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THRESHOLD EFFECTS IN INCLUSIVE TAU LEPTON DECAY

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Abstract

The role of quark masses and threshold effects in the inclusive decay of the τ lepton into hadrons is analyzed. The method of analytic perturbation theory, which avoids the problem of unphysical singularities, like the ghost pole, and gives a self-consistent description of both spacelike and timelike regions, is applied. The threshold behavior of the quark-antiquark system is described by using a new relativistic resummation factor which summarizes threshold singularities of the perturbative series of the $(\alpha_S/v)^n$ type. It is demonstrated that threshold effects reduce the value of the QCD scale parameter Λ extracted from the τ decay data.

Key-words: tau lepton, threshold effects

1. A theoretical and experimental study of processes at a low energy scale is very important in QCD because it allows one to investigate effects lying beyond the framework of the perturbative approach. At present, there is rich experimental material obtained from hadronic τ-lepton decays. The first theoretical analysis of hadronic decays of a heavy lepton was performed in 1971 [1] before the experimental discovery of the τ-lepton in 1975. Since then, the properties of the τ have been studied very intensively.

In this talk we discuss the well-known ratio of hadronic to leptonic widths for the inclusive decay of the τ-lepton, R_τ, which now known experimentally with high precision. This ratio is useful for extracting of the values of the QCD running coupling α_S at the τ mass scale and the QCD parameter Λ.

The initial theoretical expression for R_τ contains an integral over timelike momentum s

$$R_\tau = \frac{2}{\pi} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\,\Pi(s)\,, \tag{1}$$

which extends down to small s and cannot be directly calculated in the framework of the standard perturbation theory (PT). Indeed, the hadronic correlator $\Pi(s)$ is parametrized by the perturbative running coupling that has unphysical singularities and, therefore, is ill-defined in the region of small momenta. To avoid this problem, one usually applies the following procedure. The initial integral (1) is rewritten by using the Cauchy theorem in the form of a contour integral in the complex plane with the contour running around a circle with radius M_τ^2 [2]:

$$R_\tau = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=M_\tau^2} \frac{d\,z}{z} \left(1 - \frac{z}{M_\tau^2}\right)^3 \left(1 + \frac{z}{M_\tau^2}\right) D(z)\,, \tag{2}$$

where $D(z)$ is the Adler function [3]. This trick allows one, in principle, to avoid the problem of a direct calculation of the R_τ ratio. However, it should be noted that in order to perform this transformation self-consistently, it is necessary to maintain correct analytic properties of the hadronic correlator, which are violated in the framework of standard PT. The analytic approach to QCD [4] (see also [5, 6]), which we will use here, maintains needed analytic properties and allows one to perform self-consistently the procedure of analytic continuation.

We begin by representing the R_τ-ratio in the form

$$R_\tau = R_\tau^0(1+\delta_{\rm QCD}), \tag{3}$$

where R_τ^0 corresponds to the parton level description and $\delta_{\rm QCD}$ is the QCD correction. Also, we introduce QCD contributions to the imaginary part of the hadronic correlator, $r(s)$, and to the corresponding Adler function, $d(z)$: $\mathcal{R}(s) = [{\rm Im}\;\Pi(s+i\epsilon)/\pi]/R_\tau^0 \propto 1+r\,,\; D \propto 1+d\,.$ Then, one can write $\delta_{\rm QCD}$ as an integral over timelike momentum (Minkowskian region)

$$\delta_{\rm QCD} = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right) r(s)\,, \tag{4}$$

or as a contour integral in the complex plane (Euclidean region)

$$\delta_{\rm QCD} = \frac{1}{2\pi i}\oint_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) d(z)\,. \tag{5}$$

The PT description is based on the contour representation and can be developed in the following two ways. In Braaten's (Br) method [2] the quantity (5) is represented in the form of truncated power series with the expansion parameter $a_\tau \equiv \alpha_S(M_\tau^2)/\pi$. In this case the three-loop representation for $\delta_{\rm QCD}$ is

$$\delta_{\rm QCD}^{\rm Br} = a_\tau\; + r_1\,a_\tau^2 + r_2\,a_\tau^3\,, \tag{6}$$

where the coefficients r_1 and r_2 in the $\overline{\rm MS}$ scheme with three active flavors are $r_1 = 5.2023$ and $r_2 = 26.366$ [2].

