DIMENSIONAL AND DYNAMICAL ASPECTS OF THE CASIMIR EFFECT: UNDERSTANDING THE REALITY AND SIGNIFICANCE OF VACUUM ENERGY

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Zero-point fluctuations in quantum fields give rise to observable forces between material bodies, the so-called Casimir forces. In this lecture I present some results of the theory of the Casimir effect, primarily formulated in terms of Green's functions. There is an intimate relation between the Casimir effect and van der Waals forces. Applications to conductors and dielectric bodies of various shapes will be given for the cases of scalar, electromagnetic, and fermionic fields. The dimensional dependence of the effect will be described. Finally, we ask the question: Is there a connection between the Casimir effect and the phenomenon of sonoluminescence?

1 Introduction

We may identify the zero-point energy of a system of quantum fields with the vacuum expectation value of the field energy,

$$\frac{1}{2}\sum_a \hbar\omega_a = \int (d\mathbf{x})\langle T^{00}(\mathbf{x})\rangle. \tag{1}$$

In the vacuum both sides of this equality are divergent and meaningless. What is observable is the *change* in the zero-point energy when matter is introduced. In this way we can calculate the Casimir forces. For a massless scalar field, the canonical energy-momentum tensor is[1]

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi. \tag{2}$$

The vacuum expectation value may be obtained by taking derivatives of the casual Green's function:

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{i}{\hbar}\langle \mathrm{T}\phi(\mathbf{x},t)\phi(\mathbf{x}',t')\rangle. \tag{3}$$

[1] The ambiguity in defining the stress tensor is without effect. For example, the 'new-improved' traceless stress tensor gives the same Casimir energy.

Alternatively, we can calculate the stress on the material bodies. Consider the original geometry considered by Casimir, where he calculated the quantum fluctuation force between parallel, perfectly conducting plates separated by a distance a [1]. The force per unit area f on one of the plates is given in terms of the normal-normal component of the stress tensor,

$$f = \langle T_{zz} \rangle, \tag{4}$$

For electromagnetic fields, the relevant stress tensor component is

$$T_{zz} = \frac{1}{2}(H_\perp^2 - H_z^2 + E_\perp^2 - E_z^2). \tag{5}$$

We impose classical boundary conditions on the surfaces,

$$H_z = 0, \quad \mathbf{E}_\perp = 0, \tag{6}$$

and the calculation of the vacuum expectation value of the field components reduces to finding the classical TE and TM Green's functions. In general, one further has to subtract off the stress that the formalism would give if the plates were not present, the so-called volume stress, and then the result of a simple calculation, which is sketched below, is

$$f = [T_{zz} - T_{zz}(\text{vol})] = -\frac{\pi^2}{240a^4}\hbar c, \tag{7}$$

an attractive force.

The dependence on the plate separation a is, of course, completely determined by dimensional considerations. Numerically, the result is quite small,

$$f = -8.11\,\text{MeV}\,\text{fm}\,a^{-4} = -1.30 \times 10^{-27}\text{N}\,\text{m}^2\,a^{-4}, \tag{8}$$

and will be overwhelmed by electrostatic repulsion between the plates if each plate has an excess electron surface density n greater than $1/a^2$, from which it is clear that the experiment must be performed at the μm level. Nevertheless, over twenty years, many attempts to measure this force directly were made [2]. (The cited measurements include insulators as well as conducting surfaces.) Until recently, the most convincing experimental evidence came from the study of thin helium films [3]; there the corresponding Lifshitz theory [4] has been confirmed over nearly 5 orders of magnitude in the van der Waals potential (nearly two orders of magnitude in distance). Quite recently, the Casimir effect between conductors has been confirmed at the 5% level by Lamoreaux [5], and to perhaps 1% by Mohideen and Roy [6]. (In order to achieve the stated accuracy, corrections for finite conductivity, surface distortions, and perhaps temperature must be included. For a brief review see Ref. [7].)

