

## REVIEW ARTICLE

# The Casimir effect: Recent controversies and progress

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**Abstract.** The phenomena implied by the existence of quantum vacuum fluctuations, grouped under the title of the Casimir effect, are reviewed, with emphasis on new results discovered in the past four years. The Casimir force between parallel plates is rederived as the strong-coupling limit of  $\delta$ -function potential planes. The role of surface divergences is clarified. A summary of effects relevant to measurements of the Casimir force between real materials is given, starting from a geometrical optics derivation of the Lifshitz formula, and including a rederivation of the Casimir-Polder forces. A great deal of attention is given to the recent controversy concerning temperature corrections to the Casimir force between real metal surfaces. A summary of new improvements to the proximity force approximation is given, followed by a synopsis of the current experimental situation. New results on Casimir self-stress are reported, again based on  $\delta$ -function potentials. Progress in understanding divergences in the self-stress of dielectric bodies is described, in particular the status of a continuing calculation of the self-stress of a dielectric cylinder. Casimir effects for solitons, and the status of the so-called dynamical Casimir effect, are summarized. The possibilities of understanding dark energy, strongly constrained by both cosmological and terrestrial experiments, in terms of quantum fluctuations are discussed. Throughout, the centrality of quantum vacuum energy in fundamental physics is emphasized.

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## 1. Introduction

The essence of quantum physics is fluctuations. That is, knowing the position of a particle precisely means losing all knowledge about its momentum, and vice versa, and generally the product of uncertainties of a generalized coordinate  $q$  and its corresponding momentum  $p$  is bounded below:

$$\Delta q \Delta p \geq \frac{\hbar}{2}, \quad (1.1)$$

which reflects the fundamental commutation relation

$$[q, p] = i\hbar. \quad (1.2)$$

The Hamiltonian commutes with neither  $q$  nor  $p$  in general; this means that in an energy eigenstate the fluctuations in  $q$  and  $p$  are both nonzero:

$$\Delta q > 0, \quad \Delta p > 0. \quad (1.3)$$

Moreover, a harmonic oscillator has correspondingly a ground-state energy which is nonzero:

$$E_{\text{ho},n} = \hbar\omega \left( n + \frac{1}{2} \right). \quad (1.4)$$

The apparent implication of this is that a crystal, which may be thought of, roughly, as a collection of atoms held in harmonic potentials, should have a large zero-point energy at zero temperature:

$$E_{\text{ZP}} = \sum_{\text{atoms}} \frac{1}{2} \hbar\omega, \quad (1.5)$$

$\omega$  being the characteristic frequency of each potential.

The vacuum of quantum field theory may similarly be regarded as an enormously large collection of harmonic oscillators, representing the fluctuations of, for quantum electrodynamics, the electric and magnetic fields at each point in space. (Canonically, the momentum-coordinate pair correspond to the electric field and the vector potential.) Put otherwise, the QED vacuum is a sea of virtual photons. Thus the zero-point energy density of the vacuum is

$$U = \sum \frac{1}{2} \hbar\omega = 2 \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{1}{2} \hbar c |\mathbf{k}|, \quad (1.6)$$

where  $\mathbf{k}$  is the wavevector of the photon, and the factor of 2 reflects the two polarization states of the photon.

This is an enormously large quantity. If we say that the largest wavevector appearing in the integral is  $K$ , say  $\hbar c K \sim 10^{19}$  GeV, the Planck scale, then  $U \sim 10^{115}$  GeV/cm<sup>3</sup>. So it is no surprise that Dirac suggested that this zero-point energy be simply discarded, as some irrelevant constant [1] (yet he became increasingly concerned about the inconsistency of doing so throughout his life [2]). Pauli recognized that this energy surely coupled to gravity, and it would then give rise to a large cosmological constant, so large that the size of the universe could not even reach the distance to the moon

[3, 4]. This cosmological constant problem is with us to the present [5, 6]. But this was not the most perplexing issue confronting quantum electrodynamics in the 1930s.

Renormalization theory, that is, a consistent theory of quantum electrodynamics, was invented first by Schwinger [7] and then Feynman [8] in 1948; yet remarkably, across the Atlantic, Casimir in the same year predicted the direct macroscopically observable consequence of vacuum fluctuations that now bears his name [9]. This is the attraction between parallel uncharged conducting plates that has been so convincingly demonstrated by many experiments in the last few years [10]. Lifshitz and his group generalized the theory to include dielectric materials in the 1950s [11, 12, 13, 14]. There were many experiments to detect the effect in the 1950s and 1960s, but most were inconclusive, because the forces were so small, and it was very difficult to keep various interfering phenomena from washing out the effect [15]. However, there could be very little doubt of the reality of the phenomenon, since it was intimately tied to the theory of van der Waals forces between molecules, the retarded version of which had been worked out by Casimir [16] just before he discovered (with a nudge from Bohr [17]) the force between plates. Finally, in 1973, the Lifshitz theory was vindicated by an experiment by Sabisky and Anderson [18].

But by and large field theorists were unaware of the effect until Glashow's student Boyer carried out a remarkable calculation of the Casimir self-energy of a perfectly conducting spherical shell in 1968 [19]. Glashow was aware of Casimir's proposal [20] that a classical electron could be stabilized by zero-point attraction, and thought the calculation made a suitable thesis project. Boyer's result was a surprise: The zero-point force was repulsive for the case of a sphere. Davies improved on the calculation [21]; then a decade later there were two independent reconfirmations of Boyer's result, one based on multiple scattering techniques [22] (now undergoing a renaissance, for example, see [23]) and one on Green's functions techniques [24] (dubbed source theory [25]). Applications to hadronic physics followed in the next few years [26, 27, 28], and in the last two decades, there has been something of an explosion of interest in the field, with many different calculations being carried out [29, 10].

However, fundamental understanding has been very slow in coming. Why is the cosmological constant neither large nor zero? Why is the Casimir force on a sphere repulsive, when it is attractive between two plates? And is it possible to make sense of Casimir force calculations between two bodies, or of the Casimir self-energy of a single body, in terms of supposedly better understood techniques of perturbative quantum field theory [30]? As we will see, none of these questions yet has a definitive answer, yet progress has been coming. Even the temperature corrections to the Casimir effect, which were considered by Sauer [31], Mehra [32], and Lifshitz [11] in the 1950s and 1960s, have become controversial [33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. Thus recent conferences on the Casimir effect have been quite exciting events [48, 49]. It is the aim of the present review to bring the various issues into focus, and suggest paths toward the solutions of the difficulties. It is a mark of the vitality and even centrality of this field that such a review is desirable on the heels of two significant meetings on

the subject, and less than three years after the appearance of two major monographs [10, 29] on Casimir phenomena. There are in addition a number of earlier, excellent reviews [50, 51, 52], as well as more specialized treatments [53, 54, 55, 56]. Throughout this review Gaussian units are employed.

This review is organized in the following manner. In section 2 we compute Casimir energies and pressures between parallel  $\delta$  function planes, which in the limit of large coupling reproduce the results for a scalar field satisfying Dirichlet boundary conditions on those surfaces. Although these results have been described before, clarification of the nature of surface energy and divergences is provided. TM modes are also discussed here for the first time. Then, in section 3 we rederive the Lifshitz formula for the Casimir force between parallel dielectric slabs using a multiple reflection technique. The Casimir-Polder forces between two atoms, and between an atom and a plate, are rederived. After reviewing roughness and conductivity corrections, a detailed discussion of the temperature controversy is given, with the conclusion that the TE zero-mode absence must be taken seriously, which will imply that large temperature corrections should be seen experimentally. New approaches to moving beyond the proximity approximation in computing forces between nonparallel plane surface are reviewed. A discussion of the remarkable progress experimentally since 1997 is provided. In section 4 after a review of the general situation with respect to surface divergences, TE and TM forces on  $\delta$ -function spheres are described in detail, which in the limit of strong coupling reduce to the corresponding finite electromagnetic contributions. For weak coupling, Casimir energies are finite in second order in the coupling strength, but divergent in third order, a fact which has been known for several years. This mirrors the corresponding result for a dilute dielectric sphere, which diverges in third order in the deviation of the permittivity from its vacuum value. Self-stresses on cylinders are also treated, with a detailed discussion of the status of a new calculation for a dielectric cylinder, which should give a vanishing self-stress in second order in the relative permittivity. Section 5 briefly summarizes recent work on quantum fluctuation phenomena in solitonic physics, which has provided the underlying basis for much of the interest in Casimir phenomena over the years. Dynamical Casimir effects, ranging from sonoluminescence through the Unruh effect, are the subject of section 6. The presumed basis for understanding the cosmological dark energy in terms of the Casimir fluctuations is treated in section 7, where there may be a tight constraint emerging between terrestrial measurements of deviations from Newtonian gravity and the size of extra dimensions. The review ends with a summary of perspectives for the future of the field.

