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Non-Abelian Finite-Element Gauge Theory

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Abstract

We complete the formulation of the equations of motion of a non-Abelian gauge field coupled to fermions on a finite-element lattice in four space-time dimensions. This is accomplished by a straightforward iterative approach, in which successive interaction terms are added to the Dirac and Yang-Mills equations of motion, and to the field strength, in order to preserve lattice gauge invariance exactly, yielding a series in powers of ghA. Here g is the coupling constant, h is the lattice spacing, and A is the gauge potential. Gauge transformations of the potentials are determined simultaneously. The interaction terms in the equations of motion are nonlocal, and can be expressed either by an iterative formula or by a difference equation. On the other hand, the field strength is locally constructed from the potentials in terms of a path-ordered product of exponentials.

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I. INTRODUCTION

An intriguing approach to quantum field theories, called the finite element method, has been under development for the past decade. (For a recent review see [1].) This technique of putting the operator equations of motion on a lattice has a number of virtues:

- It is exactly unitary, or canonical, in the sense that the equal-time commutation relations are preserved at each lattice site.[1]
- It is the most accurate possible method of discretizing the continuum equations of motion: Typically, after N steps through the lattice, the relative error is only $O(1/N^2)$.
- Because it is an analytic technique, it is not subject to the statistical errors of Monte Carlo methods.
- It appears to resolve the fermion-doubling problem, while breaking chiral invariance with the expected axial-vector anomaly [3,4].
- Although a lattice Lagrangian does not exist in Minkowski space, it is possible to define a Hamiltonian in the sense of a time-evolution operator, which can be used to extract spectral and other information about the theory [5,6].

Very early in this development it was discovered how to formulate an Abelian gauge theory in the finite-element scheme[2] [7]. This was a nontrivial accomplishment, because,

[1]More precisely, unitarity has been proven for the following systems: quantum mechanics with one and several degrees of freedom, including fermionic; scalar field theory with an arbitrary self-interaction; and the fermionic sector of a gauge theory. As expected, anomalies are present in the gauge sector of a $U(1)$ theory, and consequently, although considerable work has been done, the unitarity of that sector remains unproven [2]. For further details see [1] and references therein.

[2]A rather different approach to the $U(1)$ gauge theory on a finite element lattice is given in [8], based on the introduction of link operators into the Dirac equation. Although in that approach

a priori, it is not obvious that the variational gauge-covariance equations are integrable. It has seemed to be much more difficult to formulate a non-Abelian theory. The essential development took place in the late 1980's [10]. There it was shown, in two space-time dimensions, how to write down the interacting Dirac equation and the Yang-Mills equations, where the interactions were determined simultaneously with the gauge transformations. The integrability of this process was again highly nontrivial. The only thing not completely determined at that time was the form of the construction of the field strength in terms of potentials. (This was given only out to fourth order in potentials.)

The generalization to four dimensions seemed straightforward. The form for the equations of motion was immediately generalizable, but only the first two terms in the field strength were given in 1990 [11]. Attention shifted to lattice QED [12].

I returned to this problem in January of this year. I computed the first three terms in the field strength in four dimensions, and then recognized the pattern in two and four dimensions. The construction is remarkably simple, and should have been recognized very early on: The field strength is simply given in terms of a path-ordered product of exponentials of the natural finite-element gauge-potential variable (link operators), with further link operators providing the necessary gauge transformations in the transverse directions. From the result, verification of gauge covariance is immediate.

The organization of this paper is very simple. In the next section we write down the free lattice equations of motion and explain how they are gauged. The full form of the Dirac

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the Dirac equation is local, an interaction is thereby introduced into the mass term, which seems somewhat unnatural. Our approach, on the other hand, is completely canonical and inductive, in the sense of gauging a local symmetry on the lattice. In particular, we *derive* the lattice gauge transformations, only after which can link operators be defined. (It would be nearly impossible to infer them *a priori*.) In any case, further investigation of Matsuyama's method will be carried out [9] to see if it can be fruitfully extended to the non-Abelian case.

equation, Yang-Mills equations, and the gauge transformation will be given. In Section III the results of the iterative calculation for the field strength through order g^2hA^3 are given, where g is the gauge coupling constant, h is the lattice spacing, and A is the vector potential. In Section IV we infer the exact form of the field strength, which is immediately verified to be gauge covariant. The continuing direction of this research program is sketched in the concluding Section V. The essential element of the proof of gauge covariance is given in the Appendix.

II. FINITE-ELEMENT EQUATIONS OF MOTION AND GAUGE INVARIANCE

The two-dimensional work has been described in detail [1,10], so we will begin immediately in four-dimensional Minkowski space. Let us start from the free Dirac equation in the continuum,

$$(i\gamma^\mu\partial_\mu + \mu)\psi = 0. \tag{2.1}$$

The linear finite-element prescription consists in replacing derivatives by forward differences, while in directions in which there is no differentiation, a forward average is employed. If we use the following notation for forward averaging,

$$x_{\overline{m}} = \frac{1}{2}(x_{m+1} + x_m), \tag{2.2}$$

the free finite-element lattice Dirac equation is

$$\frac{i\gamma^0}{h}(\psi_{\overline{\mathbf{m}},n+1} - \psi_{\overline{\mathbf{m}},n}) + \frac{i\gamma^j}{\Delta}(\psi_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}} - \psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}) + \mu\psi_{\overline{\mathbf{m}},\overline{n}} = 0, \tag{2.3}$$

where h is the temporal lattice spacing, Δ is the spatial lattice spacing, $\mathbf{m}$ represents a spatial lattice coordinate, n a temporal coordinate, and the $\perp$ indicates directions other than the one singled out.

