Exact Casimir Interaction Between Semitransparent Spheres and Cylinders

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A multiple scattering formulation is used to calculate the force, arising from fluctuating scalar fields, between distinct bodies described by δ-function potentials, so-called semitransparent bodies. (In the limit of strong coupling, a semitransparent boundary becomes a Dirichlet one.) We obtain expressions for the Casimir energies between disjoint parallel semitransparent cylinders and between disjoint semitransparent spheres. In the limit of weak coupling, we derive power series expansions for the energy, which can be exactly summed, so that explicit, very simple, closed-form expressions are obtained in both cases. The proximity force theorem holds when the objects are almost touching, but is subject to large corrections as the bodies are moved further apart.

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Multiple scattering methods for calculating Casimir (quantum vacuum) energies between bodies date back to the famous papers of Balian and Duplantier [1–3]. More recently, Emig and collaborators [4, 5] have published a series of papers, using closely related methods, to calculate numerically forces between distinct bodies, starting from periodically deformed ones [6]. The methods are developed further in papers by Bulgac, Marierski, and Wirzba [7], who obtain results for the interaction of two spheres, or a sphere and a plate (for Dirichlet boundary conditions), and by Bordag [8, 9], who has precisely quantified the first correction to the proximity force approximation (PFA) both for a cylinder and a sphere near a plane. Dalvit et al. [10, 11] use the argument principle to calculate the interaction between conducting cylinders with parallel axes when one cylinder is inside the other. Recently, there appeared papers concerning "exact" methods of calculating Casimir energies or forces between arbitrary distinct bodies by Emig, Graham, Jaffe, and Kardar [12, 13]. Most explicitly, an earlier drafted paper by Kenneth and Klich [14] appeared which shows that the basis of the latter approach lies in the Lippmann-Schwinger formulation of scattering theory [15].

We will now proceed to restate the multiple scattering technique, in a simple, straightforward way, and apply it to various situations, all characterized by δ-function potentials.

The general formula for the Casimir energy (for simplicity here we restrict attention to a massless scalar field) is [16]

$$E = \frac{i}{2\tau}\mathrm{Tr}\ln G \to \frac{i}{2\tau}\mathrm{Tr}\ln GG_0^{-1}, \tag{1}$$

where τ is the "infinite" time that the configuration exists, and G is the Green's function in the presence of a potential V satisfying (matrix notation)

$$(-\partial^2 + V)G = 1, \tag{2}$$

subject to some boundary conditions at infinity. (Details will be supplied elsewhere [17].) In the second form of Eq. (1) we have subtracted the energy of the vacuum, by inserting the free Green's function G_0, which satisfies, with the same boundary conditions as G, the free equation

$$-\partial^2 G_0 = 1. \tag{3}$$

Now we define the T-matrix (note that our definition of T differs by a factor of 2 from that in Ref. [12])

$$T = S - 1 = V(1 + G_0 V)^{-1}. \tag{4}$$

We then follow standard scattering theory [15], as reviewed in Kenneth and Klich [14]. If the potential has two disjoint parts, $V = V_1 + V_2$, it is easy to derive the following general expression for the interaction between two bodies (potentials):

$$E_{12} = -\frac{i}{2\tau}\mathrm{Tr}\ln(1 - G_0T_1G_0T_2), \tag{5}$$

where

$$T_i = V_i(1 + G_0V_i)^{-1}, \quad i = 1, 2. \tag{6}$$

This form is exactly that given by Emig et al. [12], and by Kenneth and Klich [14].

Elsewhere [17] we will show that this formulation allows us to rederive the Casimir interaction between two semitransparent plates, and the self-energy of the semitransparent sphere.