2 Dimensional Dependence

2.1 Parallel Plates

Here we wish to concentrate on dimensional dependence. For simplicity we consider a massless scalar field ϕ confined between two parallel plates in $d+1$ spatial dimensions separated by a distance a. Assume the field satisfies Dirichlet boundary conditions on the plates, that is

$$\phi(0) = \phi(a) = 0. \tag{9}$$

The Casimir force between the plates results from the zero-point energy per unit (d-dimensional) transverse area

$$u = \frac{1}{2}\sum \hbar\omega = \frac{1}{2}\sum_{n=1}^{\infty}\int \frac{d^dk}{(2\pi)^d}\sqrt{k^2 + \frac{n^2\pi^2}{a^2}}, \tag{10}$$

where we have set $\hbar = c = 1$, and introduced normal modes labeled by the positive integer n and the transverse momentum k. This may be easily evaluated by introducing a proper-time representation for the square root, and by analytically continuing from negative d we obtain

$$u = -\frac{1}{2^{d+2}\pi^{d/2+1}}\frac{1}{a^{d+1}}\Gamma\left(1 + \frac{d}{2}\right)\zeta(2+d). \tag{11}$$

which reduces to the familiar Casimir result at $d = 2$:

$$u = -\frac{\pi^2}{1440}\frac{1}{a^3}, \quad f_s = -\frac{\partial}{\partial a}u = -\frac{\pi^2}{480}\frac{1}{a^4}. \tag{12}$$

This is, as expected, 1/2 of the electromagnetic result (7).

This less-than-rigorous calculation can be put on a firm footing by a Green's function technique. Define a reduced Green's function by

$$G(x, x') = \int \frac{d^dk}{(2\pi)^d}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\int \frac{d\omega}{2\pi}e^{-i\omega(t-t')}g(z, z'), \tag{13}$$

the (interior) solution of which vanishing at $z = 0, a$, being ($\lambda^2 = \omega^2 - k^2$)

$$g(z, z') = -\frac{1}{\lambda \sin \lambda a}\sin \lambda z_< \sin \lambda(z_> - a), \tag{14}$$

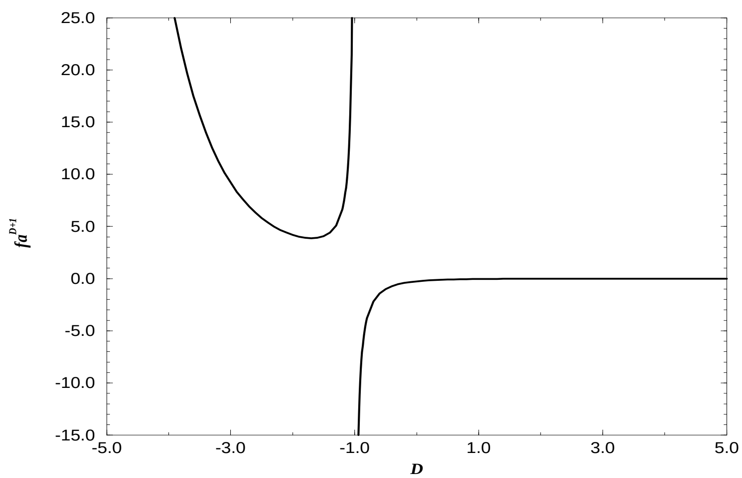

Figure 1: A plot of the Casimir force per unit area f in Eq. (16) for $-5 < D < 5$ for the case of a slab geometry (two parallel plates). Here $D = d + 1$.

where $z_>$ ($z_<$) is the greater (lesser) of z and z'. The force per unit area on the surface $z = a$ is obtained by taking the discontinuity of the normal-normal component of the stress tensor:

$$f = \langle T_{zz} \rangle \Big|_{z=z'=a-} - \langle T_{zz} \rangle \Big|_{z=z'=a+} = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{\lambda}{2} (i \cot \lambda a - 1). \quad (15)$$

This is easily evaluated by doing a complex rotation in frequency: $\omega \to i\zeta$:

$$f = -(d+1) 2^{-d-2} \pi^{-d/2-1} \frac{\Gamma\left(1 + \frac{d}{2}\right) \zeta(d+2)}{a^{d+2}}. \quad (16)$$

Evidently, Eq. (16) is the negative derivative of the Casimir energy (11) with respect to the separation between the plates: $f = -\frac{\partial u}{\partial a}$; this result has now been obtained by a completely well-defined approach, so arguments about the conceptual validity of the Casimir effect are seen to be without merit. The force per unit area, Eq. (16), is plotted in Fig. 1, where $a \to 2a$ and $d = D - 1$.

This general result was first derived by Ambjørn and Wolfram [8].

