surface (see Refs. [2, 3] and more recently Refs. [4, 5]). The most troubling aspect of these divergences is in the coupling to gravity. Gravity has its source in the local energy-momentum tensor, and such surface divergences promise serious difficulties.

As a prolegomenon to studying such issues, we here address a simpler question: How does the completely finite Casimir energy of a pair of parallel conducting plates couple to gravity? The question turns out to be surprisingly less straightforward than one might suspect! Previous authors [6–10] have given disparate answers, including gravitational forces, or gravitationally modified Casimir forces, that depend on the orientation of the Casimir apparatus with respect to the gravitational field of the earth. There are even suggestions that virtual (fluctuating) fields do not gravitate at all [11, 12]. We will here resolve some of this confusion with a convincingly calculated result consistent with the equivalence principle. That is, the renormalized Casimir energy couples to gravity just like any other energy. In our opinion, this fact is evidence that vacuum energy must be taken seriously in gravitational theory and that the problem of boundary divergences must be resolved by a better understanding of the modeling and renormalization processes.

We start by reminding the reader of the electromagnetic Casimir stress tensor between a pair of parallel perfectly conducting plates separated by a distance a, as given by Brown and Maclay [13]:

$$\langle T^{\mu\nu}\rangle = \frac{\mathcal{E}_c}{a}\,\mathrm{diag}(1,-1,-1,3), \tag{1}$$

where the third spatial direction is the direction normal to the plates. This is given in terms of the Casimir energy per unit area, $\mathcal{E}_c = -\pi^2\hbar c/(720a^3)$. Outside the plates, $\langle T^{\mu\nu}\rangle = 0$. Omitted here is a constant divergent term that is present both between and outside the plates, and also in the absence of plates, which cannot have any physical significance. Because the electromagnetic field respects conformal symmetry, there is no surface divergent term such as is present for a minimally coupled scalar field subject to Dirichlet conditions on the plates, or more generally for curved surfaces [14]. (Henceforth we will set $\hbar = c = 1$.)

Now we turn to the question of the gravitational interaction of this Casimir apparatus. It seems to us that this question can be most simply addressed through use of the gravitational definition of the energy-momentum tensor,

$$W_g \equiv \delta W_m \equiv \frac{1}{2}\int (dx)\,\sqrt{-g}\,\delta g_{\mu\nu} T^{\mu\nu}. \tag{2}$$

Following Schwinger (note the factor of 2 in the definition), for a weak field we define $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$. To first order we can ignore $\sqrt{-g}$. The gravitational energy, for a static situation, is therefore given by ($\delta W \equiv -\int dt\,\delta E$)

$$E_g \equiv \delta E_m = -\int (d\mathbf{x})\, h_{\mu\nu} T^{\mu\nu}. \tag{3}$$

We then replace $T^{\mu\nu}$ here by the one-loop expectation value of the electromagnetic stress tensor (1).

Calloni et al. [7] and Bimonte et al. [10] use the Fermi metric

$$g_{00} = -\,(1 + 2gz)\,, \quad g_{ij} = \delta_{ij}\,, \tag{4}$$

in terms of the gravitational acceleration g. This is evidently appropriate for a constant gravitational field. We will discuss its relation to the field due to the earth below. Consider a Casimir apparatus of parallel plates separated by a distance a, with transverse dimensions $L \gg a$. Let the apparatus be oriented at an angle α with respect to

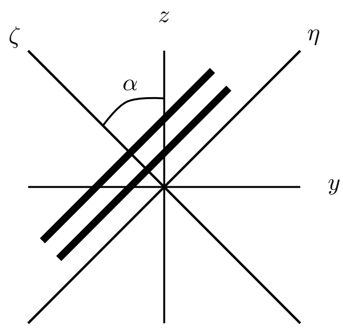

FIG. 1: Relation between two Cartesian coordinate frames: One attached to the earth (x, y, z), where $-z$ is the direction of gravity, and one attached to the parallel-plate Casimir apparatus (ζ, η, χ), where ζ is in the direction normal to the plates. The parallel plates are indicated by the heavy lines parallel to the η axis. The $x = \chi$ axis is perpendicular to the page.

the direction of the gravitational field. The Cartesian coordinate system attached to the earth is denoted by (x, y, z), where, as noted above, z is the direction of $-\mathbf{g}$. The Cartesian coordinates associated with the Casimir apparatus are (ζ, η, χ), where ζ is normal to the plates, and η and χ are parallel to the plates. (See Fig. 1.) The center of the apparatus is located at $(\zeta_0\,, \eta = 0, \chi = 0)$.