## 2. Casimir Effect Between Parallel Plates: A $\delta$ -Potential Derivation

In this section, we will rederive the classic Casimir result for the force between parallel conducting plates [9]. Since the usual Green's function derivation may be found in monographs [29], and was recently reviewed in connection with current controversies over finiteness of Casimir energies [57], we will here present a different approach, based

on  $\delta$ -function potentials, which in the limit of strong coupling reduce to the appropriate Dirichlet or Robin boundary conditions of a perfectly conducting surface, as appropriate to TE and TM modes, respectively. Such potentials were first considered by the Leipzig group [58, 59], but recently have been the focus of the program of the MIT group [60, 61, 30]. The discussion here is based on a recent paper by the author [62]. We first consider two  $\delta$ -function potentials in 1 + 1 dimensions.

### 2.1. 1 + 1 dimensions

We consider a massive scalar field (mass  $\mu$ ) interacting with two  $\delta$ -function potentials, one at  $x = 0$  and one at  $x = a$ , which has an interaction Lagrange density

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} \frac{\lambda}{a} \delta(x) \phi^2(x) - \frac{1}{2} \frac{\lambda'}{a} \delta(x - a) \phi^2(x), \quad (2.1)$$

where we have chosen the coupling constants  $\lambda$  and  $\lambda'$  to be dimensionless. (But see the following.) In the limit as both couplings become infinite, these potentials enforce Dirichlet boundary conditions at the two points:

$$\lambda, \lambda' \rightarrow \infty : \quad \phi(0), \phi(a) \rightarrow 0. \quad (2.2)$$

The Casimir energy for this situation may be computed in terms of the Green's function  $G$ ,

$$G(x, x') = i \langle T \phi(x) \phi(x') \rangle, \quad (2.3)$$

which has a time Fourier transform,

$$G(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(x, x'; \omega). \quad (2.4)$$

Actually, this is a somewhat symbolic expression, for the Feynman Green's function (2.3) implies that the frequency contour of integration here must pass below the singularities in  $\omega$  on the negative real axis, and above those on the positive real axis [63, 64]. The reduced Green's function in (2.4) in turn satisfies

$$\left[ -\frac{\partial^2}{\partial x^2} + \kappa^2 + \frac{\lambda}{a} \delta(x) + \frac{\lambda'}{a} \delta(x - a) \right] g(x, x') = \delta(x - x'). \quad (2.5)$$

Here  $\kappa^2 = \mu^2 - \omega^2$ . This equation is easily solved, with the result

$$g(x, x') = \frac{1}{2\kappa} e^{-\kappa|x-x'|} + \frac{1}{2\kappa\Delta} \left[ \frac{\lambda\lambda'}{(2\kappa a)^2} 2 \cosh \kappa|x-x'| \right. \\ \left. - \frac{\lambda}{2\kappa a} \left( 1 + \frac{\lambda'}{2\kappa a} \right) e^{2\kappa a} e^{-\kappa(x+x')} - \frac{\lambda'}{2\kappa a} \left( 1 + \frac{\lambda}{2\kappa a} \right) e^{\kappa(x+x')} \right] \quad (2.6a)$$

for both fields inside,  $0 < x, x' < a$ , while if both field points are outside,  $a < x, x'$ ,

$$g(x, x') = \frac{1}{2\kappa} e^{-\kappa|x-x'|} + \frac{1}{2\kappa\Delta} e^{-\kappa(x+x'-2a)} \left[ -\frac{\lambda}{2\kappa a} \left( 1 - \frac{\lambda'}{2\kappa a} \right) - \frac{\lambda'}{2\kappa a} \left( 1 + \frac{\lambda}{2\kappa a} \right) e^{2\kappa a} \right]. \quad (2.6b)$$

For  $x, x' < 0$ ,

$$g(x, x') = \frac{1}{2\kappa} e^{-\kappa|x-x'|} + \frac{1}{2\kappa\Delta} e^{\kappa(x+x')} \left[ -\frac{\lambda'}{2\kappa a} \left(1 - \frac{\lambda}{2\kappa a}\right) - \frac{\lambda}{2\kappa a} \left(1 + \frac{\lambda'}{2\kappa a}\right) e^{2\kappa a} \right]. \quad (2.6c)$$

Here, the denominator is

$$\Delta = \left(1 + \frac{\lambda}{2\kappa a}\right) \left(1 + \frac{\lambda'}{2\kappa a}\right) e^{2\kappa a} - \frac{\lambda\lambda'}{(2\kappa a)^2}. \quad (2.7)$$

Note that in the strong coupling limit we recover the familiar results, for example, inside

$$\lambda, \lambda' \rightarrow \infty : \quad g(x, x') \rightarrow -\frac{\sinh \kappa x_{<} \sinh \kappa(x_{>} - a)}{\kappa \sinh \kappa a}. \quad (2.8)$$

Evidently, this Green's function vanishes at  $x = 0$  and at  $x = a$ .

We can now calculate the force on one of the  $\delta$ -function points by calculating the discontinuity of the stress tensor, obtained from the Green's function (2.3) by

$$\langle T^{\mu\nu} \rangle = \left( \partial^\mu \partial^{\nu'} - \frac{1}{2} g^{\mu\nu} \partial^\lambda \partial'_\lambda \right) \frac{1}{i} G(x, x') \Big|_{x=x'}. \quad (2.9)$$

Writing a reduced stress tensor by

$$\langle T^{\mu\nu} \rangle = \int \frac{d\omega}{2\pi} t^{\mu\nu}, \quad (2.10)$$

we find inside

$$\begin{aligned} t_{xx} &= \frac{1}{2i} (\omega^2 + \partial_x \partial_{x'}) g(x, x') \Big|_{x=x'} \\ &= \frac{1}{4i\kappa\Delta} \left\{ (2\omega^2 - \mu^2) \left[ \left(1 + \frac{\lambda}{2\kappa a}\right) \left(1 + \frac{\lambda'}{2\kappa a}\right) e^{2\kappa a} + \frac{\lambda\lambda'}{(2\kappa a)^2} \right] \right. \\ &\quad \left. - \mu^2 \left[ \frac{\lambda}{2\kappa a} \left(1 + \frac{\lambda'}{2\kappa a}\right) e^{-2\kappa(x-a)} + \frac{\lambda'}{2\kappa a} \left(1 + \frac{\lambda}{2\kappa a}\right) e^{2\kappa x} \right] \right\}. \end{aligned} \quad (2.11)$$

Let us henceforth simplify the considerations by taking the massless limit,  $\mu = 0$ . Then the stress tensor just to the left of the point  $x = a$  is

$$t_{xx} \Big|_{x=a-} = -\frac{\kappa}{2i} \left\{ 1 + 2 \left[ \left( \frac{2\kappa a}{\lambda} + 1 \right) \left( \frac{2\kappa a}{\lambda'} + 1 \right) e^{2\kappa a} - 1 \right]^{-1} \right\}. \quad (2.12a)$$

From this we must subtract the stress just to the right of the point at  $x = a$ , obtained from (2.6b), which turns out to be in the massless limit

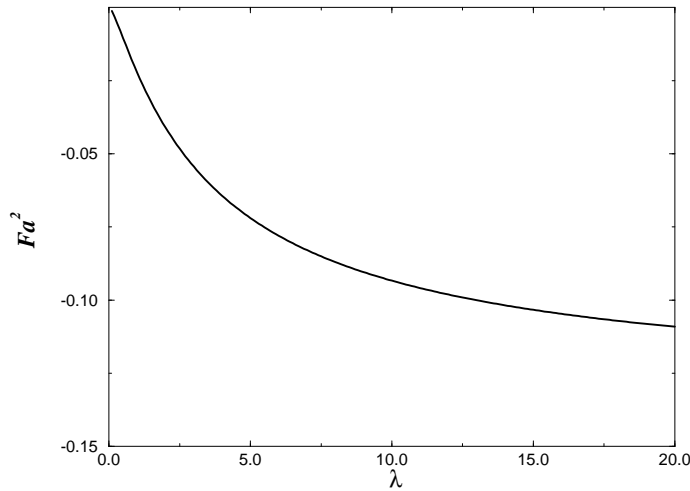
$$t_{xx} \Big|_{x=a+} = -\frac{\kappa}{2i}, \quad (2.12b)$$

which just cancels the 1 in braces in (2.12a). Thus the force on the point  $x = a$  due to the quantum fluctuations in the scalar field is given by the simple, finite expression

$$F = \langle T_{xx} \rangle \Big|_{x=a-} - \langle T_{xx} \rangle \Big|_{x=a+} = -\frac{1}{4\pi a^2} \int_0^\infty dy y \frac{1}{(y/\lambda + 1)(y/\lambda' + 1)e^y - 1}. \quad (2.13)$$

This reduces to the well-known, Lüscher result [65, 66] in the limit  $\lambda, \lambda' \rightarrow \infty$ ,

$$\lim_{\lambda=\lambda' \rightarrow \infty} F = -\frac{\pi}{24a^2}, \quad (2.14)$$



**Figure 1.** Casimir force (2.13) between two  $\delta$ -function points having strength  $\lambda$  and separated by a distance  $a$ .

and for  $\lambda = \lambda'$  is plotted in Fig. 1.