2.2 Fermion fluctuations

The effect of massless fermionic fluctuations between parallel plates embedded in three dimensional space, subject to "bag model" boundary conditions,

$$(1 + i\mathbf{n}\cdot\boldsymbol{\gamma})G\Big|_{z=0,a} = 0, \tag{17}$$

where $\mathbf{n}$ is the normal to the surface, was first calculated by Ken Johnson [9]. In place of Eq. (10), the fermionic Casimir energy for $d = 2$ is formally

$$u_F = -2\frac{1}{2}\sum_{n=0}^{\infty}\int\frac{d^2k}{(2\pi)^2}\sqrt{k^2 + \frac{(n+1/2)^2\pi^2}{a^2}}, \tag{18}$$

so once the k integral is performed the energy is proportional to

$$-2\sum_{n=0}^{\infty}(n+1/2)^3 = \frac{7}{4}\sum_{n=1}^{\infty}n^3. \tag{19}$$

Thus, the result is 7/4 times the scalar Casimir energy,

$$f_F = -\frac{7\pi^2}{1920a^4}. \tag{20}$$

The Casimir effect so implemented breaks supersymmetry. However, in a SUSY theory, if supersymmetric boundary conditions are imposed (the unconstrained components of the fermion and scalar fields are all periodic, for example), the fermionic Casimir energy will just cancel that due to the bosons. (For a simple example of how this works, see Ref. [10].)

2.3 Casimir Effect on a D-dimensional Sphere

Because of the rather mysterious dependence of the sign and magnitude of the Casimir stress on the topology and dimensionality of the bounding geometry, we have carried out a calculation of TE and TM modes bounded by a spherical shell in D spatial dimensions. We first consider massless scalar modes satisfying Dirichlet boundary conditions on the surface, which are equivalent to electromagnetic TE modes. Again we calculate the vacuum expectation value of the stress on the surface from the Green's function.

The Green's function $G(\mathbf{x},t;\mathbf{x}',t')$, Eq. (3), satisfies the inhomogeneous Klein-Gordon equation, or

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)G(\mathbf{x},t;\mathbf{x}',t') = \delta^{(D)}(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{21}$$

where ∇^2 is the Laplacian in D dimensions. We solve the above Green's function equation by dividing space into two regions, the interior of a sphere of radius a and the exterior of the sphere. On the sphere we impose Dirichlet boundary conditions

$$G(\mathbf{x}, t; \mathbf{x}', t') \mid_{|\mathbf{x}|=a} = 0. \tag{22}$$

In addition, in the interior we require that G be finite at the origin $\mathbf{x} = 0$ and in the exterior we require that G satisfy outgoing-wave boundary conditions at $|\mathbf{x}| = \infty$, that is, for a given frequency, $G \sim e^{ikr}/r$.

The radial Casimir force per unit area f on the sphere is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor:

$$f = \langle 0 | T^{rr}_{\text{in}} - T^{rr}_{\text{out}} | 0 \rangle \mid_{r=a} . \tag{23}$$

To calculate f we exploit the connection between the vacuum expectation value of the stress-energy tensor $T^{\mu\nu}(\mathbf{x}, t)$ and the Green's function at equal times $G(\mathbf{x}, t; \mathbf{x}', t)$:

$$f = \frac{1}{2i} \left[\frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(\mathbf{x}, t; \mathbf{x}', t)_{\text{in}} - \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(\mathbf{x}, t; \mathbf{x}', t)_{\text{out}} \right]_{\mathbf{x}=\mathbf{x}',\ |\mathbf{x}|=a} . \tag{24}$$

Adding the interior and the exterior contributions, and performing the usual imaginary frequency rotation, we obtain the expression for the stress [11]:

$$f = -\sum_{n=0}^{\infty} w_n(D) \int_0^\infty dx\, x \frac{d}{dx} \ln \left(I_{n-1+\frac{D}{2}}(x) K_{n-1+\frac{D}{2}}(x) x^{2-D} \right), \tag{25}$$

$$w_n(D) = \frac{(n - 1 + \frac{D}{2}) \Gamma(n + D - 2)}{2^{D-1} \pi^{\frac{D+1}{2}} a^{D+1} n! \, \Gamma\left(\frac{D-1}{2}\right)}. \tag{26}$$

It is easy to check that this reduces to the known case at $D = 1$, for there the series truncates—only $n = 0$ and 1 contribute, and we easily find

$$f = -\frac{\pi}{96 a^2}, \tag{27}$$

which agrees with Eq. (16) for $d = 0$ and $a \to 2a$.

In general, we proceed as follows:

- Analytically continue to $D < 0$, where the sum (25) converges, although the integrals become complex.

- Add and subtract the leading asymptotic behavior of the integrals.

- Continue back to $D > 0$, where everything is now finite.

The results of numerical evaluations for the total stress F on the sphere are as shown in Fig. 2.