Now we calculate the gravitational energy from Eq. (3):

$$E_g = \int (d\mathbf{x})\, gzT^{00} = \frac{g\mathcal{E}_c}{a}\, L^2 a\, \zeta_0 \cos\alpha, \tag{5}$$

up to an additive constant, independent of ζ_0. (Any constant added to the gravitational potential gz simply shifts the value of this constant.) Thus, the gravitational force per area $A = L^2$ on the apparatus is

$$\frac{F}{A} = -\frac{1}{A}\frac{\partial E_g}{\partial z_0} = -g\mathcal{E}_c \equiv -\frac{\epsilon}{2a}\mathcal{E}_c\,, \tag{6}$$

because $z_0 = \zeta_0 \cos\alpha$. Note that on the earth's surface, the dimensionless number $\epsilon \equiv 2ga/c^2$ is very small. For a plate separation of 1μm, $\epsilon = 2 \times 10^{-22}$, so the considerations here may be only of theoretical interest. (However, Calloni et al. [7] discuss the possibility of experimentally observing this force.)

It is a bit simpler to use the energy formula (3) to calculate the force by considering the variation in the gravitational energy directly, that is,

$$\delta E_g = -\int (d\mathbf{x})\, \delta h_{\mu\nu} T^{\mu\nu}. \tag{7}$$

It is easy to verify that this gives the correct force on a mass point, $\mathbf{F} = m\mathbf{g}$. If we use this formula to calculate the gravitational force per area on the rigid Casimir apparatus, by considering a virtual displacement upward by an amount δz_0, we find the same α-independent result found in Eq. (6).

Alternatively, we can start from the definition of the gravitational field [15],

$$\delta W_g = \int (dx)\, \delta T^{\mu\nu} h_{\mu\nu}\,, \tag{8}$$

which can again be checked to yield the correct force on a mass point. For the constant field (4) the force on a Casimir apparatus is obtained from the change in the energy density T^{00}; that is, recalling that $z_0 = \zeta_0 \cos\alpha$,

$$\delta T^{00} = \frac{\mathcal{E}_c\, \delta z_0}{a}\frac{1}{\cos\alpha}[\delta(\zeta - \zeta_0 - a/2) - \delta(\zeta - \zeta_0 + a/2)], \tag{9}$$

where the δ functions arise from the step functions at the boundaries. This yields from Eq. (8) the same result (6).

Our answer is consistent with the principle of equivalence, and with the second analysis of Jaekel and Reynaud [16], who state that the inertia of Casimir energy (at least in two dimensions) is E_c/c^2. However, it is only $\frac{1}{4}$ that found by Bimonte et al. [10], which is also the first force formula [Eqs. (7) and (8)] provided by Calloni et al. [7]. Our Eq. (6) does, however, reproduce the second formula [Eq. (9)] given in Ref. [7], which those authors describe as the one that should be observable. We discuss this situation further below.

We now digress to consider whether the constant-field approximation (4) is adequate for an apparatus suspended above the earth or some other pointlike mass. Should we instead use the perturbation of the Schwarzschild metric? One might expect that the resulting curvature corrections are of order $\frac{L}{R} \ll 1$, at worst, relative to the main term, where R is the earth's radius. The point, however, is that naive attempts to do the calculation in curved space change the answer by factors like 2, and also differ among themselves, and the resolutions of the fallacies are sufficiently instructive to justify our belaboring the point.

The Schwarzschild metric in isotropic coordinates [17] and for weak fields ($GM/r \ll 1$) is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) d\mathbf{r}^2. \tag{10}$$

If we expand a short distance z above the earth's surface, we find the nonzero components of the gravitational field to be $h_{00} = h_{11} = h_{22} = h_{33} = GM/R - gz$, in terms of the acceleration of gravity, $g = GM/R^2$. (It is important to recognize that the constant GM/R is irrelevant in the following, and that correspondingly the results do not depend on the origin of z.) The virtue of isotropic coordinates is that the spatial line element (apart from an overall factor) has the usual Cartesian form $d\mathbf{r}^2 = dx^2 + dy^2 + dz^2$. Now when we compute the gravitational energy from Eq. (7) each component of the Casimir stress tensor contributes with equal weight:

$$\delta E_g = gA\, \delta z_0 \int_{\zeta_0 - a/2}^{\zeta_0 + a/2} d\zeta\, (T^{00} + T^{11} + T^{22} + T^{33}), \tag{11}$$

which gives the force

$$-\frac{1}{A}\frac{\delta E_g}{\delta z_0} = \frac{F}{A} = -2gaT^{00} = -2g\mathcal{E}_c\,, \qquad (12)$$

since $T = T^\lambda{}_\lambda = 0$. This is twice the previous result (6). Note that again the result is independent of α. The same result is obtained if we start from Eq. (8).