The method proposed by Le Diberder and Pich (LP) [7] uses the PT expansion of the d-function

$$d(z) = a(z) + d_1 a^2(z) + d_2 a^3(z)\,, \tag{7}$$

where in the $\overline{\rm MS}$-scheme $d_1^{\overline{\rm MS}} = 1.6398$ and $d_2^{\overline{\rm MS}} = 6.3710$ [8] for three active quarks. The PT running coupling $a(z)$ is obtained from the renormalization group equation with the three-loop β-function. The substitution of Eq. (7) into Eq. (5) leads to the following non-power representation

$$\delta_{\rm QCD}^{\rm LP} = A^{(1)}(a) + d_1\,A^{(2)}(a) + d_2\,A^{(2)}(a) \tag{8}$$

with

$$A^{(n)}(a) = \frac{1}{2\pi i}\oint_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) a^n(z)\,. \tag{9}$$

As noted above, transformation to the contour representation (5) requires the existence of certain analytic properties of the correlator: namely, it must be an analytic function in the complex z-plane with a cut along the positive real axis. The correlator parametrized, as usual, by the PT running coupling does not have this virtue. Moreover, the conventional renormalization group method determines the running coupling in the spacelike region, whereas the initial expression (1) contains an integration over timelike momentum, and there is the question of how to parametrize a quantity defined for timelike momentum transfers [9]. To perform this procedure self-consistently, it is important to maintain correct analytic properties of the hadronic correlator [10, 11, 12]. Because of this failure of analyticity, Eqs. (4) and (5) are not equivalent in the framework of PT and, if one remains within PT, it is difficult to estimate the errors introduced by this transformation. However, using analytic perturbation theory (APT) [5, 13] it is possible to resolve these problems.[1] Important features of APT are: (i) this approach maintains the correct analytic properties and leads to a self-consistent procedure of analytic continuation from the spacelike to the timelike region; (ii) it has much improved convergence properties and turns out to be stable with respect to higher-loop corrections; (iii) renormalization scheme dependence of the results obtained within this method is reduced dramatically.

In the framework of the analytic approach, the functions $d(z)$ and $r(s)$ are expressed in terms of the effective spectral function $\rho(\sigma)$ [4, 11]

$$d(z) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma - z}\,\rho(\sigma)\,, \qquad r(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\rho(\sigma)\,. \tag{10}$$

The spectral function is defined as the imaginary part of the perturbative[2] approximation to $d_{\rm pt}(z)$ on the physical cut. At the three-loop level, it is

$$\rho(\sigma) = \varrho_0(\sigma) + d_1\varrho_1(\sigma) + d_2\varrho_2(\sigma)\,, \qquad \varrho_n(\sigma) = {\rm Im}[a_{\rm pt}^{n+1}(\sigma + i\epsilon)]\,. \tag{11}$$

Using Eq. (4) or equivalently Eq. (5), we obtain the QCD correction to the R_τ-ratio in terms of $\rho(\sigma)$

$$\delta_{\rm an} = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma}\rho(\sigma) - \frac{1}{\pi}\int_0^{M_\tau^2}\frac{d\sigma}{\sigma}\left(1 - \frac{\sigma}{M_\tau^2}\right)^3\left(1 + \frac{\sigma}{M_\tau^2}\right)\rho(\sigma)\,. \tag{12}$$

This expression can be written down as the non-power expansion

$$\delta_{\rm an} = \delta^{(0)} + d_1\,\delta^{(1)} + d_2\,\delta^{(2)}\,. \tag{13}$$

The function $\varrho_0(\sigma)$ in Eq. (11) defines the analytic spacelike, $a_{\rm an}(z)$, and timelike (s-channel), $\tilde{a}_s(s)$, running couplings as follows

$$a_{\rm an}(z) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma - z}\,\varrho_0(\sigma)\,, \qquad \tilde{a}_s(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\varrho_0(\sigma). \tag{14}$$

Note, unlike the one-loop PT running coupling, $a_{\rm pt}^{(1)}(z) = (2/b)/\ln(-z/\Lambda^2)$, the analytic running coupling has no unphysical ghost pole and, therefore, possesses the correct analytic properties, arising from Källén-Lehmann analyticity reflecting the general principles of the theory. In the one-loop approximation it is [4]

$$a_{\rm an}^{(1)}(z) \;=\; a_{\rm pt}^{(1)}(z) + \frac{2}{b}\frac{\Lambda^2}{\Lambda^2 + z}\,. \tag{15}$$

[1]The nonperturbative a-expansion technique in QCD [14] also leads to a well-defined procedure of analytic continuation [10].