Note the following features for the scalar modes:

- Poles occur at $D = 2n$, $n = 1, 2, 3, \ldots$.
- Branch points occur at $D = -2n$, $n = 0, 1, 2, 3, \ldots$, and the stress is complex for $D < 0$.
- The stress vanishes at negative even integers, $F(-2n) = 0$, $n = 1, 2, 3, \ldots$, but is nonzero at $D = 0$: $F(0) = -1/2a^2$.
- The case of greatest physical interest, $D = 3$, has a finite stress, but one which is much smaller than the corresponding electrodynamic one: $F(3) = +0.0028168/a^2$. (This result was confirmed in Ref. [12].)

The TM modes are modes which satisfy mixed boundary conditions on the surface,

$$\frac{\partial}{\partial r} r^{D-2} G(\mathbf{x}, t; \mathbf{x}', t')\bigg|_{|\mathbf{x}|=r=a} = 0, \tag{28}$$

The results are qualitatively similar, and are also shown in Fig. 2. In particular, removing the $n = 0$ contribution from the sum of the TE and TM contributions, we recover the repulsive Boyer result in three dimensions [13],

$$E_{\text{sphere}} = \frac{0.092353}{2a}. \tag{29}$$

For the 3-sphere, the fermionic Casimir energy subject to the boundary condition (17) is a factor of two smaller [14]

$$E_F = \frac{0.0204}{a}. \tag{30}$$

2.4 Cylinders

A similar calculation of the electromagnetic Casimir effect of a perfectly conducting infinite right circular cylinder in three dimensions was performed by DeRaad and me twenty years ago. The calculation is rather more involved, and the regularization of the divergences more subtle. The result for the Casimir energy per unit length, or the force per unit area, is [15]

$$\mathcal{E} = \pi a^2 f = -0.01356/a^2. \tag{31}$$

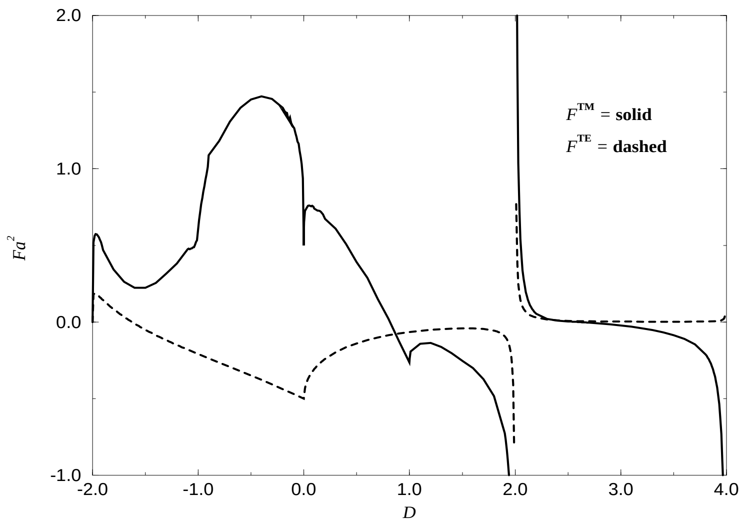

Figure 2: A plot of the TM and TE (Dirichlet) Casimir stress for $-2 < D < 4$ on a spherical shell. For $D < 2$ ($D < 0$) the stress $F^{\rm TM}$ ($F^{\rm TE}$) is complex and we have plotted the real part.

Unlike the three-dimensional sphere, the cylinder experiences an attractive Casimir stress. Two recent calculations have confirmed this result using zeta-function techniques [16, 17].

3 Force between dielectric slabs

Over 40 years ago, Lifshitz and collaborators [4] worked out the corresponding forces between dielectric slabs. Imagine we have a permittivity which depends on z as follows:

$$\epsilon(z) = \begin{cases} \epsilon_1, & z < 0, \\ \epsilon_3, & 0 < z < a, \\ \epsilon_2, & a < z. \end{cases} . \tag{32}$$

Then the Lifshitz force between the bodies at zero temperature is given by ($\kappa^2 = k^2 + \epsilon\zeta^2$, ζ is the imaginary frequency)

$$f^{T=0}_{\rm Casimir} = -\frac{1}{8\pi^2}\int_0^\infty d\zeta \int_0^\infty dk^2\, 2\kappa_3 \left(d^{-1} + d'^{-1}\right) . \tag{33}$$

Here the denominators are given by, for the electric (TM) Green's function,

$$d = \frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1}\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2} e^{2\kappa_3 a} - 1. \tag{34}$$

The magnetic (TE) Green's function has the same form as the electric one but with the replacement

$$\kappa \to \kappa/\epsilon \equiv \kappa', \tag{35}$$

except in the exponentials; the corresponding denominator is denoted by d'. From this, we can obtain the finite temperature expression immediately by the substitution

$$\zeta^2 \to \zeta_n^2 = 4\pi^2 n^2/\beta^2, \tag{36}$$

$$\int_0^\infty \frac{d\zeta}{2\pi} \to \frac{1}{\beta}\sum_{n=0}^{\infty}{}', \tag{37}$$

the prime being a reminder to count the $n = 0$ term with half weight. (For a fuller discussion of temperature dependence, see Ref. [18].)