Recently, Sundberg and Jaffe [67] have used their background field method to calculate the Casimir force due to fermion fields between two  $\delta$ -function spikes in  $1 + 1$  dimension. Apart from quibbles about infinite energies, in the limit  $\lambda \rightarrow \infty$  they recover the same result as for scalar, (2.14), which is as expected [68], since in the ideal limit the relative factor between scalar and spinor energies is  $2(1 - 2^{-D})$  in  $D$  spatial dimensions, i.e.,  $7/4$  for three dimensions and  $1$  for one.

We can also compute the energy density. In this simple massless case, the calculation appears identical, because  $t_{xx} = t_{00}$  (reflecting the conformal invariance of the free theory). The energy density is constant [(2.11) with  $\mu = 0$ ] and subtracting from it the  $a$ -independent part that would be present if no potential were present, we immediately see that the total energy is  $E = Fa$ , so  $F = -\partial E/\partial a$ . This result differs significantly from that given in Refs. [61, 60, 69], which is a divergent expression in the massless limit, not transformable into the expression found by this naive procedure. However, that result may be easily derived from the following expression for the total energy,

$$\begin{aligned} E &= \int (d\mathbf{r}) \langle T^{00} \rangle = \frac{1}{2i} \int (d\mathbf{r}) (\partial^0 \partial'^0 - \nabla^2) G(x, x') \Big|_{x=x'} \\ &= \frac{1}{2i} \int (d\mathbf{r}) \int \frac{d\omega}{2\pi} 2\omega^2 \mathcal{G}(\mathbf{r}, \mathbf{r}), \end{aligned} \quad (2.15)$$

if we integrate by parts and omit the surface term. Integrating over the Green's functions in the three regions, given by (2.6a), (2.6b), and (2.6c), we obtain for  $\lambda = \lambda'$ ,

$$E = \frac{1}{2\pi a} \int_0^\infty dy \frac{1}{1 + y/\lambda} - \frac{1}{4\pi a} \int_0^\infty dy y \frac{1 + 2/(y + \lambda)}{(y/\lambda + 1)^2 e^y - 1}, \quad (2.16)$$

where the first term is regarded as an irrelevant constant ( $\lambda/a$  is constant), and the second is the same as that given by equation (70) of Ref. [60] upon integration by parts.

The origin of this discrepancy with the naive energy is the existence of a surface contribution to the energy. Because  $\partial_\mu T^{\mu\nu} = 0$ , we have, for a region  $V$  bounded by a surface  $S$ ,

$$0 = \frac{d}{dt} \int_V (d\mathbf{r}) T^{00} + \oint_S dS_i T^{0i}. \quad (2.17)$$

Here  $T^{0i} = \partial^0 \phi \partial^i \phi$ , so we conclude that there is an additional contribution to the energy,

$$E_s = -\frac{1}{2i} \int d\mathbf{S} \cdot \nabla G(x, x') \Big|_{x'=x} \quad (2.18a)$$

$$= -\frac{1}{2i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum \frac{d}{dx} g(x, x') \Big|_{x'=x}, \quad (2.18b)$$

where the derivative is taken at the boundaries (here  $x = 0, a$ ) in the sense of the outward normal from the region in question. When this surface term is taken into account the extra terms in (2.16) are supplied. The integrated formula (2.15) automatically builds in this surface contribution, as the implicit surface term in the integration by parts. (These terms are slightly unfamiliar because they do not arise in cases of Neumann or Dirichlet boundary conditions.) See Fulling [70] for further discussion. That the surface energy of an interface arises from the volume energy of a smoothed interface is demonstrated in Ref. [62], and elaborated in section 2.4.

It is interesting to consider the behavior of the force or energy for small coupling  $\lambda$ . It is clear that, in fact, (2.13) is not analytic at  $\lambda = 0$ . (This reflects an infrared divergence in the Feynman diagram calculation.) If we expand out the leading  $\lambda^2$  term we are left with a divergent integral. A correct asymptotic evaluation leads to the behavior

$$F \sim \frac{\lambda^2}{4\pi a^2} (\ln 2\lambda + \gamma), \quad E \sim -\frac{\lambda^2}{4\pi a} (\ln 2\lambda + \gamma - 1), \quad \lambda \rightarrow 0. \quad (2.19)$$

This behavior indeed was anticipated in earlier perturbative analyses. In Ref. [57] the general result was given for the Casimir energy for a  $D$  dimensional spherical  $\delta$ -function potential (a factor of  $1/4\pi$  was inadvertently omitted)

$$E = -\frac{\lambda^2}{\pi a} \frac{\Gamma\left(\frac{D-1}{2}\right) \Gamma(D-3/2) \Gamma(1-D/2)}{2^{1+2D} [\Gamma(D/2)]^2}. \quad (2.20)$$

This possesses an infrared divergence as  $D \rightarrow 1$ :

$$E^{(D=1)} = \frac{\lambda^2}{4\pi a} \Gamma(0), \quad (2.21)$$

which is consistent with the nonanalytic behavior seen in (2.19).

## 2.2. Parallel Planes in 3 + 1 Dimensions

It is trivial to extract the expression for the Casimir pressure between two  $\delta$  function planes in three spatial dimensions, where the background lies at  $x = 0$  and  $x = a$ . We



merely have to insert into the above a transverse momentum transform,

$$G(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{(d\mathbf{k})}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')_{\perp}} g(x, x'; \kappa), \quad (2.22)$$

where now  $\kappa^2 = \mu^2 + k^2 - \omega^2$ . Then  $g$  has exactly the same form as in (2.6a)–(2.6c). The reduced stress tensor is given by, for the massless case,

$$t_{xx} = \frac{1}{2}(\partial_x \partial_{x'} - \kappa^2) \frac{1}{i} g(x, x') \Big|_{x=x'}, \quad (2.23)$$

so we immediately see that the attractive pressure on the planes is given by ( $\lambda = \lambda'$ )

$$P = -\frac{1}{32\pi^2 a^4} \int_0^{\infty} dy y^3 \frac{1}{(y/\lambda + 1)^2 e^y - 1}, \quad (2.24)$$

which coincides with the result given in Refs. [30, 71]. The leading behavior for small  $\lambda$  is

$$P^{\text{TE}} \sim -\frac{\lambda^2}{32\pi^2 a^4}, \quad \lambda \ll 1, \quad (2.25a)$$

while for large  $\lambda$  it approaches half of Casimir's result [9] for perfectly conducting parallel plates,

$$P^{\text{TE}} \sim -\frac{\pi^2}{480a^4}, \quad \lambda \gg 1. \quad (2.25b)$$

The Casimir energy per unit area again might be expected to be

$$\mathcal{E} = -\frac{1}{96\pi^2 a^3} \int_0^{\infty} dy \frac{y^3}{(y/\lambda + 1)^2 e^y - 1} = \frac{1}{3} \frac{P}{a}, \quad (2.26)$$

because then  $P = -\frac{\partial}{\partial a} \mathcal{E}$ . In fact, however, it is straightforward to compute the energy density  $\langle T^{00} \rangle$  in the three regions,  $x < 0$ ,  $0 < x < a$ , and  $a < x$ , and then integrate it over  $x$  to obtain the energy/area, which differs from (2.26) because, now, there exists transverse momentum. We also must include the surface term (2.18a), which is of opposite sign, and of double magnitude, to the  $k^2$  term. The net extra term is

$$\mathcal{E}' = \frac{1}{48\pi^2 a^3} \int_0^{\infty} dy y^2 \frac{1}{1 + y/\lambda} \left[ 1 - \frac{y/\lambda}{(y/\lambda + 1)^2 e^y - 1} \right]. \quad (2.27)$$

If we regard  $\lambda/a$  as constant (so that the strength of the coupling is independent of the separation between the planes) we may drop the first, divergent term here as irrelevant, being independent of  $a$ , because  $y = 2\kappa a$ , and then the total energy is

$$\mathcal{E} = -\frac{1}{96\pi^2 a^3} \int_0^{\infty} dy y^3 \frac{1 + 2/(\lambda + y)}{(y/\lambda + 1)^2 e^y - 1}, \quad (2.28)$$

which coincides with the massless limit of the energy first found by Bordag *et al* [58], and given in Refs. [30, 71]. As noted in section 2.1, this result may also readily be derived through use of (2.15). When differentiated with respect to  $a$ , (2.28), with  $\lambda/a$  fixed, yields the pressure (2.24).