We should be able to obtain the same result using the original Schwarzchild coordinates, where $h_{00} = -gz$, $h_{\rho\rho} = -gz$, and all other components of $h_{\mu\nu}$ are zero. However, now if we use the first method (7), the result is $F/A = -4g\mathcal{E}_c\cos^2\alpha$, so now the force depends on the orientation of the apparatus. Even if $\alpha = 0$, the magnitude differs from Eq. (12) by an additional factor of 2. (It thereby fortuitously agrees with the result in Ref. [10] for that angle.)

What is going on here? The reason we get different answers in different coordinate systems is that our starting point (3) is not gauge-invariant. Under a coordinate redefinition, which for weak fields is a gauge transformation of $h_{\mu\nu}$ [15], $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,$, where ξ_μ is a vector field, Eq. (3) is invariant only if the stress tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$ (in the weak-field context). Otherwise, there is a change in the action, $\Delta W = -2\int(dx)\xi_\nu\partial_\mu T^{\mu\nu}$. Now in our case (where we make the finite size of the plates explicit, but ignore edge effects on $T^{\mu\nu}$ because $L \gg a$)

$$T^{\mu\nu} = \frac{\mathcal{E}_c}{a}\,\mathrm{diag}(1,-1,-1,3)\theta(\zeta-\zeta_0+a/2)\theta(a/2-\zeta+\zeta_0)\theta(\eta+L/2)\theta(L/2-\eta)\theta(\chi+L/2)\theta(L/2-\chi). \qquad (13)$$

Taking the divergence of Eq. (13) gives corresponding δ functions on the surfaces and leads immediately to the change in the energy obtained from ΔW as

$$\Delta E_g = \frac{6\mathcal{E}_c}{a}\int d\eta d\chi\,[\xi_\zeta(\zeta_0-a/2,\eta,\chi)-\xi_\zeta(\zeta_0+a/2,\eta,\chi)] - \frac{2\mathcal{E}_c}{a}\int d\zeta d\chi\,[\xi_\eta(\zeta,-L/2,\chi)-\xi_\eta(\zeta,L/2,\chi)] - (\eta\leftrightarrow\chi). \qquad (14)$$

This transformation entirely accounts algebraically for the difference between the force in isotropic and Schwarzchild coordinates, but it does not yet explain physically why there are two different answers, nor tell us which, if either, is correct.

There seem to be two possible ways to proceed. First, it is clear that the energy-momentum tensor of the complete physical situation must be conserved, and therefore the expression (3) would be gauge-invariant if we included a physical mechanism holding the plates apart against the Casimir attraction. That road probably leads to complicated, model-dependent calculations. The alternative is to find a physical basis for believing that one coordinate system is more realistic than another. Fortunately, that problem apparently has a natural solution. The crux of the difficulty is that the relations between coordinate increments and physical distances depend upon the distance from the gravitating center in the most common coordinate systems.

Of course, a perfect coordinate system is not possible in a curved space, but the kind that comes closest to representing distances accurately all along a timelike worldline is a Fermi coordinate system, the general-relativistic extrapolation of an inertial coordinate frame. Such a system has been given by Marzlin [18] for a resting observer in the field of any static mass distribution. Here we give a simple rederivation for the case at hand. Starting from the isotropic metric (10), first eliminate the constant term by rescaling the coordinates, $t \to \left(1+\frac{GM}{R}\right)t$, $\mathbf{r} \to \left(1-\frac{GM}{R}\right)\mathbf{r}$, and expand to first order:

$$ds^2 = -(1+2gz)dt^2 + (1-2gz)d\mathbf{r}^2. \qquad (15)$$

But we need $\mathbf{r}$ to measure physical displacements even when $z \neq 0$, so we write

$$x = x' + gx'z', \qquad y = y' + gy'z',$$
$$z = z' + \tfrac{g}{2}(z'^2 - x'^2 - y'^2). \qquad (16)$$

Then to first order in coordinates we obtain the Fermi metric (4):

$$ds^2 = -(1+2gz)dt^2 + d\mathbf{r}'^2. \qquad (17)$$

The corresponding gravitational force is therefore given by Eq. (6), after all!