3.1 Relation to van der Waals force

If the central medium is tenuous, $\epsilon - 1 \ll 1$, and is surrounded by vacuum, for large distances $a \gg \lambda_c$, where λ_c is a characteristic wavelength of the medium, we can expand the above general formula and obtain a dispersion-free result:

$$f \approx -\frac{23(\epsilon - 1)^2}{640\pi^2 a^4}. \tag{38}$$

For this regime, this should be derivable from the sum of van der Waals forces, obtained from an intermolecular potential of the form

$$V = -\frac{B}{r^\gamma}. \tag{39}$$

We do this by computing the energy (N = density of molecules)

$$E = -\frac{1}{2}BN^2 \int_0^a dz \int_0^a dz' \int \frac{(d\mathbf{r}_\perp)(d\mathbf{r}'_\perp)}{[(\mathbf{r}_\perp - \mathbf{r}'_\perp)^2 + (z - z')^2]^{\gamma/2}}. \tag{40}$$

If we disregard the infinite self-interaction terms (analogous to dropping the volume energy terms in the Casimir calculation), we get

$$f = -\frac{\partial}{\partial a}\frac{E}{A} = -\frac{2\pi BN^2}{(2-\gamma)(3-\gamma)}\frac{1}{a^{\gamma-3}}. \tag{41}$$

So then, upon comparison with (38), we set $\gamma = 7$ and in terms of the polarizability,

$$\alpha = \frac{\epsilon - 1}{4\pi N}, \tag{42}$$

we find

$$B = \frac{23}{4\pi}\alpha^2, \tag{43}$$

or, equivalently, we recover the Casimir-Polder retarded dispersion potential [19],

$$V = -\frac{23}{4\pi}\frac{\alpha^2}{r^7}, \tag{44}$$

whereas for short distances ($a \ll \lambda_c$) we recover the London potential [20],

$$V = -\frac{3}{\pi}\frac{1}{r^6}\int_0^\infty d\zeta\, \alpha(\zeta)^2. \tag{45}$$

Given the divergences of the above calculation, and the essentially one-dimensional restriction, it is of interest to consider a tenuous dielectric sphere. The theory of the Casimir energy for a dielectric ball was first worked out by me 20 years ago [21]. The general expression is of course quite complicated

$$E = -\frac{1}{4\pi a}\int_{-\infty}^\infty dy\, e^{iy\delta}\sum_{l=1}^\infty (2l+1)x\frac{d}{dx}\ln S_l, \tag{46}$$

where

$$S_l = [s_l(x')e_l'(x) - s_l'(x')e_l(x)]^2 - \xi^2[s_l(x')e_l'(x) + s_l'(x')e_l(x)]^2, \tag{47}$$

where the s_l, e_l are spherical Bessel functions of imaginary argument, the quantity ξ is

$$\xi = \frac{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} - 1}{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} + 1}, \tag{48}$$

where ϵ', μ' represent the permittivity and permeability in the interior, the corresponding unprimed quantities refer to the exterior, the time-splitting regularization parameter is denoted by δ, and

$$x = |y|\sqrt{\epsilon\mu}, \quad x' = |y|\sqrt{\epsilon'\mu'}. \tag{49}$$

It is easy to check that this result reduces to that for a perfectly conducting spherical shell [13] if we set the speed of light inside and out the same, $\sqrt{\epsilon\mu} =$

$\sqrt{\epsilon'\mu'}$, as well as set $\xi = 1$. However, if the speed of light is different in the two regions, the result is no longer finite, but quartically divergent, and indeed the Schwinger result [22] follows for that leading divergent term.