Gauge transformations are introduced just as in the continuum. The Dirac equation (2.3) is invariant under the infinitesimal global phase transformation

$$\delta\psi_{\mathbf{m},n} = ig\delta\omega\psi_{\mathbf{m},n}. \tag{2.4}$$

However, if $\delta\omega$ depends on position, invariance is spoiled. Let us define a local transformation so that the mass term in (2.3) transforms covariantly:

$$\delta\psi_{\overline{\mathbf{m}},\overline{n}} = ig\delta\omega_{\mathbf{m},n}\psi_{\overline{\mathbf{m}},\overline{n}}. \tag{2.5}$$

The forward differences in (2.3) do not transform covariantly; to achieve covariance we must add interaction terms. The interaction is with a vector potential transforming by the lattice analog of

$$A_\mu \to A_\mu + \delta^{(0)}A_\mu, \quad \delta^{(0)}A_\mu = \partial_\mu\omega, \tag{2.6}$$

that is,

$$\delta^{(0)}(A_0)_{\mathrm{m}} = \frac{1}{h}(\delta\Lambda_{\overline{\mathbf{m}},n+1} - \delta\Lambda_{\overline{\mathbf{m}},n}), \quad \delta^{(0)}(A_j)_{\mathrm{m}} = \frac{1}{\Delta}(\delta\Lambda_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}} - \delta\Lambda_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}), \tag{2.7}$$

where m represents the four-vector coordinate $(\mathbf{m}, n)$ and the superscript (0) is a reminder that this is the first term in a series of variations. We will take the connection between $\delta\omega$ and $\delta\Lambda$ to be the finite-element one:

$$\delta\omega_{\mathrm{m}} = \delta\Lambda_{\overline{\mathrm{m}}}. \tag{2.8}$$

Then, using the boundary conditions (where M is the number of lattice sites in any spatial direction)

$$\psi_{m_j+M,\mathbf{m}_\perp,n} = (-1)^{M+1}\psi_{m_j,\mathbf{m}_\perp,n}, \quad \delta\psi_{m_j+M,\mathbf{m}_\perp,n} = (-1)^{M+1}\delta\psi_{m_j,\mathbf{m}_\perp,n}, \tag{2.9}$$

and the initial condition

$$\delta\psi_{\overline{\mathbf{m}},0} = \frac{ig}{2}(\delta\omega_{\mathbf{m},0} + \delta\omega_{\mathbf{m},-1})\psi_{\overline{\mathbf{m}},0}, \tag{2.10}$$

it is easy to see that the noncovariance of

$$\frac{i\gamma^0}{h}(\psi_{\overline{\mathbf{m}},n+1} - \psi_{\overline{\mathbf{m}},n}) + \frac{i\gamma^j}{\Delta}(\psi_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}} - \psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}) \tag{2.11}$$

will be cancelled if the following interaction terms are added to the left side of (2.3):

$$I^{(1)}_{\mathbf{m}} = \frac{i\gamma^j}{\Delta} ig\Delta \sum_{m'_j=m_j+1}^{m_j+M} (-1)^{m_j+m'_j} (A_j)_{\overline{m'_j-1},\mathbf{m}_\perp,n} \psi_{m'_j,\overline{\mathbf{m}}_\perp,\overline{n}} \tag{2.12}$$

and

$$K^{(1)}_{\mathbf{m}} = -2\frac{i\gamma^0}{h} igh {\sum_{n'=0}^{n}}' (-1)^{n+n'} (A_0)_{\mathbf{m},\overline{n'-1}} \psi_{\overline{\mathbf{m}},n'}, \tag{2.13}$$

where the prime on the summation sign indicates that the $n'=0$ term is counted with half weight. The variations to which the interaction terms are subjected are, from (2.7),

$$\delta^{(0)}(A_j)_{\overline{m_j-1},\mathbf{m}_\perp,n} = \frac{1}{\Delta}(\delta\omega_{m_j,\mathbf{m}_\perp,n} - \delta\omega_{m_j-1,\mathbf{m}_\perp,n}) \tag{2.14a}$$

$$\delta^{(0)}(A_0)_{\mathbf{m},\overline{n-1}} = \frac{1}{h}(\delta\omega_{\mathbf{m},n} - \delta\omega_{\mathbf{m},n-1}). \tag{2.14b}$$

It will be noted that the form of the interaction term with the scalar potential involves only the fields at the current and earlier times. This feature persists, and enables one to solve the equations by time-stepping through the lattice. It is easy to verify [7,10] that the Dirac equation with (2.12) and (2.13) included reduces to the correct continuum limit. [In this verification, the form of the initial condition (2.10) is crucial.]

However, we are not finished, because we have not varied the interaction terms (2.12) and (2.13) with respect to ψ, (2.5), nor achieved covariance of the interaction term. This can be accomplished by adding new interaction terms $I^{(2)}$ and $K^{(2)}$, and additional gauge transformations, denoted by $\delta^{(1)}$. A straightforward calculation reveals that

$$\delta_\psi I^{(1)}_{\mathbf{m}} = ig\delta\omega_{\mathbf{m}} I^{(1)}_{\mathbf{m}} - \delta^{(1)} I^{(1)}_{\mathbf{m}} - \delta^{(0)} I^{(2)}_{\mathbf{m}}. \tag{2.15}$$