[2]To distinguish APT and PT cases, we will use subscripts "an" and "pt".

Table 1: *Relative contributions of higher-loop terms in different methods.*

Method	Expansion terms
APT	$1 \,+\, \delta_{\rm an} \;=\; 1 \,+\, 0.167 \,+\, 0.021 \,+\, 0.002$
PT (LP)	$1 \,+\, \delta_{\rm pt}^{\rm LP} \;=\; 1 \,+\, 0.148 \,+\, 0.030 \,+\, 0.012$
PT (Br)	$1 \,+\, \delta_{\rm pt}^{\rm Br} \;=\; 1 \,+\, 0.104 \,+\, 0.056 \,+\, 0.030$

The nonperturbative (non-logarithmic) term, which has appeared in the analytic running coupling, does not change the ultraviolet limit of the theory and thus the APT and the PT approaches coincide with each other in the asymptotic region of high energies.

The one-loop s-channel running coupling[3] is given by [11]

$$\tilde{a}_s^{(1)} \;=\; \frac{2}{b}\left[\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{\ln s/\Lambda^2}{\pi}\right)\right] \,. \tag{16}$$

Thus, the APT description can be equivalently phrased either on the basis of the original expression (4), which involves the Minkowskian quantity $r(s)$, or on the contour representation (5), which involves the Euclidean quantity $d(z)$. The difference between the PT and APT contributions to the R_τ can be transparently shown by the one-loop relation: $\delta_{\rm an} = \delta_{\rm pt} - (2/b)\Lambda^2/M_\tau^2 + {\rm O}(\Lambda^4/M_\tau^4)$. The additional term, which is 'invisible' in the perturbative expansion, turns out to be important numerically [16, 13].

In the case of massless quarks, the APT analysis of the inclusive τ decay on the three-loop level has been performed in [17]. In Table 1 we compare this result with the perturbative calculations performed by Braaten's and Le Diberder–Pich methods. This table demonstrates that the APT expansion has much improved convergence properties as compared with different PT approximations. Note also that the renormalization scheme dependence of APT results is dramatically reduced [17, 18].

2. The R_τ-ratio can be separated experimentally in the form of three parts

$$R_\tau \;=\; R_{\tau,V} \,+\, R_{\tau,A} \,+\, R_{\tau,S} \,. \tag{17}$$

The terms $R_{\tau,V}$ and $R_{\tau,A}$ are contributions coming from the non-strange hadronic decays associated with vector (V) and axial-vector (A) quark currents respectively, and $R_{\tau,S}$ contains strange decays (S).

Within the perturbative approximation with massless quarks the vector and axial-vector contributions to R_τ coincide with each other

$$R_{\tau,V} = R_{\tau,A} = \frac{3}{2}|V_{ud}|^2(1 \,+\, \delta_{\rm QCD}) \,. \tag{18}$$

However, the experimental measurements [19, 20] shown that these components are not equal to each other. The corresponding difference is associated with non-perturbative QCD effects which are usually described in the form of power corrections [2, 19, 20]. The experimental data for the isovector spectral function [19, 20] have been used in [21] to extract the Adler D_V-function which we show as the dashed line in Fig. 1. The experimental

[3] As has been argued from general principles, the behavior of the effective couplings in the spacelike and the timelike domains cannot be the same [15].

D-function turns out to be a smooth function without any traces of resonance structure.[4] One can expect that this object more precisely reflects the quark-hadron duality and, therefore, is convenient for comparing theoretical predictions with experimental data.[5] Note here that any finite order of the operator product expansion fails to describe the infrared tail of the D-function. We will apply the analytic approach and study the role of quark masses and threshold effects by comparing our results with experimental data for the D_V-function.

3. The convenient way to incorporate quark mass effects is to use an approximate expression [24] which here can be written as

$$\mathcal{R}(s) = T(v)\ [1 + g(v)r(s)]\,\Theta(s - 4m^2)\,, \tag{19}$$

where

$$T(v) = v\frac{3 - v^2}{2}\,,\ \ g(v) = \frac{4\pi}{3}\left[\frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)\right]\,,\ \ v = \sqrt{1 - \frac{4m^2}{s}}\,, \tag{20}$$

and $m = m_u = m_d$ denotes the effective mass of u and d quarks which we take here to be equal each other.