Although, in general, this expression is not finite, there are several methods of isolating the divergences, at least if the ball is tenuous, ($\epsilon - 1 \ll 1$), and the finite repulsive observable Casimir energy is [23]

$$E_{\rm Cas} = \frac{23}{1536\pi a}(\epsilon - 1)^2. \tag{50}$$

What is most remarkable about this result is that it coincides with the van der Waals energy calculated two years earlier for this nontrivial geometry. That is, starting from the Casimir-Polder potential (44) we summed the pairwise potentials between molecules making up the media. A sensible way to regulate this calculation is dimensional continuation, similar to that described above. That is, we evaluate the integral

$$E_{\rm vdW} = -\frac{23}{8\pi}\alpha^2 N^2 \int d^D r\, d^D r' (r^2 + r'^2 - 2rr'\cos\theta)^{-\gamma/2}, \tag{51}$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$, by first regarding $D > \gamma$ so the integral exists. The integral may be done exactly in terms of gamma functions, which when continued to $D = 3$, $\gamma = 7$ yields Eq. (50) [24].

Thus there can hardly be any doubt that the Casimir effect, in the tenuous limit, coincides with the van der Waals attraction between molecules. This seems to go some way toward providing understanding of this zero-point fluctuation phenomenon. But the subject is not closed. Two years ago Romeo and I demonstrated that for a dilute cylinder the van der Waals energy is *zero* [25, 17]. Presumably the same holds for the Casimir energy, in order $(\epsilon - 1)^2$, but a demonstration of that is not yet at hand. (It turns out to be quite difficult to compute the Casimir effect for a dielectric cylinder.) Remarkably, for a dilute cylinder with constant speed of light inside and out, $\epsilon\mu =$ constant, it has been demonstrated that the Casimir energy vanishes, that is, the energy is of order ξ^4 [17, 26], but that would seem to be a completely different case.

4 "Dynamical Casimir effect" and relevance to sonoluminescence

Sonoluminescence refers to that remarkable phenomenon in which a small bubble of air injected into a container of water and suspended in a node of a strong acoustic standing wave emits light. More precisely, if it is driven with a standing wave of about 20,000 Hz at an overpressure of about 1 atm, the bubble expands and contracts in concert with the wave, from a maximum radius $\sim 10^{-3}$ cm to a minimum radius of $\sim 10^{-4}$ cm. Exactly at minimum radius roughly 1 million optical photons are emitted, for a total energy liberated of 10 MeV. For a review of the experimental situation as of a few years ago, see Ref. [27].

Julian Schwinger, informed of these experiments by Putterman, immediately assumed the effect derived from a dynamical version of the Casimir effect, and published in the last few years of his life a series of papers in the PNAS-USA attempting to account for the observations in this way [28]. Subsequently a number of other have jumped on this bandwagon [29, 30, 31].

The problem is that the "dynamical Casimir effect" remains largely unknown.[2] Schwinger and his followers had to rely on the known results for the Casimir effect with static boundary conditions. Two possible avenues then appeared:

- One could employ the adiabatic approximation, which does not seem unreasonable, since the time scale for emission of the photons, $\sim 10^{-11}$ s, is far longer than the characteristic optical time scale, $\sim 10^{-15}$ s. Schwinger [28], and Carlson et al. [30], then used the divergent bulk energy term

$$E_{\text{bulk}} = \frac{4\pi a^3}{3} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{1}{2} k \left(1 - \frac{1}{n}\right). \tag{52}$$

  If a reasonable cutoff is inserted, the required 10 MeV of energy is indeed present.

My objection to this is that it now seems unequivocal that the Casimir energy for a (dilute) dielectric ball is given by the finite expression (50). The divergent expression (52) is properly absorbed in a renormalization of the material properties. If Eq. (50) is used, the Casimir energy is *ten orders of magnitude too small to be relevant.*

- The second avenue, followed by Schwinger [28] and Liberati et al. [31], is to use the instantaneous or sudden approximation, in which the static bubble simply disappears. Then the photon production rate can be

[2]Of course, radiation from a moving mirror is well studied [32]. For cavities, little beyond perturbation theory, valid for "small but arbitrary dynamical changes," hardly relevant to the profound changes seen in sonoluminescence, is known. For recent references see Ref. [33].

calculated from the overlap of the two static configurations, through the Bogoliubov coefficients. Again, reasonable agreement with observations is reported.