Here the first term on the right is the required covariance term, the second term involves a new variation of $\mathbf{A}$,

$$\delta^{(1)}(A_j)_{\overline{m'_j-1},\mathbf{m}_\perp,n} = ig\frac{1}{2}[\delta\omega_{m_j,\mathbf{m}_\perp,n} + \delta\omega_{m_j-1,\mathbf{m}_\perp,n}, (A_j)_{\overline{m'_j-1},\mathbf{m}_\perp,n}], \tag{2.16}$$

and the third term is the $\delta^{(0)}$ variation of a new interaction term

$$I^{(2)}_{\mathbf{m}} = -\frac{i\gamma^j}{2\Delta}(ig\Delta)^2 \sum_{m'_j=m_j+1}^{m_j+M} \sum_{m''_j=m'_j+1}^{m'_j+M-1} (-1)^{m_j+m''_j}(A_j)_{\overline{m'_j-1},\mathbf{m}_\perp,n}(A_j)_{\overline{m''_j-1},\mathbf{m}_\perp,n}\psi_{m''_j,\overline{\mathbf{m}}_\perp,\overline{n}}. \quad (2.17)$$

Similarly, we find the new variation of A_0,

$$n \geq 1: \quad \delta^{(1)}(A_0)_{\mathbf{m},\overline{n-1}} = \frac{ig}{2}[\delta\omega_{\mathbf{m},n} + \delta\omega_{\mathbf{m},n-1},(A_0)_{\mathbf{m},\overline{n-1}}], \quad (2.18a)$$

$$n = 0: \quad \delta^{(1)}(A_0)_{\mathbf{m},\overline{-1}} = \frac{ig}{2}[\delta\omega_{\mathbf{m},0} + \delta\omega_{\mathbf{m},-1},(A_0)_{\mathbf{m},\overline{-1}}] + \frac{ig}{4}[\delta\omega_{\mathbf{m},0} - \delta\omega_{\mathbf{m},-1},(A_0)_{\mathbf{m},\overline{-1}}], \quad (2.18b)$$

and a new interaction term involving the scalar potential,

$$K^{(2)}_{\mathbf{m}} = -2\frac{i\gamma^0}{h}(igh)^2 \sum_{n'=0}^{n}{}' \sum_{n''=0}^{n'}{}''(-1)^{n+n'}(A_0)_{\mathbf{m},\overline{n'-1}}(A_0)_{\mathbf{m},\overline{n''-1}}\psi_{\overline{\mathbf{m}},n''}. \quad (2.19)$$

Here the double prime on the second summation sign means that both the first and last terms are counted with half weight, and that if the upper limit is zero, the sum vanishes.

Of course, the $\delta^{(1)}$ variations (2.16) and (2.18) precisely reduce to the usual non-Abelian transformations of a gauge field in the continuum, and the new interaction terms (2.17) and (2.19) vanish in the continuum limit.

This iterative procedure continues indefinitely. Fortunately, it exponentiates. Perhaps the easiest way to express $I^{(N)}$, where N is the order in $g\Delta A$, is by an inductive formula. Let us adopt a shorthand notation for ease of writing the results. Let $I = \gamma^j I_j$, and because I_j depends locally on all coordinates except m_j, let us suppress those other coordinates, call $m_j = m$, and simply write I_m to stand for $(I_j)_{m_j,\mathbf{m}_\perp,n}$. Then we find

$$N \geq 1: \quad I^{(N)}_m = \frac{1}{2}\sum_{m'=m+1}^{m+M}(-1)^{m+m'}\sum_{k=1}^{N}\frac{1}{k!}(-ig\Delta\mathcal{A}_{m'})^k I^{(N-k)}_{m'}, \quad (2.20)$$

where

$$\mathcal{A}_m = (A_j)_{\overline{m_j-1},\mathbf{m}_\perp,n} \quad (2.21)$$

and we define $I^{(0)}_m = -2i\psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}/\Delta$. The gauge transformations of the potentials are given by nested commutators:

$$k \neq 1: \quad \delta^{(k)}\mathcal{A}_m = \frac{(ig\Delta)^k}{\Delta}\frac{\mathcal{B}_k}{k!}\underbrace{[\cdots[\delta\omega_m - \delta\omega_{m-1}, \mathcal{A}_m], \cdots, \mathcal{A}_m]}_{k \text{ nested commutators}}, \tag{2.22a}$$

$$\delta^{(1)}\mathcal{A}_m = \frac{ig}{2}[\delta\omega_m + \delta\omega_{m-1}, \mathcal{A}_m], \tag{2.22b}$$

where $\mathcal{B}_k$ is the kth Bernoulli number. The required covariance statement

$$\delta_\psi I_m^{(N)} + \sum_{k=0}^{N} \delta^{(k)} I_m^{(N-k+1)} = ig\delta\omega_m I_m^{(N)} \tag{2.23}$$

is proved in the Appendix of [10]. By summing (2.20) over all N we derive the sum equation satisfied by the full interaction term involving the vector potential, $I = \sum_{N=1}^{\infty} I^{(N)}$:

$$I_m = \frac{1}{2}\sum_{m'=m+1}^{m+M}(-1)^{m+m'}\left(e^{-ig\Delta\mathcal{A}_{m'}} - 1\right)\left(I_{m'} + I_{m'}^{(0)}\right), \tag{2.24}$$

from which a difference equation can be immediately derived:

$$I_m + e^{ig\Delta\mathcal{A}_m} I_{m-1} = \frac{2i}{\Delta}\left(1 - e^{ig\Delta\mathcal{A}_m}\right)\psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}. \tag{2.25}$$

Of course, I transforms in the required manner ($\delta_A = \sum_{k=0}^{\infty}\delta^{(k)}$)

$$\delta_\psi I_m + \delta_A I_m - \delta^{(0)} I_m^{(1)} = ig\delta\omega_m I_m \tag{2.26}$$

where the third variation on the left-hand side is just that necessary to cancel the noncovariance of the free Dirac equation (2.3).