2 + 1 SPATIAL DIMENSIONS

We now proceed to apply this method to the interaction between bodies, starting with a 2 + 1 dimensional version, which allows us to describe, for example, cylinders with parallel axes. Let the distance between the centers of the bodies be R. Then we perform a Fourier analysis of the reduced Green's function, defined by

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \int \frac{dk_z}{2\pi} e^{ik_z(z-z')} g_0(\mathbf{R}_\perp + \mathbf{r}'_\perp - \mathbf{r}_\perp), \tag{7}$$

where the reduced Green's function has the expansion (as long as the two potentials do not overlap)

$$g_0 = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I_{m'}(\kappa r') e^{-im'\phi'} \tilde{g}^0_{m,m'}(\kappa R), \tag{8}$$

where $\omega = i\zeta$ and $\kappa^2 = k_z^2 + \zeta^2$. The Fourier-Bessel transform of the reduced Green's function is

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{(-1)^{m'}}{2\pi} K_{m-m'}(\kappa R). \tag{9}$$

Thus we can derive an expression for the interaction between two bodies, in terms of discrete matrices,

$$\frac{E_{\rm int}}{L} = \frac{1}{8\pi^2} \int d\zeta\, dk_z \ln\det\left(1 - \tilde{g}^0 t_1 \tilde{g}^{0\top} t_2\right), \tag{10}$$

where $\top$ denotes transpose, and where the t matrix elements are given by

$$t_{mm'} = \int (d\mathbf{r}_\perp) \int (d\mathbf{r}'_\perp) I_m(\kappa r) e^{-im\phi} I_{m'}(\kappa r') e^{im'\phi'} T. \tag{11}$$

Consider, as an example, two parallel semitransparent cylinders, of radii a and b, respectively, lying outside each other, described by the potentials

$$V_1 = \lambda_1 \delta(r-a), \quad V_2 = \lambda_2 \delta(r'-b), \tag{12}$$

with the separation R between the axes satisfying $R > a+b$. It is easy to work out the scattering matrix:

$$(t_1)_{mm'} = 2\pi\lambda_1 a \delta_{mm'} \frac{I_m^2(\kappa a)}{1 + \lambda_1 a I_m(\kappa a) K_m(\kappa a)}. \tag{13}$$

Then the Casimir energy E per unit length L is

$$\frac{E}{L} = \frac{1}{4\pi} \int_0^\infty d\kappa\, \kappa \operatorname{tr}\ln(1-A), \tag{14}$$

where $A = B(a)B(b)$, in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R) \frac{\lambda_1 a I_{m'}^2(\kappa a)}{1 + \lambda_1 a I_{m'}(\kappa a) K_{m'}(\kappa a)}. \tag{15}$$

As a check, it is easy to reproduce the result derived by Bordag [8] for a cylinder in front of a plane, using an evident image method.

In weak coupling, the formula (14) for the interaction energy between the cylinders is

$$\frac{E}{L} = -\frac{\lambda_1\lambda_2 ab}{4\pi R^2} \sum_{m,m'=-\infty}^{\infty} \times \int_0^\infty dx\, x\, K_{m+m'}^2(x) I_m^2(xa/R) I_{m'}^2(xb/R). \tag{16}$$

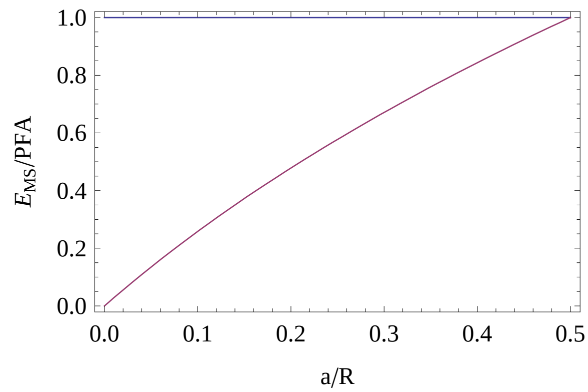

FIG. 1: Plotted is the ratio of the exact interaction energy (19) of two weakly-coupled cylinders to the proximity force approximation (20) as a function of the cylinder radius a for $a = b$.