My objection here is that impossibly short time scales are required. For the sudden approximation to make sense, the time scale τ of collapse must be small compared to 10^{-15} s. But in fact, the overall collapse time scale is 10^{-4} s, and the emission time scale is probably 10^{-11} s. An estimate of what one expects with reasonable numbers may be obtained from the Unruh temperature [34]

$$T = \frac{A}{2\pi}, \tag{53}$$

where A is the acceleration of the surface. Estimating A by a/τ^2, a being some relevant bubble radius, and recognizing that the observations are consistent with a photon temperature of order 20,000 K, we estimate the required time scale by

$$\tau^2 \sim \frac{a}{2\pi T}\frac{\hbar}{c} \sim \frac{10^{-4}\text{cm} \times 2 \times 10^{-5}\text{eV cm}}{(10 \text{ eV})(3 \times 10^{10}\text{cm/s})^2} \sim 10^{-31}\,\text{s}, \quad \tau \sim 10^{-15}\,\text{s}, \tag{54}$$

which once again seems physically unrealizable.[3]

5 Conclusions

A great many isolated facts about the Casimir effect are now known:

- The dimensional dependence for planes[4] and hyperspheres is now known.
- The equivalence of the van der Waals force and the Casimir force for dilute media is definitively established.
- Moreover, important applications of the Casimir effect to many fields, such as cosmology,[5] have been made.

However, we have very little understanding of the effects.

[3]The same conclusion is reached if the Larmor formula for the radiated power, $P = (2/3)(e^2/c^3)A^2$, the leading approximation in Ref. [33], is used. See also Ref. [35]

[4]Although results have been given for rectangular cavities, for example by [8], these results include only interior modes, and are thus suspect.

[5]A recent example is Ref. [10].

- We cannot predict, *a priori*, the sign of the effect.
- There are divergences occurring, for example with the dielectric ball, the nature of which is not well understood.
- Physically, why does the Casimir energy for a sphere in even dimensions diverge?
- Most importantly, can one make progress in understanding the dynamical Casimir effect?

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[1] H. B. G. Casimir, *Proc. K. Ned. Akad. Wet.* **51** (1948) 793.

[2] B. V. Deriagin and I. I. Abrikosova, *Zh. Eksp. Teor. Fiz.* **30** (1956) 993; **31** (1956) 3 [English transl.: *Soviet Phys. JETP* **3** (1957) 819; **4** (1957) 2]; B. V. Deriagin, I. Abrikosova, and E. M. Lifshitz, *Quart. Rev.* **10** (1968) 295; A. Kitchener and A. P. Prosser, *Proc. Roy. Soc. (London) A* **242** (1957) 403; M. Y. Sparnaay, *Physica* **24** (1958) 751; W. Black, J. G. V. de Jongh, J. T. G. Overbeck, and M. J. Sparnaay, *Trans. Faraday Soc.* **56** (1960) 1597; A. van Silfhout, *Proc. Kon. Ned. Akad. Wetensch. B* **69** (1966) 501; D. Tabor and R. H. S. Winterton, *Nature* **219** (1968) 1120; D. Tabor and R. H. S. Winterton, *Proc. Roy. Soc. (London) A* **312** (1969) 435; R. H. S. Winterton, *Contemp. Phys.* **11** (1970) 559; J. N. Israelachivili and D. Tabor, *Proc. Roy. Soc. (London) A* **331** (1972) 19.

[3] E. S. Sabisky and C. H. Anderson, *Phys. Rev. A* **7** (1973) 790.

[4] E. M. Lifshitz, *Zh. Eksp. Teor. Fiz.* **29** (1956) 94 [English transl.: *Soviet Phys. JETP* **2** (1956) 73]; I. D. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, *Usp. Fiz. Nauk* **73** (1961) 381 [English transl.: *Soviet Phys. Usp.* **4** (1961) 153].

[5] S. K. Lamoreaux, *Phys. Rev. Lett.* **78** (1997) 5; **81** (1998) 5475(E); *Phys. Rev. A* **59** (1999) R3149.

[6] U. Mohideen and A. Roy. *Phys. Rev. Lett.* **81** (1998) 4549; A. Roy, C.-Y. Lin, and U. Mohideen, *Phys. Rev. D* **60** (1999) R111101.

[7] M. Bordag, B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, in *The Casimir Effect 50 Years Later*, ed. M. Bordag (World Scientific, Singapore, 1999), p. 39.

[8] J. Ambjørn and S. Wolfram, *Ann. Phys. (N.Y.)* **147** (1983) 1.

[9] K. A. Johnson, *Acta Phys. Pol.* **B6** (1975) 865.

[10] I. Brevik, K. A. Milton, S. D. Odintsov, and K. E. Osetrin, *Phys. Rev. D* **62** (2000) 064005.

[11] C. M. Bender and K. A. Milton, *Phys. Rev. D* **50** (1994) 6547.

[12] S. Leseduarte and A. Romeo, *Europhys. Lett.* **34** (1996) 79; *Ann. Phys. (N.Y.)* **250** (1996) 448.