A description of quark-antiquark systems near threshold requires us to take into account the resummation factor which summarizes the threshold singularities of the perturbative series of the $(\alpha_S/v)^n$ type. In a nonrelativistic approximation, this is the well known Sommerfeld-Sakharov factor [25, 26]. For a systematic relativistic analysis of quark-antiquark systems, it is essential from the very beginning to have a relativistic generalization of this factor. Moreover, it is important to take into account the difference between the Coulomb potential in the case of QED and the quark-antiquark potential in the case of QCD. This QCD relativistic factor has been proposed in [27] to have the form

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}\,, \qquad X(\chi) = \frac{4\pi\,\alpha_S}{3\sinh\chi}\,, \tag{21}$$

where χ is the rapidity which related to s by $2m\cosh\chi = \sqrt{s}$. The relativistic resummation factor (21) reproduces both the expected nonrelativistic and ultrarelativistic limits and corresponds to a QCD-like quark-antiquark potential.

The threshold resummation factor leads to the following modification of the expression (19)

$$\mathcal{R}_V(s) = T(v)\ \left[S(\chi) - \frac{1}{2}X(\chi) + g(v)\,r(s)\right]\Theta(s - 4m^2)\,, \tag{22}$$

which we use to calculate the vector part of R_τ

$$R_{\tau,V} = \frac{1}{2}\,R_\tau^0\int_0^{M_\tau^2}\frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2\left(1 + \frac{2s}{M_\tau^2}\right)\mathcal{R}_V(s)\,. \tag{23}$$

The vector part of the Adler function is

$$D_V(Q^2) = Q^2\int_0^\infty ds\,\frac{\mathcal{R}_V(s)}{(s + Q^2)^2}\,. \tag{24}$$

[4] The D-function obtained in [22] from the data for electron-positron annihilation into hadrons has a similar property.

[5] The Minkowskian and Euclidean characteristics of the process of electron-positron annihilation into hadrons have been considered in [23].

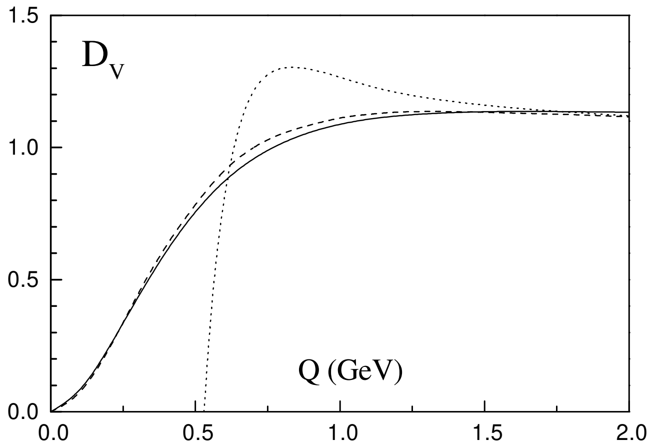

Figure 1: *The vector D-function for the τ decay. The solid curve is the APT result. The experimental curve (dashed line) corresponding to the ALEPH data and the perturbative result with power corrections (dotted line) are taken from [21].*

In Fig. 1 we plot the D-function obtained in the APT approach (solid curve) by using the value of the quark masses $m_u = m_d = 250\,\mathrm{MeV}$ and $m_s = 400\,\mathrm{MeV}$. Practically the same values of the quark masses were used in [28, 29]. These values are close to the constituent quark masses and incorporate some nonperturbative effects. The shape of the infrared tail of the D-function is sensitive to the value of these masses. The scale parameter Λ in the $\overline{\mathrm{MS}}$ renormalization scheme is $\Lambda = 370$ MeV which corresponds to the experimental value $R_{\tau,V}^{expt} = 1.78$ [19]. The experimental curve (dashed line) and the curve which corresponds to the perturbative result with power corrections (dotted line) are taken from [21].

We have presented the description of the 'light' vector D-function based on the analytic approach in QCD which is not in conflict with the general principles of the theory. The conventional method of approximating this function as a sum of perturbative terms and power corrections cannot describe the low energy scale region. We have shown that within the APT approach, taking into account mass and threshold effects, it is possible to obtain good agreement with experimental data down to the lowest energy scale. Moreover, we have found that threshold resummation is very important for the problem considered here. The effect of the QCD relativistic S-factor is a reduction of the value of the QCD scale parameter Λ extracted from the τ-data, which turned out to be too large within the massless analysis [17] compared with high-energy data. Thus, our analysis demonstrates the important role played in τ-lepton physics by both the analytic properties and the threshold resummation.

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