In just the same way we can derive analogous expressions for the full interactions and transformations of the scalar potential. Again we suppress all but the time coordinate n, and write the interaction term as $K \to \gamma^0 K$ in what follows. The only complication has to do with the initial conditions. The order N interaction term is given by

$$K_n^{(N)} = -\sum_{n'=1}^{n}(-1)^{n+n'}\sum_{k=1}^{N}\frac{(-igh)^k}{k!}\mathcal{A}_{n'}^k K_{n'}^{(N-k)} + \frac{(igh)^N}{2^N N!}(-1)^n \mathcal{A}_0^N K_0^{(0)}, \tag{2.27}$$

where $K_n^{(0)} = -2i\psi_{\overline{\mathbf{m}},n}/h$. The required transformation law

$$\delta_\psi K_n^{(N)} + \sum_{k=0}^{N}\delta^{(k)} K_n^{(N-k+1)} = ig\delta\omega_n K_n^{(N)} \tag{2.28}$$

is satisfied provided the transformations are given by

$$k \neq 1: \quad \delta^{(k)}\mathcal{A}_n = \frac{(igh)^k}{h}\frac{\mathcal{B}_k}{k!}\underbrace{[\cdots[\delta\omega_n - \delta\omega_{n-1}, \mathcal{A}_n], \cdots, \mathcal{A}_n]}_{k \text{ nested commutators}}, \quad n \neq 0, \tag{2.29a}$$

$$\delta^{(1)}\mathcal{A}_n = \frac{ig}{2}[\delta\omega_n + \delta\omega_{n-1}, \mathcal{A}_n], \quad n \neq 0, \tag{2.29b}$$

$$k \neq 1: \quad \delta^{(k)}\mathcal{A}_0 = \frac{(-igh)^k}{h}\frac{\mathcal{B}_k}{2^k k!}\underbrace{[\cdots[\delta\omega_0 - \delta\omega_{-1}, \mathcal{A}_0], \cdots, \mathcal{A}_0]}_{k \text{ nested commutators}}, \tag{2.29c}$$

$$\delta^{(1)}\mathcal{A}_0 = \frac{ig}{2}[\delta\omega_0 + \delta\omega_{-1}, \mathcal{A}_0] - ig\frac{\mathcal{B}_1}{2}[\delta\omega_0 - \delta\omega_{-1}, \mathcal{A}_0]. \tag{2.29d}$$

A sum and difference equation may be immediately derived for the full interaction term involving the scalar potential $K_n = \sum_{N=1}^{\infty} K_n^{(N)}$:

$$K_n = -\sum_{n'=1}^{n}(-1)^{n+n'}\left(e^{-igh\mathcal{A}_{n'}} - 1\right)\left(K_{n'} + K_{n'}^{(0)}\right) + (-1)^n\left(e^{igh\mathcal{A}_0/2} - 1\right)K_0^{(0)}, \tag{2.30}$$

and

$$K_n + e^{igh\mathcal{A}_n}K_{n-1} = \frac{2i}{h}\left(1 - e^{igh\mathcal{A}_n}\right)\psi_{\overline{\mathbf{m}},n}. \tag{2.31}$$

The full lattice Dirac equation is given by (2.3), (2.20), and (2.27):

$$\gamma^0\left[\frac{i}{h}(\psi_{\overline{\mathbf{m}},n+1} - \psi_{\overline{\mathbf{m}},n}) + K_{\mathbf{m},n}\right] + \gamma^j\left[\frac{i}{\Delta}(\psi_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}} - \psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}) + (I_j)_{\mathbf{m},n}\right] + \mu\psi_{\overline{\mathbf{m}},\overline{n}} = 0. \tag{2.32}$$

We have already emphasized that this equation gives $\psi_{\mathbf{m},n+1}$ in terms of fields at time n and earlier, so that this difference equation may be solved by time stepping through the lattice.

We have given so much detail on the Dirac equation because this is the hard part of the calculation. The crucial point is that we have now determined not merely the interaction terms in the Dirac equation, but the precise form of the gauge transformations on the lattice. We must now use that information to construct the gauge sector of the theory. (If one starts directly with the gauge sector, it is much less clear how to proceed.)

Let us assume that we can construct a gauge-covariant field strength from the potentials. (That construction will be given in the next two sections.) That is, we have a field strength defined on the lattice which transforms under an infinitesimal gauge transformation according to

$$\delta(F_{\mu\nu})_{\overline{\mathbf{m}}} = ig[\delta\omega_{\mathbf{m}}, (F_{\mu\nu})_{\overline{\mathbf{m}}}]. \tag{2.33}$$

(This is precisely analogous to (2.5).) This is a difference equation for the transformations of the field strength at the lattice sites. Because we require boson fields to be periodic, we must henceforward assume that M is odd. (Recall the periodicity condition (2.9).) For the inital condition we adopt the analog of (2.10),

$$\delta(F_{0i})_{\overline{\mathbf{m}},0} = \frac{ig}{2}[\delta\omega_{\mathbf{m},0} + \delta\omega_{\mathbf{m},-1}, (F_{0i})_{\overline{\mathbf{m}},0}]. \tag{2.34}$$