In the limit of strong coupling, we obtain

$$\lim_{\lambda \rightarrow \infty} \mathcal{E} = -\frac{\pi^2}{1440a^3}, \quad (2.29)$$

which is exactly one-half the energy found by Casimir for perfectly conducting plates [9]. Evidently, in this case, the TE modes (calculated here) and the TM modes (calculated in the following subsection) give equal contributions.

## 2.3. TM Modes

To verify this claim, we solve a similar problem with boundary conditions that the derivative of  $g$  is continuous at  $x = 0$  and  $a$ ,

$$\left. \frac{\partial}{\partial x} g(x, x') \right|_{x=0, a} \text{ is continuous,} \quad (2.30a)$$

but the function itself is discontinuous,

$$g(x, x') \Big|_{x=a-}^{x=a+} = \lambda a \left. \frac{\partial}{\partial x} g(x, x') \right|_{x=a}, \quad (2.30b)$$

and similarly at  $x = 0$ . These boundary conditions reduce, in the limit of strong coupling, to Neumann boundary conditions on the planes, appropriate to electromagnetic TM modes:

$$\lambda \rightarrow \infty : \quad \left. \frac{\partial}{\partial x} g(x, x') \right|_{x=0, a} = 0. \quad (2.30c)$$

It is completely straightforward to work out the reduced Green's function in this case. When both points are between the planes,  $0 < x, x' < a$ ,

$$g(x, x') = \frac{1}{2\kappa} e^{-\kappa|x-x'|} + \frac{1}{2\kappa\tilde{\Delta}} \left\{ \left( \frac{\lambda\kappa a}{2} \right)^2 2 \cosh \kappa(x-x') + \frac{\lambda\kappa a}{2} \left( 1 + \frac{\lambda\kappa a}{2} \right) \left[ e^{\kappa(x+x')} + e^{-\kappa(x+x'-2a)} \right] \right\}, \quad (2.31a)$$

while if both points are outside the planes,  $a < x, x'$ ,

$$g(x, x') = \frac{1}{2\kappa} e^{-\kappa|x-x'|} + \frac{1}{2\kappa\tilde{\Delta}} \frac{\lambda\kappa a}{2} e^{-\kappa(x+x'-2a)} \left[ \left( 1 - \frac{\lambda\kappa a}{2} \right) + \left( 1 + \frac{\lambda\kappa a}{2} \right) e^{2\kappa a} \right], \quad (2.31b)$$

where the denominator is

$$\tilde{\Delta} = \left( 1 + \frac{\lambda\kappa a}{2} \right)^2 e^{2\kappa a} - \left( \frac{\lambda\kappa a}{2} \right)^2. \quad (2.32)$$

It is easy to check that in the strong-coupling limit, the appropriate Neumann boundary condition (2.30c) is recovered. For example, in the interior region,  $0 < x, x' < a$ ,

$$\lim_{\lambda \rightarrow \infty} g(x, x') = \frac{\cosh \kappa x_{<} \cosh \kappa(x_{>} - a)}{\kappa \sinh \kappa a}. \quad (2.33)$$

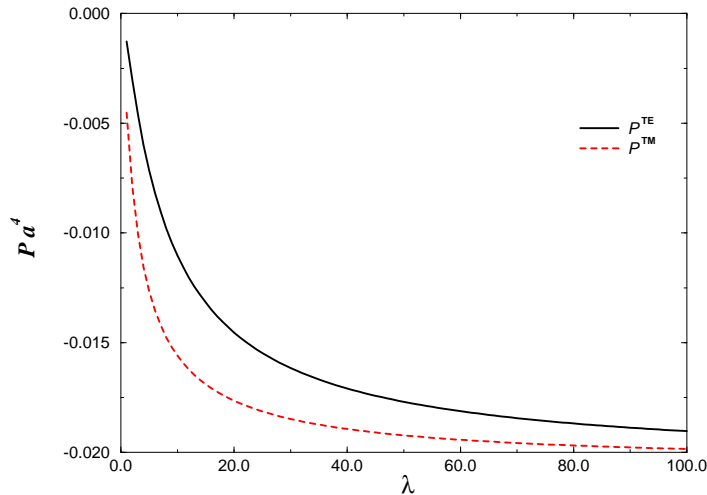
Now we can compute the pressure on the plane by computing the  $xx$  component of the stress tensor, which is given by (2.23),

$$t_{xx} = \frac{1}{2i} (-\kappa^2 + \partial_x \partial'_x) g(x, x') \Big|_{x=x'}. \quad (2.34)$$

The action of derivatives on exponentials is very simple, so we find

$$t_{xx} \Big|_{x=a-} = \frac{1}{2i} \left[ -\kappa - \frac{2\kappa}{\tilde{\Delta}} \left( \frac{\lambda\kappa a}{2} \right)^2 \right], \quad (2.35a)$$

$$t_{xx} \Big|_{x=a+} = -\frac{1}{2i} \kappa, \quad (2.35b)$$



**Figure 2.** TE and TM Casimir pressures between  $\delta$ -function planes having strength  $\lambda$  and separated by a distance  $a$ .

so the flux of momentum deposited in the plane  $x = a$  is

$$t_{xx}\Big|_{x=a-} - t_{xx}\Big|_{x=a+} = \frac{i\kappa}{\left(\frac{2}{\lambda\kappa a} + 1\right)^2 e^{2\kappa a} - 1}, \quad (2.36)$$

and then by integrating over frequency and transverse momentum we obtain the pressure:

$$P^{\text{TM}} = -\frac{1}{32\pi^2 a^4} \int_0^\infty dy y^3 \frac{1}{\left(\frac{4}{\lambda y} + 1\right)^2 e^y - 1}. \quad (2.37)$$

In the limit of weak coupling, this behaves as follows:

$$P^{\text{TM}} \sim -\frac{15}{64\pi^2 a^4} \lambda^2, \quad (2.38)$$

which is to be compared with (2.25a). In strong coupling, on the other hand, it has precisely the same limit as the TE contribution, (2.25b), which confirms the expectation given at the end of the previous subsection. Graphs of the two functions are given in Fig. 2.

For calibration purposes we give the Casimir pressure in practical units between ideal perfectly conducting parallel plates at zero temperature:

$$P = -\frac{\pi^2}{240a^4} \hbar c = -\frac{1.30 \text{ mPa}}{(a/1\mu\text{m})^4}. \quad (2.39)$$

#### 2.4. Surface energy as bulk energy of boundary layer

Here we show that the surface energy can be interpreted as the bulk energy of the boundary layer. We do this by considering a scalar field in 1 + 1 dimensions interacting

with the background

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{2}\phi^2\sigma, \quad (2.40)$$

where

$$\sigma(x) = \begin{cases} h, & -\frac{\delta}{2} < x < \frac{\delta}{2}, \\ 0, & \text{otherwise,} \end{cases} \quad (2.41)$$

with the property that  $h\delta = 1$ . The reduced Green's function satisfies

$$\left[ -\frac{\partial^2}{\partial x^2} + \kappa^2 + \lambda\sigma(x) \right] g(x, x') = \delta(x - x'). \quad (2.42)$$

This may be easily solved in the region of the slab,  $-\frac{\delta}{2} < x < \frac{\delta}{2}$ ,

$$g(x, x') = \frac{1}{2\kappa'} \left\{ e^{-\kappa'|x-x'|} + \frac{1}{\hat{\Delta}} \left[ (\kappa'^2 - \kappa^2) \cosh \kappa'(x+x') \right. \right. \\ \left. \left. + (\kappa' - \kappa)^2 e^{-\kappa'\delta} \cosh \kappa'(x-x') \right] \right\}. \quad (2.43)$$