Now we can use the method described above to transform the energy in isotropic coordinates to that in Fermi coordinates. We use Eq. (14) to compute the additional gravitational energy, in terms of the gauge field ξ_μ that carries us from isotropic coordinates to Fermi coordinates,

$$h^F_{\mu\nu} = h^I_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \qquad (18)$$

Here $h^I_{00} = -gz$, $h^I_{ij} = -gz\delta_{ij}\,$, $h^F_{00} = -gz$, $h^F_{ij} = 0$,

$h_{0i}^{I,F}=0$. The gauge field turns out to be

$$\begin{aligned}\xi_\zeta &= \frac{1}{2}g\left(\frac{1}{2}\zeta^2\cos\alpha+\zeta\eta\sin\alpha\right)+f(\eta,\chi),\\ \xi_\eta &= \frac{1}{2}g\left(\zeta\eta\cos\alpha+\frac{1}{2}\eta^2\sin\alpha\right)+g(\zeta,\chi),\\ \xi_\chi &= \frac{1}{2}g\left(\zeta\cos\alpha+\eta\sin\alpha\right)\chi+h(\zeta,\eta),\end{aligned} \tag{19}$$

where the functions f, g, and h are irrelevant. Now from Eq. (14) we obtain a $\Delta E_g(z_0)$ that yields an additional force, $\Delta F/A=g\mathcal{E}_c$. When this is added to the isotropic force (12), we obtain

$$\frac{F^I+\Delta F}{A}=-2g\mathcal{E}_c+g\mathcal{E}_c=-g\mathcal{E}_c=\frac{F^F}{A}, \tag{20}$$

as given in Eq. (6).

The conceptual reason why other coordinates give different answers is that under the virtual displacement involved in defining $\frac{\partial}{\partial z_0}$ one is stretching the apparatus as well as moving it. Correcting for the spurious changes in L and a restores the Fermi result in all cases. The importance of distinguishing a from the physical gap was noted by Sorge [9] in studying a related problem.

As noted above, Calloni et al. [7] find a result 4 times ours, which is the only result from Ref. [7] cited in the later paper [10] with which it shares authors. However, Ref. [7] states that that force formula has two parts, in the ratio of 3/1, and that only the smaller piece is "Newtonian," or "to be tested against observation." Our understanding of what that means is the following. Start from Eq. (2) and consider a general coordinate transformation, $x'^\mu = x^\mu+\delta x^\mu$, so that $g'_{\mu\nu}(x')=g_{\mu\nu}(x)+\delta g_{\mu\nu}(x)$, where

$$\delta g_{\mu\nu}=\delta x^\lambda\partial_\lambda g_{\mu\nu}+g_{\alpha\nu}\frac{\partial\delta x^\alpha}{\partial x^\mu}+g_{\mu\beta}\frac{\partial\delta x^\beta}{\partial x^\nu}. \tag{21}$$

For a rigid translation, δx^λ is a constant, so only the first term in Eq. (21) is present, which gives the result (6). However, if we do not make this restriction, we obtain from Eq. (2) (after integration by parts) a surface-term correction to the force:

$$\int_\Omega (dx)\,\sqrt{-g}f_\lambda=\frac{\delta W_m}{\delta x^\lambda}-\int_{\partial\Omega}d\Sigma_\nu\,\sqrt{-g}T^\nu{}_\lambda, \tag{22}$$

where the force vector density is [7]

$$\sqrt{-g}f_\lambda=-\partial_\nu\left(\sqrt{-g}T^\nu{}_\lambda\right)+\frac{1}{2}\sqrt{-g}T^{\mu\nu}\partial_\lambda g_{\mu\nu}. \tag{23}$$

Note that the surface term identically cancels the first term in f_λ. Now if Ω refers to all space, the surface term vanishes (as we have shown explicitly). But if Ω is just the space volume between the plates, and we include this correction for the Fermi metric (4) for which $\sqrt{-g}=1+gz$, we obtain an additional term $-3g\mathcal{E}_c\cos\alpha$. Adding this to the previous result (6), we obtain the result of Ref. [10] if $\alpha=0$. However, in general the result depends on the angle between the apparatus and the vertical. Is this consistent with the equivalence principle (the scalar nature of mass)? A similar angle dependence will now occur with the isotropic Schwarzchild metric. Why should one trust the formula (23) over the more fundamental variational principle when boundaries are present? Omission of the surface term resolves the discrepancy, giving the equivalence-principle result (6).

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