Our starting point for describing gauge fields coupled to a fermionic current is the "free" Yang-Mills equation

$$\frac{1}{h}\sum_{\nu\neq\mu}\left[(F^{\mu\nu})_{m_\nu+1,\overline{m}_\mu,\overline{\mathbf{m}}_\perp} - (F^{\mu\nu})_{m_\nu,\overline{m}_\mu,\overline{\mathbf{m}}_\perp}\right] = (j^\mu)_{\mathbf{m}}. \tag{2.35}$$

Here, for ease of notation, we have not destinguished between the space and time lattice: h or Δ should be used as appropriate. A gauge covariant candidate for the fermionic current is (T is the generator of the gauge group)

$$j^\mu_{\mathbf{m}} = g\overline{\psi}_{\overline{\mathbf{m}}}T\gamma^\mu\psi_{\overline{\mathbf{m}}}, \tag{2.36}$$

because, then, according to (2.5), the right side of (2.35) transforms covariantly:

$$\delta j^\mu_{\mathbf{m}} = ig[\delta\omega_{\mathbf{m}}, j^\mu_{\mathbf{m}}]. \tag{2.37}$$

It is the difference in (2.35) that spoils gauge covariance. But now it is immediately obvious that the previous analysis for the Dirac equation applies. Multiplication simply gets replaced by commutation. We simply must add to the left side of (2.35) the interaction term

$$\mathcal{I}^\mu = \sum_{\nu\neq\mu}\mathcal{K}^\nu, \quad \mathcal{K}^\nu_{\mathbf{m}} = \mathcal{K}_n \tag{2.38}$$

(that is, in $\mathcal{K}^\nu$ the only nonlocal dependence is in $m_\nu = n$), where $\mathcal{K}^\nu$ satisfies the difference equation

$$\mathcal{K}_n e^{igh\mathcal{A}_n} + e^{igh\mathcal{A}_n}\mathcal{K}_{n-1} = -\frac{2}{h}\left[e^{igh\mathcal{A}_n}, (F^{\mu\nu})_{m_\nu,\overline{\mathrm{m}}_\perp}\right]. \tag{2.39}$$

The solution to this difference equation can be immediately written down by comparison with the corresponding terms in the Dirac equation (multiplication being replaced by commutation) and is given explicitly in [1,10]. In particular, the inductive formulæ are

$$\mathcal{K}_m^{(N)} = \frac{1}{2}\sum_{m'=m+1}^{m+M}(-1)^{m+m'}\sum_{k=1}^{N}\frac{(-ig\Delta)^k}{k!}\underbrace{[\mathcal{A}_{m'},[\mathcal{A}_{m'},[\cdots,[\mathcal{A}_{m'},\mathcal{K}_{m'}^{(N-k)}]\cdots]]]}_{k\text{ nested commutators}}, \tag{2.40}$$

for the $\nu = j$ interaction, and

$$\begin{aligned}\mathcal{K}_n^{(N)} = &-\sum_{n'=1}^{n}(-1)^{n+n'}\sum_{k=1}^{N}\frac{(-igh)^k}{k!}\underbrace{[\mathcal{A}_{n'},[\mathcal{A}_{n'},[\cdots,[\mathcal{A}_{n'},\mathcal{K}_{n'}^{(N-k)}]\cdots]]]}_{k\text{ nested commutators}}\\ &+\frac{(igh)^N}{2^N N!}(-1)^n\underbrace{[\mathcal{A}_0,[\mathcal{A}_0,[\cdots,[\mathcal{A}_0,\mathcal{K}_0^{(0)}]\cdots]]]}_{N\text{ nested commutators}},\end{aligned} \tag{2.41}$$

for the $\nu = 0$ interaction.

III. ITERATIVE CONSTRUCTION OF THE FIELD STRENGTH TENSOR

The finite-element prescription for the construction of the field strength in terms of potentials, in lowest order, is simply the lattice curl:

$$(F_{\mu\nu}^{(0)})_{\overline{\mathrm{m}}} = \frac{1}{h}[(A_\nu)_{m_\mu+1,\overline{\mathrm{m}}_\perp} - (A_\nu)_{m_\mu,\overline{\mathrm{m}}_\perp}] - \frac{1}{h}[(A_\mu)_{m_\nu+1,\overline{\mathrm{m}}_\perp} - (A_\mu)_{m_\nu,\overline{\mathrm{m}}_\perp}]. \tag{3.1}$$

(In this section, for simplicity, we set $h = \Delta$; the distinction can be readily recovered if desired.) Of course, this is invariant under the transformation $\delta^{(0)}$, (2.14). Next, we apply the transformation $\delta^{(1)}$, (2.16) and (2.18); remarkably enough, we are able to write the result in the form

$$\delta^{(1)}(F_{\mu\nu}^{(0)})_{\overline{\mathrm{m}}} = ig[\delta\omega_{\mathrm{m}}, (F_{\mu\nu}^{(0)})_{\overline{\mathrm{m}}}] - \delta^{(0)}(F_{\mu\nu}^{(1)})_{\overline{\mathrm{m}}}, \tag{3.2}$$

that is, the difference from the required covariance (2.33) is a total variation of a new interaction term. To present this interaction most simply, let us introduce some further notation. We use the averaged potential (2.21), which we write here as

$$\mathcal{A}^\nu_{klmn} = \frac{1}{2}\left(A^\nu_{klmn} + A^\nu_{klmn-1}\right), \tag{3.3}$$

where henceforward we order the coordinate indices κ, λ, μ, ν. (Here $n = m_\nu$ is not necessarily the time coordinate.) In terms of this, we define a "field strength" variable local in the transverse coordinates (κ, λ)

$$(\mathcal{F}^{(0)}_{\mu\nu})_{klmn} = \frac{1}{h}\left[(\mathcal{A}_\nu)_{klm+1n+1} - (\mathcal{A}_\nu)_{klmn+1} - (\mathcal{A}_\mu)_{klm+1n+1} + (\mathcal{A}_\mu)_{klm+1n}\right]. \tag{3.4}$$

Under the $\delta^{(0)}$ variation (2.14),

$$\delta^{(0)}(\mathcal{A}_\nu)_{klmn} = \frac{1}{h}(\delta\omega_{klmn} - \delta\omega_{klmn-1}), \tag{3.5}$$

$\mathcal{F}^{(0)}_{\mu\nu}$ is invariant.