Here  $\kappa' = \sqrt{\kappa^2 + \lambda h}$ , and

$$\hat{\Delta} = 2\kappa\kappa' \cosh \kappa'\delta + (\kappa^2 + \kappa'^2) \sinh \kappa'\delta. \quad (2.44)$$

This result may also easily be derived from the multiple reflection formulas given in section 3.1, and agrees with that given by Graham and Olum [72]. The energy of the slab now is obtained by integrating the energy density

$$t^{00} = \frac{1}{2i} (\omega^2 + \partial_x \partial_{x'} + \lambda h) g \Big|_{x=x'} \quad (2.45)$$

over frequency and the width of the slab. This gives the vacuum energy of the slab

$$E_s = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \frac{1}{2\kappa' \hat{\Delta}} \left[ (\kappa' - \kappa)^2 (-\kappa^2 - \kappa'^2 + \lambda h) e^{-\kappa'\delta} \delta \right. \\ \left. + (\kappa'^2 - \kappa^2) (-\kappa^2 + \kappa'^2 + \lambda h) \frac{\sinh \kappa'\delta}{\delta} \right]. \quad (2.46)$$

If we now take the limit  $\delta \rightarrow 0$  and  $h \rightarrow \infty$  so that  $h\delta = 1$ , we immediately obtain

$$E_s = \frac{1}{2\pi} \int_0^{\infty} d\kappa \frac{\lambda}{\lambda + 2\kappa}, \quad (2.47)$$

which precisely coincides with one-half the constant term in (2.16), with  $\lambda$  there replaced by  $\lambda a$  here.

There is no surface term in the total Casimir energy as long as the slab is of finite width, because we may easily check that  $\frac{d}{dx} g|_{x=x'}$  is continuous at the boundaries  $\pm \frac{\delta}{2}$ . However, if we only consider the energy internal to the slab we encounter not only the energy (2.15) but a surface term from the integration by parts. It is only this boundary term that gives rise to  $E_s$ , (2.47), in this way of proceeding.

Further insight is provided by examining the local energy density. In this we follow the work of Graham and Olum [72, 73]. However, let us proceed here with more generality, and consider the stress tensor with an arbitrary conformal term,

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi + \lambda h \phi^2) - \alpha (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2, \quad (2.48)$$

in  $d + 2$  dimensions,  $d$  being the number of transverse dimensions. Applying the corresponding differential operator to the Green's function (2.43), introducing polar coordinates in the  $(\zeta, k)$  plane, with  $\zeta = \kappa \cos \theta$ ,  $k = \kappa \sin \theta$ , and

$$\langle \sin^2 \theta \rangle = \frac{d}{d+1}, \quad (2.49)$$

we get the following form for the energy density within the slab,

$$T^{00} = \frac{2^{-d-2}\pi^{-(d+1)/2}}{\Gamma((d+3)/2)} \int_0^\infty \frac{d\kappa \kappa^d}{\kappa' \hat{\Delta}} \left\{ (\kappa'^2 - \kappa^2) [(1-4\alpha)(1+d)\kappa'^2 - \kappa^2] \cosh 2\kappa'x - (\kappa' - \kappa)^2 e^{-\kappa'\delta} \kappa^2 \right\}. \quad (2.50)$$

From this we can calculate the behavior of the energy density as the boundary is approached from the inside:

$$T^{00} \sim \frac{\Gamma(d+1)\lambda h}{2^{d+4}\pi^{(d+1)/2}\Gamma((d+3)/2)} \frac{1-4\alpha(d+1)/d}{(\delta-2|x|)^d}, \quad |x| \rightarrow \delta/2. \quad (2.51)$$

For  $d = 2$  for example, this agrees with the result found in Ref. [72] for  $\alpha = 0$ :

$$T^{00} \sim \frac{\lambda h}{96\pi^2} \frac{(1-6\alpha)}{(\delta/2 - |x|)^d}, \quad |x| \rightarrow \frac{\delta}{2}. \quad (2.52)$$

Note that, as we expect, this surface divergence vanishes for the conformal stress tensor [74], where  $\alpha = d/4(d+1)$ . (There will be subleading divergences if  $d > 2$ .)

We can also calculate the energy density on the other side of the boundary, from the Green's function for  $x, x' < -\delta/2$ ,

$$g(x, x') = \frac{1}{2\kappa} \left[ e^{-\kappa|x-x'|} - e^{\kappa(x+x'+\delta)} (\kappa'^2 - \kappa^2) \frac{\sinh \kappa'\delta}{\hat{\Delta}} \right], \quad (2.53)$$

and the corresponding energy density is given by

$$T^{00} = -\frac{d(1-4\alpha(d+1)/d)}{2^{d+2}\pi^{(d+1)/2}\Gamma((d+3)/2)} \int_0^\infty d\kappa \kappa^{d+1} \frac{1}{\hat{\Delta}} (\kappa'^2 - \kappa^2) e^{2\kappa(x+\delta/2)} \sinh \kappa'\delta, \quad (2.54)$$

which vanishes if the conformal value of  $\alpha$  is used. The divergent term, as  $x \rightarrow -\delta/2$ , is just the negative of that found in (2.51). This is why, when the total energy is computed by integrating the energy density, it is finite for  $d < 2$ , and independent of  $\alpha$ . The divergence encountered for  $d = 2$  may be handled by renormalization of the interaction potential [72]. In the limit as  $h \rightarrow \infty$ ,  $h\delta = 1$ , we recover the divergent expression (2.47) for  $d = 0$ , or in general

$$\lim_{h \rightarrow \infty} E_s = \frac{1}{2^{d+2}\pi^{(d+1)/2}\Gamma((d+3)/2)} \int_0^\infty d\kappa \kappa^d \frac{\lambda}{\lambda + 2\kappa}. \quad (2.55)$$

Therefore, surface divergences have an illusory character.

For further discussion on surface divergences, see section 4.1.

### 3. Casimir Effect Between Real Materials

#### 3.1. The Lifshitz Formula Revisited

As a prolegomena to the derivation of the Lifshitz formula for the Casimir force between parallel dielectric slabs, let us note that the results in the previous section may be easily derived geometrically, in terms of multiple reflections. Suppose we have translational invariance in the  $y$  and  $z$  directions, so in terms of reduced Green's functions, everything is one-dimensional. Suppose at  $x = 0$  and  $x = a$  we have discontinuities giving rise to reflection and transmission coefficients. That is, if we only had the  $x = 0$  interface, the reduced Green's function would have the form

$$g(x, x') = \frac{1}{2\kappa} \left( e^{-\kappa|x-x'|} + r e^{-\kappa(x+x')} \right), \quad (3.1a)$$

for  $x, x' > 0$ , while for  $x' > 0 > x$ ,

$$g(x, x') = \frac{1}{2\kappa} t e^{-\kappa(x'-x)}. \quad (3.1b)$$

Similarly, if we only had the interface at  $x = a$ , we would have similarly defined reflection and transmission coefficients  $r'$  and  $t'$ . Transmission and reflection coefficients defined for a wave incident from the left instead of the right will be denoted with tildes. If both interfaces are present, we can calculate the Green's function in the region to the right of the rightmost interface  $x, x' > a$  in the form

$$g(x, x') = \frac{1}{2\kappa} \left( e^{-\kappa|x-x'|} + R e^{-\kappa(x+x'-2a)} \right), \quad (3.2a)$$

where  $R$  may be easily computed by summing multiple reflections:

$$\begin{aligned} R &= r' + t' e^{-\kappa a} r e^{-\kappa a} \tilde{t}' + t' e^{-\kappa a} r e^{-\kappa a} \tilde{r}' e^{-\kappa a} r e^{-\kappa a} \tilde{t}' + \dots \\ &= r' + \frac{r t' \tilde{t}'}{e^{2\kappa a} - r \tilde{r}'}. \end{aligned} \quad (3.2b)$$

For the TE  $\delta$ -function potential (2.1),  $r = \tilde{r} = -(1 + 2\kappa a/\lambda)^{-1}$ , and  $t = \tilde{t} = 1 + r$ , and we immediately recover the result (2.6b). But the same formula applies to electromagnetic modes in a dielectric medium with two parallel interfaces, where the permittivity is

$$\varepsilon(x) = \begin{cases} \varepsilon_1, & x < 0, \\ \varepsilon_3, & 0 < x < a, \\ \varepsilon_2, & a < x. \end{cases} \quad (3.3)$$

In that case [75]

$$r = \frac{\kappa_3 - \kappa_1}{\kappa_3 + \kappa_1}, \quad r' = \frac{\kappa_2 - \kappa_3}{\kappa_2 + \kappa_3}, \quad \tilde{r}' = -r', \quad (3.4a)$$

and

$$t' = 1 + r', \quad \tilde{t}' = 1 - r', \quad (3.4b)$$

where  $\kappa_i^2 = k^2 - \omega^2 \varepsilon_i$ . Substituting these expressions into (3.2b) we obtain

$$R = \frac{\kappa_2 - \kappa_3}{\kappa_2 + \kappa_3} + \frac{4\kappa_2\kappa_3}{\kappa_3^2 - \kappa_2^2} \frac{1}{\frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1} \frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2} e^{2\kappa_3 a} - 1}, \quad (3.5)$$

which coincides with the formula (3.16) given in Ref. [29].