It is straightforward to develop a power series in a/R for the interaction between semitransparent cylinders. One merely exploits the small argument expansion for the modified Bessel functions $I_m(xa/R)$ and $I_{m'}(xb/R)$. The result is amazingly simple:

$$\frac{E}{L} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \frac{1}{2} \sum_{n=0}^\infty \left(\frac{a}{R}\right)^{2n} P_n\left(\frac{b}{a}\right), \tag{17}$$

where in terms of the binomial coefficients

$$P_n\left(\frac{b}{a}\right) = \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{b}{a}\right)^{2k}. \tag{18}$$

Remarkably, it is possible to perform the sums [18], so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\frac{E}{L} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2} \left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{-1/2}. \tag{19}$$

We note that in the limit $R - a - b = d \to 0$, d being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case,

$$V(d) = -\frac{\lambda_1\lambda_2}{32\pi} \sqrt{\frac{2ab}{R}} \frac{1}{d^{1/2}}, \quad d \ll a, b. \tag{20}$$

In Figs. 1–2 we compare the exact energy (19) with the proximity force approximation (20). Evidently, the former approach the latter when the sum of the radii $a+b$ of the cylinders approaches the distance R between their centers. The rate of approach is linear (with slope 3/2) for the equal radius case, but with slope $b/4a$ when $a \ll b$.

3-DIMENSIONAL FORMALISM

The three-dimensional formalism is very similar. Again, details will be supplied in Ref. [17]. Let us proceed

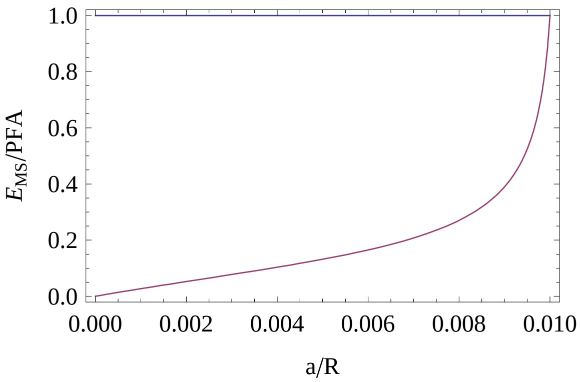

FIG. 2: Plotted is the ratio of the exact interaction energy (19) of two weakly-coupled cylinders to the proximity force approximation (20) as a function of the cylinder radius a for $b/a = 99$.

to write down the expression for the interaction between two semitransparent cylinders:

$$E = \frac{1}{4\pi}\int_0^\infty d\zeta\,\mathrm{tr}\ln(1-A), \tag{21}$$

where the matrix

$$A_{lm,l'm'} = \delta_{m,m'}\sum_{l''} B_{ll''m}(a)B_{l''l'm}(b) \tag{22}$$

is given in terms of the quantities [the three-j symbols (Wigner coefficients) here vanish unless $l+l'+l''$ is even]

$$B_{ll'm}(a) = \frac{i\sqrt{\pi}}{\sqrt{2\zeta R}} i^{-l-l'}\sqrt{(2l+1)(2l'+1)}\sum_{l''}(2l''+1)\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m & 0\end{pmatrix}\frac{K_{l''+1/2}(\zeta R)\lambda_1 a I^2_{l'+1/2}(\zeta a)}{1+\lambda_1 a I_{l'+1/2}(\zeta a)K_{l'+1/2}(\zeta a)}. \tag{23}$$

For strong coupling, this result reduces to that found by Bulgac et al. [7] for Dirichlet spheres, and recently generalized by Emig et al. [13] for Robin boundary conditions.

For weak coupling, a major simplification results because ot the orthogonality property ($l \le l'$),

$$\sum_{m=-l}^{l}\begin{pmatrix} l & l' & l'' \\ m & -m & 0\end{pmatrix}\begin{pmatrix} l & l' & l''' \\ m & -m & 0\end{pmatrix} = \delta_{l''l'''}\frac{1}{2l''+1}. \tag{24}$$

Then the formula for the energy of interaction between the two spheres is

$$E = -\frac{\lambda_1 a\lambda_2 b}{4R}\int_0^\infty\frac{dx}{x}\sum_{ll'l''}(2l+1)(2l'+1)(2l''+1)\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}^2 K^2_{l''+1/2}(x)I^2_{l+1/2}(xa/R)I^2_{l'+1/2}(xb/R). \tag{25}$$

There is no infrared divergence because for small x the product of Bessel functions goes like $x^{2(l+l'-l'')+1}$, and $l'' \le l + l'$ because of the triangle property of the $3j$-symbols.