[13] T. H. Boyer, *Phys. Rev.* **174** (1968) 1764. Boyer's result has been confirmed by many authors, including B. Davies, *J. Math. Phys.* **13** (1972) 1324; R. Balian and B. Duplantier, *Ann. Phys. (N.Y.)* **112** (1978) 165; K. A. Milton, L. DeRaad, Jr., and J. Schwinger, *Ann. Phys. (N.Y.)* **115** (1978) 388.

[14] K. Johnson, unpublished; K. A. Milton, *Ann. Phys. (N.Y.)* **150** (1983) 432.

[15] L. L. DeRaad, Jr. and K. A. Milton, *Ann. Phys. (N.Y.)* **136** (1981) 229.

[16] P. Gosdzinsky and A. Romeo, *Phys. Lett. B* **441** (1998) 265.

[17] K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, *Phys. Rev. D* **59** (1999) 105009.

[18] K. A. Milton, in *Applied Field Theory*, ed. C. Lee, H. Min, and Q-H. Park (Chungbum, Seoul, 1999), p. 1, hep-th/9901011.

[19] H. B. G. Casimir and D. Polder, *Phys. Rev.* **73** (1948) 360.

[20] F. London, *Z. Phys.* **63** (1930) 245.

[21] K. A. Milton, *Ann. Phys. (N.Y.)* **127** (1980) 49.

[22] J. Schwinger, *Lett. Math. Phys.* **24** (1992) 59, 227; *Proc. Natl. Acad. Sci. USA* **89** (1992) 4091, 11118. For a history of Schwinger's developments, see K. A. Milton in *The Casimir Effect 50 Years Later*, ed. M. Bordag (World Scientific, Singapore, 1999), p. 20.

[23] I. Brevik, V. N. Marachevsky, and K. A. Milton, *Phys. Rev. Lett.* **82** (1999) 3948; G. Barton, *J. Phys. A* **32** (1999) 525; M. Bordag, K. Kirsten, and D. Vassilevich, *Phys. Rev. D* **59** (1999) 085011; J. S. Høye and I. Brevik, *J. Stat. Phys.* **100** (2000) 223.

[24] K. A. Milton and Y. J. Ng, *Phys. Rev. E* **57** (1998) 5504.

[25] A. Romeo, private communication.

[26] V. V. Nesterenko and I. G. Pirozhenko, *Phys. Rev. D* **60** (2000) 125007.

[27] B. P. Barber, R. A. Hillier, R. Löfstedt, S. J. Putterman, and K. Weniger, *Phys. Rep.* **281** (1997) 65.

[28] J. Schwinger, *Proc. Natl. Acad. Sci. USA* **90** (1993) 958, 2105, 4505, 7285; **91** (1994) 6473.

[29] C. Eberlein, *Phys. Rev. Lett.* **76** (1996) 3842.

[30] C. E. Carlson, C. Molina-París, J. Perez-Mercader, and M. Visser, *Phys. Lett.* **B395** (1997) 76; *Phys. Rev. D* **56** (1997) 1262; C. Molina-París and M. Visser, *Phys. Rev. D* **56** (1997) 6629.

[31] S. Liberati, M. Visser, F. Belgiorno, and D. W. Sciama, *Phys. Rev. Lett.* **83** (1999) 678; *Phys. Rev. D* **61** (2000) 085023; **61** (2000) 085024; S. Liberati, F. Belgiorno, M. Visser, and D. W. Sciama, *J. Phys. A* **33** (2000) 2251; F. Belgiorno, S. Liberati, M. Visser, and D. W. Sciama, Phys. Lett. A **271** (2000) 308.

[32] G. T. Moore, *J. Math. Phys.* **11** (1970) 2679; E. Sassaroli, Y. N. Srivastava, and A. Widom, *Phys. Rev. A* **50** (1994) 1027; S. A. Fulling and P. C. W. Davies, *Proc. R. Soc. London*, Ser. A **348** (1976) 393; L. H. Ford and A. Vilenkin, *Phys. Rev. A* **25** (1982) 2569; for example.

[33] R. Schützhold, G. Plunien, and G. Soff, *Phys. Rev. A* **57** (1998) 2311; **58** (1998) 1783; G. Plunien, R. Schützhold, and G. Soff, *Phys. Rev. Lett.* **84** (2000) 1882.

[34] P. C. W. Davies, *J. Phys. A* **8** (1975) 365; W. G. Unruh, *Phys. Rev. D* **14** (1976) 870; N. D. Birrell and P. C. W. Davies, *Quantum Field Theory in Curved Space* (Cambridge University Press, Cambridge, 1982); S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, 1989).

[35] K. A. Milton and Y. J. Ng, *Phys. Rev. E* **55** (1997) 4207.