However, to calculate most readily the force between the slabs, we need the corresponding formula for the reduced Green's function between the interfaces. This may also be readily derived by multiple reflections:

$$\begin{aligned}
g(x, x') &= \frac{1}{2\kappa} \left[ e^{-\kappa|x-x'|} + \tilde{r}' e^{-\kappa(2a-x-x')} + r\tilde{r}' e^{-\kappa(2a-x'+x)} + r\tilde{r}'^2 e^{-\kappa(4a-x-x')} \right. \\
&\quad \left. + r^2\tilde{r}'^2 e^{-\kappa(4a+x-x')} + \dots \right. \\
&\quad \left. + r e^{-\kappa(x'+x)} + r\tilde{r}' e^{-\kappa(2a+x'-x)} + r^2\tilde{r}' e^{-\kappa(2a+x'+x)} + r^2\tilde{r}'^2 e^{-\kappa(4a+x'-x)} + \dots \right] \\
&= \frac{1}{2\kappa} \left\{ e^{-\kappa|x-x'|} + \frac{1}{e^{2\kappa a} - r\tilde{r}'} \left[ 2r\tilde{r}' \cosh \kappa(x-x') + \tilde{r}' e^{\kappa(x+x')} + r e^{-\kappa(x+x'-2a)} \right] \right\}.
\end{aligned} \tag{3.6}$$

Indeed, this reduces to (2.6a) when the appropriate reflection coefficients are inserted. The pressure on the planes may be computed from the discontinuity in the stress tensor, or

$$t_{xx} \Big|_{x=a-} - t_{xx} \Big|_{x=a+} = \frac{1}{2i} (-\kappa^2 + \partial_x \partial_{x'}) g(x, x') \Big|_{x=x'=a+}^{x=x'=a-} = \frac{i\kappa}{r\tilde{r}' e^{2\kappa a} - 1}, \tag{3.7}$$

from which the  $\delta$ -potential results (2.12a) and (2.12b) follow immediately. For the case of parallel dielectric slabs the TE modes therefore contribute the following expression for the pressure $\ddagger$ :

$$P^{\text{TE}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{(d\mathbf{k})}{(2\pi)^2} \frac{i\kappa_3}{\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2} \frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1} e^{2\kappa_3 a} - 1}. \tag{3.8}$$

The contribution from the TM modes are obtained by the replacement

$$\kappa \rightarrow \kappa' = \frac{\kappa}{\varepsilon}, \tag{3.9}$$

except in the exponentials [75]. This gives for the force per unit area at zero temperature

$$P_{\text{Casimir}}^{T=0} = -\frac{1}{4\pi^2} \int_0^\infty d\zeta \int_0^\infty dk^2 \kappa_3 (d^{-1} + d'^{-1}), \tag{3.10}$$

with the denominators here being  $[\kappa_i = \sqrt{k^2 + \zeta^2 \varepsilon_i(i\zeta)}]$

$$d = \frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1} \frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2} e^{2\kappa_3 a} - 1, \quad d' = \frac{\kappa'_3 + \kappa'_1}{\kappa'_3 - \kappa'_1} \frac{\kappa'_3 + \kappa'_2}{\kappa'_3 - \kappa'_2} e^{2\kappa_3 a} - 1, \tag{3.11}$$

which correspond to the TE and TM Green's functions, respectively. This is the celebrated Lifshitz formula [11, 12, 13, 14], which we shall discuss further in the following subsections. We merely note here that if we take the limit  $\varepsilon_{1,2} \rightarrow \infty$ , and set  $\varepsilon_3 = 1$ , we recover Casimir's result for the attractive force between parallel, perfectly conducting plates (2.39).

Henkel *et al* [76] have computed the Casimir force at short distances ( $\sim 1$  nm) from interactions between polaritons. Their result agrees with the Lifshitz formula with the plasma formula (3.33) employed, see Ref. [77, 78].

$\ddagger$  For the case of dielectric slabs, the propagation constant  $\kappa$  is different on the two sides; we omit the term corresponding to the free propagator, however. In the energy, the omitted terms are proportional to the volume of each slab, and therefore correspond to the volume or bulk energy of the material.

### 3.2. The Relation to van der Waals Forces

Now suppose the central slab consists of a tenuous medium and the surrounding medium is vacuum, so that the dielectric constant in the slab differs only slightly from unity,

$$\epsilon - 1 \ll 1. \quad (3.12)$$

Then, with a simple change of variable,

$$\kappa = \zeta p, \quad (3.13)$$

we can recast the Lifshitz formula (3.10) into the form

$$P \approx -\frac{1}{32\pi^2} \int_0^\infty d\zeta \zeta^3 [\epsilon(\zeta) - 1]^2 \int_1^\infty \frac{dp}{p^2} [(2p^2 - 1)^2 + 1] e^{-2\zeta p a}. \quad (3.14)$$

If the separation of the surfaces is large compared to the wavelength characterizing  $\epsilon$ ,  $a\zeta_c \gg 1$ , we can disregard the frequency dependence of the dielectric constant, and we find

$$P \approx -\frac{23(\epsilon - 1)^2}{640\pi^2 a^4}. \quad (3.15)$$

For short distances,  $a\zeta_c \ll 1$ , the approximation is

$$P \approx -\frac{1}{32\pi^2} \frac{1}{a^3} \int_0^\infty d\zeta (\epsilon(\zeta) - 1)^2. \quad (3.16)$$

These formulas are identical with the well-known forces found for the complementary geometry in Ref. [79].

Now we wish to obtain these results from the sum of van der Waals forces, derivable from a potential of the form

$$V = -\frac{B}{r^\gamma}. \quad (3.17)$$

We do this by computing the energy ( $\mathcal{N}$  = density of molecules)

$$E = -\frac{1}{2} B \mathcal{N}^2 \int_0^a dz \int_0^a dz' \int (d\mathbf{r}_\perp)(d\mathbf{r}'_\perp) \frac{1}{[(\mathbf{r}_\perp - \mathbf{r}'_\perp)^2 + (z - z')^2]^{\gamma/2}}. \quad (3.18)$$

If we disregard the infinite self-interaction terms (analogous to dropping the volume energy terms in the Casimir calculation), we get [79, 80]

$$P = -\frac{\partial E}{\partial a} \frac{1}{A} = -\frac{2\pi B \mathcal{N}^2}{(2 - \gamma)(3 - \gamma)} \frac{1}{a^{\gamma-3}}. \quad (3.19)$$

So then, upon comparison with (3.15), we set  $\gamma = 7$  and in terms of the polarizability,

$$\alpha = \frac{\epsilon - 1}{4\pi \mathcal{N}}, \quad (3.20)$$

we find

$$B = \frac{23}{4\pi} \alpha^2, \quad (3.21)$$

or, equivalently, we recover the retarded dispersion potential of Casimir and Polder [16],

$$V = -\frac{23}{4\pi} \frac{\alpha^2}{r^7}, \quad (3.22)$$



whereas for short distances we recover from (3.16) the London potential [81],

$$V = -\frac{3}{\pi} \frac{1}{r^6} \int_0^\infty d\zeta \alpha(\zeta)^2. \quad (3.23)$$

Recent, nonperturbative approaches to Casimir-Polder forces include that of Buhmann *et al* [82].