In terms of these new variables, the lowest-order field strength is obtained simply by averaging in the transverse coordinates:

$$(F^{(0)}_{\mu\nu})_{\overline{\mathrm{m}}} = (\mathcal{F}^{(0)}_{\mu\nu})_{\overline{kl}mn}. \tag{3.6}$$

Because the first-order variation is given by (2.16) and (2.18) (except for $\mathcal{A}_0$ at the initial time),

$$\delta^{(1)}(\mathcal{A}_\nu)_{klmn} = \frac{ig}{2}[\delta\omega_{klmn} + \delta\omega_{klmn-1}, (\mathcal{A}_\nu)_{klmn}], \tag{3.7}$$

the interaction term found in (3.2) can now be written as

$$\begin{aligned}(F^{(1)}_{\mu\nu})_{\overline{\mathrm{m}}} = (\mathcal{F}^{(1)}_{\mu\nu})_{\overline{kl}mn} - \frac{igh}{4}\{&[(\mathcal{A}_\kappa)_{k+1lmn}, (\mathcal{F}^{(0)}_{\mu\nu})_{k+1lmn}] + [(\mathcal{A}_\lambda)_{kl+1mn}, (\mathcal{F}^{(0)}_{\mu\nu})_{kl+1mn}] \\ &+ \frac{1}{2}[(\mathcal{A}_\kappa)_{k+1l+1mn}, (\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}] + \frac{1}{2}[(\mathcal{A}_\lambda)_{k+1l+1mn}, (\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}] \\ &+ \frac{1}{2}[(\mathcal{A}_\kappa)_{k+1lmn}, (\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}] + \frac{1}{2}[(\mathcal{A}_\lambda)_{kl+1mn}, (\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]\},\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}(\mathcal{F}^{(1)}_{\mu\nu})_{klmn} = \frac{ig}{2}\{&[(\mathcal{A}_\mu)_{klm+1n}, (\mathcal{A}_\mu)_{klm+1n+1}] - [(\mathcal{A}_\nu)_{klmn+1}, (\mathcal{A}_\nu)_{klm+1n+1}] \\ &- [(\mathcal{A}_\mu)_{klm+1n+1}, (\mathcal{A}_\nu)_{klm+1n+1}] + [(\mathcal{A}_\mu)_{klm+1n}, (\mathcal{A}_\nu)_{klmn+1}] \\ &- [(\mathcal{A}_\mu)_{klm+1n}, (\mathcal{A}_\nu)_{klm+1n+1}] - [(\mathcal{A}_\mu)_{klm+1n+1}, (\mathcal{A}_\nu)_{klmn+1}]\}.\end{aligned} \tag{3.9}$$

It will be noted that (3.9) is exactly the form of the first-order field strength found in two dimensions [10].

We continue, by finding the second-order field strength from the covariance equation

$$\delta^{(2)}(F^{(0)}_{\mu\nu})_{\overline{\mathrm{m}}}+\delta^{(1)}(F^{(1)}_{\mu\nu})_{\overline{\mathrm{m}}}+\delta^{(0)}(F^{(2)}_{\mu\nu})_{\overline{\mathrm{m}}}=ig[\delta\omega_{\mathrm{m}},(F^{(1)}_{\mu\nu})_{\overline{\mathrm{m}}}]. \tag{3.10}$$

Here $(n>0)$

$$\delta^{(2)}(\mathcal{A}_\nu)_{klmn}=-\frac{g^2h}{12}[[\delta\omega_{klmn}-\delta\omega_{klmn-1},(\mathcal{A}_\nu)_{klmn}],(\mathcal{A}_\nu)_{klmn}], \tag{3.11}$$

from (2.22) and (2.29). The result of a long calculation is remarkably simple:

$$\begin{aligned}(F^{(2)}_{\mu\nu})_{\overline{\mathrm{m}}}=(\mathcal{F}^{(2)}_{\mu\nu})_{\overline{k}lmn}&-\frac{igh}{4}\{[(\mathcal{A}_\kappa)_{k+1lmn},(\mathcal{F}^{(1)}_{\mu\nu})_{k+1lmn}]+[(\mathcal{A}_\lambda)_{kl+1mn},(\mathcal{F}^{(1)}_{\mu\nu})_{kl+1mn}]\\
&+\frac12[(\mathcal{A}_\kappa)_{k+1l+1mn},(\mathcal{F}^{(1)}_{\mu\nu})_{k+1l+1mn}]+\frac12[(\mathcal{A}_\lambda)_{k+1l+1mn},(\mathcal{F}^{(1)}_{\mu\nu})_{k+1l+1mn}]\\
&+\frac12[(\mathcal{A}_\kappa)_{k+1lmn},(\mathcal{F}^{(1)}_{\mu\nu})_{k+1l+1mn}]+\frac12[(\mathcal{A}_\lambda)_{kl+1mn},(\mathcal{F}^{(1)}_{\mu\nu})_{k+1l+1mn}]\}\\
-\frac{g^2h^2}{8}&\{[(\mathcal{A}_\lambda)_{kl+1mn},[(\mathcal{A}_\lambda)_{kl+1mn},(\mathcal{F}^{(0)}_{\mu\nu})_{kl+1mn}]]\\
&+[(\mathcal{A}_\kappa)_{k+1lmn},[(\mathcal{A}_\kappa)_{k+1lmn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1lmn}]]\\
&+\frac12[(\mathcal{A}_\kappa)_{k+1l+1mn},[(\mathcal{A}_\kappa)_{k+1l+1mn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]]\\
&+\frac12[(\mathcal{A}_\lambda)_{k+1l+1mn},[(\mathcal{A}_\lambda)_{k+1l+1mn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]]\\
&+\frac12[(\mathcal{A}_\kappa)_{k+1lmn},[(\mathcal{A}_\kappa)_{k+1lmn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]]\\
&+\frac12[(\mathcal{A}_\lambda)_{kl+1mn},[(\mathcal{A}_\lambda)_{kl+1mn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]]\\
&+[(\mathcal{A}_\lambda)_{kl+1mn},[(\mathcal{A}_\kappa)_{k+1l+1mn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]]\\
&+[(\mathcal{A}_\kappa)_{k+1lmn},[(\mathcal{A}_\lambda)_{k+1l+1mn},(\mathcal{F}^{(0)}_{\mu\nu})_{k+1l+1mn}]]\},\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}(\mathcal{F}^{(2)}_{\mu\nu})_{klmn}=-\frac{g^2h}{12}\{&[B_1,[B_1,C_1]]-[C_1,[C_1,B_1]]-[B_1,[B_1,C_0]]+[C_1,[C_1,B_0]]\\
&+2[B_1,[B_0,C_1]]-2[C_1,[C_0,B_1]]+2[B_0,[B_1,C_1]]-2[C_0,[C_1,B_1]]\\
&-2[B_1,[B_0,C_0]]+2[C_1,[C_0,B_0]]+4[B_0,[B_1,C_0]]-4[C_0,[C_1,B_0]]\\
&+[B_0,[B_0,C_1]]-[C_0,[C_0,B_1]]-[B_0,[B_0,C_0]]+[C_0,[C_0,B_0]]\\
&-[B_0,[B_0,B_1]]+[C_0,[C_0,C_1]]-[B_1,[B_0,B_1]]+[C_1,[C_0,C_1]]\}.\end{aligned} \tag{3.13}$$

Here we have adopted one final bit of condensed notation:

$$B_1 = (\mathcal{A}_\mu)_{klm+1n+1}, \quad B_0 = (\mathcal{A}_\mu)_{klm+1n}, \quad C_1 = (\mathcal{A}_\nu)_{klm+1n+1}, \quad C_0 = (\mathcal{A}_\nu)_{klmn+1}. \quad (3.14)$$

One additional explicit term in $\mathcal{F}$, for $\mathcal{F}^{(3)}_{\mu\nu}$, was given in [10].

It is quite clear that this process may be continued indefinitely. We will now see that these terms may be summed exactly.

IV. EXACT FIELD STRENGTH

At first glance, the pattern for the construction of the $\mathcal{F}_{\mu\nu} = \sum_{N=1}^{\infty} \mathcal{F}^{(N)}_{\mu\nu}$ seems a bit mysterious. However, upon a bit of inspection, the terms given here and in [10] are entirely reproduced by the expansion in h of the following path-ordered product of exponentials (link variables) around the finite element (plaquette):

$$e^{-ih^2 g(\mathcal{F}_{\mu\nu})_{klmn}} = Pe^{-ig\oint A\cdot dl} = e^{-ihgB_0} e^{-ihgC_1} e^{ihgB_1} e^{ihgC_0}. \quad (4.1)$$

This is precisely the form conventionally used in lattice gauge theory (implicitly in Wilson's original paper [13], and explicitly, for example, in [14]). Of course, here, the appropriately averaged finite-element potentials are employed.[3] That this is exactly correct emerges immediately when we recognize the effect of the gauge transformations on the link operators ($\delta = \sum_{N=0}^{\infty} \delta^{(N)}$):

$$\delta e^{-igh(\mathcal{A}_\nu)_{klmn+1}} = ig \left\{ \delta\omega_{klmn} e^{-igh(\mathcal{A}_\nu)_{klmn+1}} - e^{-igh(\mathcal{A}_\nu)_{klmn+1}} \delta\omega_{klmn+1} \right\}, \quad (4.2)$$

and the corresponding adjoint equation. The proof is given in the Appendix. It then follows immediately from (4.1) that

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[3]That this is nontrivial can be seen from the remarks in [14]: "This identification of $A_\mu(n)$ with the vector potential is only strictly correct in the continuum limit," and that only as the lattice spacing goes to zero do the link variables transform equivalently to (4.2).

$$\delta(\mathcal{F}_{\mu\nu})_{klmn} = ig[\delta\omega_{klmn}, (\mathcal{F}_{\mu\nu})_{klmn}]. \tag{4.3}$$