Again, it is straightforward to carry out a power series expansion in a/R, which turns out to have a simple form

$$E = -\frac{\lambda_1 a\lambda_2 b}{8R}\frac{ab}{R^2}\sum_{n=0}^\infty\frac{1}{n+1}\sum_{m=0}^{n}\binom{2n+2}{2m+1}\times\left(\frac{a}{R}\right)^{2n}\left(\frac{b}{a}\right)^{2m}. \tag{26}$$

Once more, it can be recognized as the following closed form:

$$E = \frac{\lambda_1 a\lambda_2 b}{16\pi R}\ln\left(\frac{1-\left(\frac{a}{R}+\frac{b}{R}\right)^2}{1-\left(\frac{a}{R}-\frac{b}{R}\right)^2}\right). \tag{27}$$

Again, when $d = R - a - b \ll a, b$, the proximity force theorem is reproduced:

$$V(d) \sim \frac{\lambda_1\lambda_2 ab}{16\pi R}\ln(d/R), \quad d \ll a, b. \tag{28}$$

However, as Figs. 3, 4 demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is

$$\frac{E}{V} = 1 + \frac{\ln[(1+\alpha)^2/2\alpha]}{\ln d/R}, \quad d \ll a, b, \tag{29}$$

for $b/a = \alpha$. Truncating the power series (26) at $n = 100$ would only begin to show the approach to the PFA limit. The error in using the PFA between spheres can be very substantial.

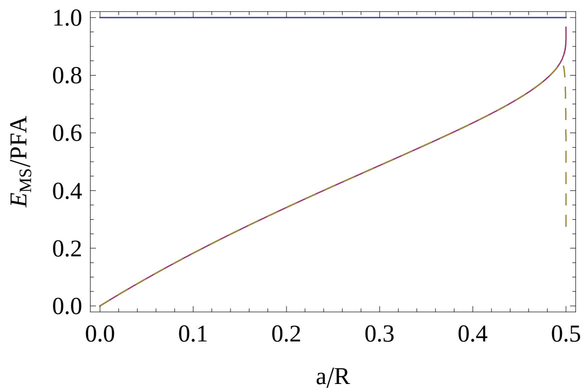

FIG. 3: Plotted is the ratio of the exact interaction energy (27) of two weakly-coupled spheres to the proximity force approximation (28) as a function of the cylinder radius a for $a = b$. Shown also is the power series expansion (26), truncated at $n = 100$, indicating that it is necessary to include very high powers to capture the proximity force limit.

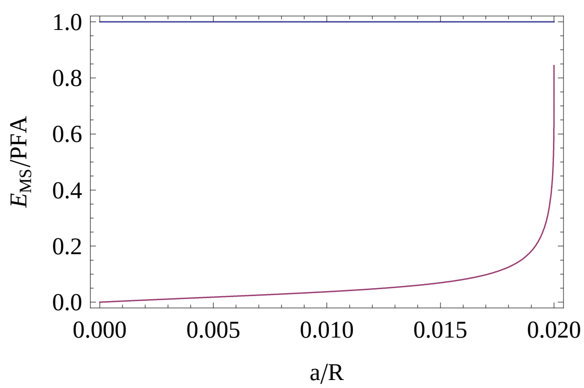

FIG. 4: Plotted is the ratio of the exact interaction energy (27) of two weakly-coupled spheres to the proximity force approximation (28) as a function of the cylinder radius a for $b/a = 49$.

CONCLUSION

We have used standard multiple scattering techniques to calculate the Casimir interaction between two semi-transparent (δ-function) spheres and between two semi-transparent parallel cylinders. When the coupling constant is weak, we are able to sum the power series expansion in a/R exactly, and obtain a closed form for the Casimir interaction energy. This energy reduces to the proximity force limit when the bodies are very close together, but in general, the PFA does a poor job in describing the interaction. These exact results represent the first known exact closed-form results for the Casimir interaction between two bodies which are not plane surfaces. More details and examples will be given in Ref. [17].

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