*3.2.1. Force Between a Molecule and a Plate* One can also calculate the force between a polarizable molecule, with electric polarizability  $\alpha(\omega)$ , and a dielectric slab. A simple, gauge-invariant way of doing this starts from the variational form [79, 29]

$$\delta W = - \int_{-\infty}^{\infty} dt \delta E = -\frac{i}{2} \int (dx) \delta \varepsilon(x) \Gamma_{kk}(x, x), \quad (3.24)$$

where  $\delta \varepsilon(\mathbf{r}) = 4\pi\alpha(\omega)\delta(\mathbf{r} - \mathbf{R})$ ,  $\mathbf{R}$  denoting the position of the molecule. Here  $\mathbf{\Gamma}$  is the electromagnetic Green's dyadic, defined by

$$\mathbf{\Gamma}(\mathbf{r}, \mathbf{r}') = i\langle \mathbf{E}(\mathbf{r}) \mathbf{E}(\mathbf{r}') \rangle. \quad (3.25)$$

In terms of the reduced Green's function, defined by (2.22), then

$$\delta E = \frac{i}{2} 4\pi \int \frac{d\omega}{2\pi} \frac{d^2k}{(2\pi)^2} \alpha(\omega) g_{kk}(x, x; \omega, \mathbf{k}). \quad (3.26)$$

It is easily seen how the trace of the reduced Green's function can be expressed in terms of the reduced TE and TM Green's functions,

$$g_{kk} = \left( \omega^2 g^{\text{TE}} + \frac{k^2}{\varepsilon \varepsilon'} g^{\text{TM}} + \frac{1}{\varepsilon} \frac{\partial}{\partial x} \frac{1}{\varepsilon'} \frac{\partial}{\partial x'} g^{\text{TM}} \right) \Big|_{x=x'}. \quad (3.27)$$

For a single interface, the Green's functions to the right of a dielectric slab situated in the half-space  $x < 0$  are given by (3.1a) with the reflection coefficients in the vacuum

$$r^{\text{TE}} = \frac{\kappa - \kappa_1}{\kappa + \kappa_1}, \quad r^{\text{TM}} = \frac{\kappa - \kappa_1/\varepsilon_1}{\kappa + \kappa_1/\varepsilon_1}, \quad (3.28)$$

where  $\kappa^2 = k^2 + \zeta^2$  and  $\kappa_1^2 = k^2 + \zeta^2 \varepsilon_1$ . In this way, we immediately obtain the energy between a dielectric slab (permittivity  $\varepsilon_1$ ) and a polarizable molecule a distance  $Z$  from it:

$$E_{\text{slab,mol}} = -\frac{1}{16\pi^2} \int_0^\infty d\zeta 4\pi\alpha(\zeta) \int_0^\infty dk^2 \frac{1}{\kappa} e^{-2\kappa Z} \left[ -\zeta^2 \frac{\kappa - \kappa_1}{\kappa + \kappa_1} + (2k^2 + \zeta^2) \frac{\varepsilon_1 \kappa - \kappa_1}{\varepsilon_1 \kappa + \kappa_1} \right]. \quad (3.29)$$

If the separation between the plate and the molecule is large, we expect that we may neglect the frequency dependence of the polarizability,  $\alpha(\zeta) \rightarrow \alpha(0)$ . There are then two simple limits. If we take  $\varepsilon_1 \rightarrow \infty$  we are describing a perfectly conducting plane, in which case we immediately obtain the result first given by Casimir and Polder [16]

$$E_{\text{metal,mol}} = -\frac{3\alpha(0)}{8\pi Z^4}. \quad (3.30)$$

On the other hand, we could consider a tenuous medium,  $(\varepsilon_1 - 1) \ll 1$ , in which case

$$E_{\text{dilute,mol}} = -\frac{23}{160\pi} \frac{\alpha(0)(\varepsilon_1 - 1)}{Z^4}. \quad (3.31)$$

The latter should be, as in the previous subsection, interpretable as the sum of pairwise van der Waals interactions between the external molecule and the molecules which make up the slab, given by the Casimir-Polder interaction (3.22). The net energy then is

$$-\frac{23}{4\pi}\alpha\mathcal{N}\int_Z^\infty dz\int_0^\infty d\rho\rho\int_0^{2\pi}d\phi\frac{\alpha(0)}{(z^2+\rho^2)^{7/2}}=-\frac{23}{4\pi}\alpha\frac{\mathcal{N}2\pi}{20}\frac{\alpha(0)}{Z^4}, \quad (3.32)$$

which coincides with (3.31) when (3.20) is used.

The force between a molecule and a plate has been measured by Sukenik et al. [83], who actually verified the force between a molecule and two plates [84] at the roughly 10% level. Recently, this result has been questioned (at about the same level of accuracy) by Bordag [85], who argued that a subtle error involving the quantization of gauge fields in the presence of boundaries was made by Casimir and Polder [16] and subsequent workers. The fact that the result can be given an unambiguous gauge-invariant derivation, and that it is closely related to the Lifshitz formula and the retarded dispersion van der Waals force suggests that this critique is invalid. (Bordag now concedes that the usual result is valid for “thick” plates, where the normal component of  $\mathbf{E}$  is given by the surface charge density.)

For a recent rederivation of (3.30) see Hu *et al* [86]. A very recent paper by Babb, Klimchitskaya, and Mostepanenko [87] gives a rederivation of the Casimir-Polder energy (3.30) in the retarded limit, and finds no support for Bordag’s modification. They then go on to discuss the dynamical polarizability and thermal corrections for real materials, and find substantial (35%) corrections at short distances  $\sim 100$  nm.

In this connection we might also mention the work of Noguez and Román-Velázquez [88], who calculate the force between a sphere and a plate made of dissimilar materials in the non-retarded limit (see also van Kampen [89] and Gerlach [90]) in terms of multipolar interactions. They find significant deviations from the proximity approximation (section 3.5), which says that there is no difference between the force between a sphere made of material A and a plate made of material B and the reversed situation, when the separation is comparable or large compared to the radius of the sphere, and that under the above-mentioned A-B interchange the forces change by up to 6%. See also Ref. [91, 92].

Ford and Sopova [93, 94] consider Casimir forces between small metal spheres and dielectric (and conducting) plates, modeled by a plasma dispersion relation

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2}. \quad (3.33)$$

The electric dipole approximation used requires  $a\omega_p \ll 1$ , that is, the radius of the sphere  $a$  must be in the 10–100 nm range. The force is oscillatory, being alternatively attractive and repulsive as a function of the height  $Z$  of the atom above the plate. Thus levitation in the earth’s gravitational field might be possible, for  $Z \sim 1$   $\mu\text{m}$ .

### 3.3. Roughness and Conductivity Corrections

**3.3.1. Roughness Corrections** No real material surface is completely smooth. Even beyond the atomic level, there will be regions of higher and lower elevations. Insofar as these are plateaus large compared to the separation between the disjoint surfaces, the corrections can be easily incorporated by use of the proximity approximation (see section 3.5 below). This is nothing other than the naively obvious statement that if  $P(a)$  is the force per unit area between two parallel plates separated by a distance  $a$ , the average force per area between rough surfaces made up of large plateaus and valleys, with the perpendicular distance between two adjacent points on the two surfaces in terms of transverse coordinates  $(x, y)$  being  $a(x, y)$ , is

$$P = \frac{1}{A} \int dx dy P(a(x, y)). \quad (3.34)$$

In Ref. [95], for example, an equivalent expression is used directly with data obtained by topography of the surfaces using an atomic force microscope. Traditionally, a stochastic estimate has been used. Let the separations  $a$  be distributed around the mean  $a_0$  according to a Gaussian, with the probability of finding separation  $a$  being given by

$$p(a) = \frac{1}{\sqrt{\pi}\delta a} e^{-(a-a_0)^2/(\delta a)^2}. \quad (3.35)$$

We will assume  $\delta a \ll a_0$ . Then,  $\langle a \rangle = a_0$ ,  $\langle (a - a_0)^2 \rangle = \frac{1}{2}(\delta a)^2$ , and in general

$$\begin{aligned} \langle a^\alpha \rangle &= \int_0^\infty da a^\alpha p(a) = \frac{1}{\sqrt{\pi}\delta a} \int_{-\infty}^\infty da e^{-a^2/(\delta a)^2} (a + a_0)^\alpha \\ &= a_0^\alpha \left[ 1 + \frac{\alpha(\alpha-1)}{2} \frac{1}{2} \frac{(\delta a)^2}{a_0^2} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!} \frac{3}{4} \frac{(\delta a)^4}{a_0^4} + \dots \right], \end{aligned} \quad (3.36)$$

The force between a sphere and a plate depends on the closest distance  $d$  between them like  $d^{-3}$ , see (3.78) below, so the stochastic estimate for the roughness correction in that case, in terms of the mean-square fluctuation amplitude  $A = \delta a/\sqrt{2}$ , is

$$F_{\text{sph-pl,rough}} = F_{\text{sph-pl}} \left[ 1 + 6 \left( \frac{A}{d} \right)^2 + 45 \left( \frac{A}{d} \right)^4 + \dots \right]. \quad (3.37)$$

A much more detailed discussion may be found in Ref. [10]. It must be appreciated that the approximate treatment based on the proximity approximation is invalid for short wavelength deformations [96].