Now, we immediately recognize the structure of the full field strength $F_{\mu\nu} = \sum_{N=0}^{\infty} F_{\mu\nu}^{(N)}$, the first three terms of which are given in (3.6), (3.8), and (3.12):

$$\begin{aligned}(F_{\mu\nu})_{\overline{klmn}} = \frac{1}{4}\Big\{&(\mathcal{F}_{\mu\nu})_{klmn} + e^{-igh(\mathcal{A}_\lambda)_{kl+1mn}}(\mathcal{F}_{\mu\nu})_{kl+1mn}e^{igh(\mathcal{A}_\lambda)_{kl+1mn}} \\ &+ e^{-igh(\mathcal{A}_\kappa)_{k+1lmn}}(\mathcal{F}_{\mu\nu})_{k+1lmn}e^{igh(\mathcal{A}_\kappa)_{k+1lmn}} \\ &+ \frac{1}{2}e^{-igh(\mathcal{A}_\lambda)_{kl+1mn}}e^{-igh(\mathcal{A}_\kappa)_{k+1l+1mn}}(\mathcal{F}_{\mu\nu})_{k+1l+1mn}e^{igh(\mathcal{A}_\kappa)_{k+1l+1mn}}e^{igh(\mathcal{A}_\lambda)_{kl+1mn}} \\ &+ \frac{1}{2}e^{-igh(\mathcal{A}_\kappa)_{k+1lmn}}e^{-igh(\mathcal{A}_\lambda)_{k+1l+1mn}}(\mathcal{F}_{\mu\nu})_{k+1l+1mn}e^{igh(\mathcal{A}_\lambda)_{k+1l+1mn}}e^{igh(\mathcal{A}_\kappa)_{k+1lmn}}\Big\}.\end{aligned} \tag{4.4}$$

That $F_{\mu\nu}$ transforms covariantly according to (2.33) follows immediately from (4.2). That is, the field strength is the average of the "transversely-local" field strength $\mathcal{F}_{\mu\nu}$ over the four corners of the transverse finite element, with gauge factors inserted appropriately to gauge-transform the field strength back to the origin ($klmn$).

Finally, we note that at the initial time $F_{\mu\nu}$ involves $(\mathcal{A}_0)_{\mathbf{m},1}$ and not $(\mathcal{A}_0)_{\mathbf{m},0}$, so that the initial variations (2.29c) and (2.29d) never come into play.

V. CONCLUSIONS

We have thus finished the task of giving a complete set of gauge-invariant equations of motion for Yang-Mills fields coupled to fermions. This should enable us to apply finite-element methods of solution which have so far been applied to scalar field theories [15] and to electrodynamics [12,4]. New methods are being developed as well [6]. It is therefore reasonable to hope that we may be able to use this technique to shed new light on difficult problems in non-Abelian theories such as QCD.

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APPENDIX: PROOF OF GAUGE TRANSFORMATIONS OF LINK OPERATORS

The proof of (4.2) follows essentially from the proof of covariance given in the Appendix of [10]. We have, for example, from (2.22) (we are using the notation (3.14) and only display the values of the μ, ν indices) ($n > 0$)

$$\delta^{(k)} e^{ighB_0} = \sum_{N=0}^{\infty} \frac{(igh)^N}{N!} \sum_{l=0}^{N-1} B_0^l \frac{(igh)^k}{h} \frac{\mathcal{B}_k}{k!} \underbrace{[\cdots[\delta\omega_{m+1n} - \delta\omega_{mn}, B_0], \cdots, B_0]}_{k \text{ nested commutators}} B_0^{N-l-1}, \quad k \neq 1, \tag{A1}$$

and

$$\delta^{(1)} e^{ighB_0} = \sum_{N=0}^{\infty} \frac{(igh)^N}{N!} \sum_{l=0}^{N-1} B_0^l \frac{ig}{2} [\delta\omega_{m+1n} + \delta\omega_{mn}, B_0] B_0^{N-l-1}, \tag{A2}$$

so the total variation of the link variable is

$$\begin{aligned} \delta e^{ighB_0} = \sum_{k=0}^{\infty} \delta^{(k)} e^{ighB_0} &= ig[\delta\omega_{mn}, e^{ighB_0}] \\ &+ \frac{1}{h} \sum_{k=0}^{\infty} (-1)^k \sum_{N=1}^{\infty} \sum_{l=0}^{N-1} B_0^l \frac{(igh)^{N+k}}{N!} \frac{\mathcal{B}_k}{k!} \underbrace{[\cdots[\delta\omega_{m+1n} - \delta\omega_{mn}, B_0], \cdots, B_0]}_{k \text{ nested commutators}} B_0^{N-l-1} \end{aligned} \tag{A3}$$

The nested commutator is simply given by

$$\underbrace{[\cdots[\delta\omega, B], \cdots, B]}_{k \text{ nested commutators}} = \sum_{s=0}^{k} (-1)^s \begin{pmatrix} k \\ s \end{pmatrix} B^s \delta\omega B^{k-s}. \tag{A4}$$

Then the sum on the right hand side of (A3) reduces to

$$\frac{1}{h}\sum_{r=0}^{\infty}\sum_{p=0}^{r}(igh)^{r+1}B_0^p(\delta\omega_{m+1n}-\delta\omega_{mn})B_0^{r-p}$$
$$\times\sum_{k=0}^{r}(-1)^k\frac{\mathcal{B}_k}{k!}\frac{1}{(r+1-k)!}\sum_{s=p-r+k}^{p}(-1)^s\begin{pmatrix}k\\ s\end{pmatrix}=ig(\delta\omega_{m+1n}-\delta\omega_{mn})e^{ighB_0}, \tag{A5}$$

where the last two summations over s and k are carried out using identity (A3) of [10]. Thus we have established the desired result:

$$\delta e^{ighB_0}=ig(\delta\omega_{m+1n}e^{ighB_0}-e^{ighB_0}\delta\omega_{mn}), \tag{A6}$$

which is the adjoint of (4.2).

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