**3.3.2. Finite Conductivity** Another interesting result, important for the recent experiments [97, 98, 99, 100], is the correction for an imperfect conductor, where for frequencies above the infrared, an adequate representation for the dielectric constant is [75] that given by the plasma model (3.33) where the plasma frequency is, in Gaussian units

$$\omega_p^2 = \frac{4\pi e^2 N}{m}, \quad (3.38)$$

where  $e$  and  $m$  are the charge and mass of the electron, and  $N$  is the number density of free electrons in the conductor. A simple calculation shows, at zero temperature [101, 79],

$$P \approx -\frac{\pi^2}{240a^4} \left[ 1 - \frac{8}{3\sqrt{\pi}} \frac{1}{ea} \left( \frac{m}{N} \right)^{1/2} \right]. \quad (3.39)$$

If we define a penetration parameter, or skin depth, by  $\delta = 1/\omega_p$ , we can write the force per area for parallel plates out to fourth order as [102, 50, 103, 10]

$$P \approx -\frac{\pi^2}{240a^4} \left[ 1 - \frac{16}{3} \frac{\delta}{a} + 24 \frac{\delta^2}{a^2} - \frac{640}{7} \left( 1 - \frac{\pi^2}{210} \right) \frac{\delta^3}{a^3} + \frac{2800}{9} \left( 1 - \frac{163\pi^2}{7350} \right) \frac{\delta^4}{a^4} \right], \quad (3.40)$$

while using the proximity force theorem (see section 3.5), to convert pressures between parallel plates to forces between a lens of radius  $R$  and a plate,

$$\mathcal{F}_{n-1} = \frac{2\pi R}{n-1} a P_n, \quad (3.41)$$

for a term in the pressure going like  $P_n \propto a^{-n}$ , the force between a spherical surface and a plate separated by a distance  $d$  is

$$\mathcal{F} \approx -\frac{\pi^3 R}{360d^3} \left[ 1 - 4 \frac{\delta}{d} + \frac{72}{5} \frac{\delta^2}{d^2} - \frac{320}{7} \left( 1 - \frac{\pi^2}{210} \right) \frac{\delta^3}{d^3} + \frac{400}{3} \left( 1 - \frac{163\pi^2}{7350} \right) \frac{\delta^4}{d^4} \right]. \quad (3.42)$$

Lambrecht, Jaekel, and Reynaud [104] analyzed the Casimir force between mirrors with arbitrary frequency-dependent reflectivity, and found that it is always smaller than that between perfect reflectors.

We might also mention here the interesting suggestion that repulsive Casimir forces might exist [105] between parallel plates. This harks back to an old suggestion of Boyer [106], that repulsion will occur between two plates, one of which is a perfect electrical conductor,  $\varepsilon \rightarrow \infty$ , and the other a perfect magnetic conductor,  $\mu \rightarrow \infty$ ,

$$P = \frac{7}{8} \frac{\pi^2}{240} \frac{1}{a^4}. \quad (3.43)$$

However, it appears that it will prove very difficult to observe such effects in the laboratory [107]. Klich [108] now seems to agree with this assessment.

### 3.4. Thermal Corrections

The discussion in this subsection is adapted from that in Refs. [109, 110]. We begin by reviewing how temperature effects are incorporated into the expression for the force between parallel dielectric (or conducting) plates separated by a distance  $a$ . To obtain the finite temperature Casimir force from the zero-temperature expression, one conventionally makes the following substitution in the imaginary frequency,

$$\zeta \rightarrow \zeta_m = \frac{2\pi m}{\beta}, \quad (3.44a)$$

and replaces the integral over frequencies by a sum,

$$\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \rightarrow \frac{1}{\beta} \sum_{m=-\infty}^{\infty}. \quad (3.44b)$$

This reflects the requirement that thermal Green's functions be periodic in imaginary time with period  $\beta$  [111]. Suppose we write the finite-temperature pressure as [for the explicit form, see (3.10) and (3.58) below]

$$P^T = \sum'_{m=0} f_m, \quad (3.45)$$

where the prime on the summation sign means that the  $m = 0$  term is counted with half weight. To get the low temperature limit, one can use the Euler-Maclaurin (EM) sum formula,

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(k) dk + \frac{1}{2}f(0) - \sum_{q=1}^{\infty} \frac{B_{2q}}{(2q)!} f^{(2q-1)}(0), \quad (3.46)$$

where  $B_n$  is the  $n$ th Bernoulli number. This means here, with half-weight for the  $m = 0$  term,

$$P^T = \int_0^{\infty} f(m) dm - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0). \quad (3.47)$$

It is noteworthy that the terms involving  $f(0)$  cancel in (3.47). The reason for this is that the EM formula equates an integral to its trapezoidal-rule approximation plus a series of corrections; thus the  $1/2$  for  $m = 0$  in (3.45) is built in automatically. For perfectly conducting plates separated by vacuum [see the  $\lambda \rightarrow \infty$  limit of (2.24) or (2.37), or the  $\epsilon_{1,2} \rightarrow \infty$  limit of (3.10) with  $\epsilon_3 = 1$ ]

$$f(x) = -\frac{2}{\pi\beta} \int_{2\pi x/\beta}^{\infty} \kappa^2 d\kappa \frac{1}{e^{2\kappa a} - 1}. \quad (3.48)$$

Of course, the integral in (3.47) is just the inverse of the finite-temperature prescription (3.44b), and gives the zero-temperature result. The only nonzero odd derivative occurring is

$$f'''(0) = -\frac{16\pi^2}{\beta^4}, \quad (3.49)$$

which gives a Stefan's law type of term, seen in (3.53) below.

The problem is that the EM formula only applies if  $f(m)$  is continuous. If we follow the argument of Ref. [35, 36, 44, 112], and take the  $\epsilon_{1,2} \rightarrow \infty$  limit of (3.10) at the end§ ( $\epsilon_{1,2}$  are the permittivities of the two parallel dielectric slabs), this is not the case, and for the TE mode

$$f_0 = 0, \quad (3.50a)$$

$$f_m = -\frac{\zeta(3)}{4\pi\beta a^3}, \quad 0 < \frac{2\pi a m}{\beta} \ll 1. \quad (3.50b)$$

Then we have to modify the argument as follows:

$$P^T = \sum'_{m=0} f_m = \sum_{m=1}^{\infty} f_m = \sum'_{m=0} \tilde{f}_m - \frac{1}{2}\tilde{f}_0, \quad (3.51)$$

§ This is contrary to the ‘‘Schwinger’’ prescription advocated in Refs. [79, 29], in which the perfect-conductor limit is taken before the zero-mode is extracted.

where  $\tilde{f}_m$  is defined by continuity,

$$\tilde{f}_m = \begin{cases} f_m, & m > 0, \\ \lim_{m \rightarrow 0} f_m, & m = 0. \end{cases} \quad (3.52)$$

Then by using the EM formula,

$$\begin{aligned} P^T &= \frac{\beta}{2\pi} \int_0^\infty d\zeta f(\zeta) + \frac{\zeta(3)}{8\pi\beta a^3} - \frac{\pi^2}{45} \frac{1}{\beta^4} \\ &= -\frac{\pi^2}{240a^4} \left[ 1 + \frac{16}{3} \left( \frac{a}{\beta} \right)^4 \right] + \frac{\zeta(3)}{8\pi a^3} T, \quad aT \ll 1. \end{aligned} \quad (3.53)$$

The same result for the low-temperature limit is extracted through use of the Poisson sum formula, as, for example, discussed in Ref. [29]. Let us refer to these results, with the TE zero mode excluded, as the modified ideal metal model (MIM). The conventional result for an ideal metal (IM), obtained first by Lifshitz [11, 13] and by Sauer [31] and Mehra [32], is given by (3.53) with the linear term in  $T$  omitted.

Exclusion of the TE zero mode will reduce the linear dependence at high temperature by a factor of two,

$$P_{\text{IM}}^T \sim -\frac{\zeta(3)}{4\pi a^3} T, \quad P_{\text{MIM}}^T \sim -\frac{\zeta(3)}{8\pi a^3} T, \quad aT \gg 1, \quad (3.54)$$

but this is not observable by present experiments. The observable consequence, however, is that it adds a linear term at low temperature, which is given in (3.53), up to exponentially small corrections [29].

There are apparently two serious problems with the result (3.53):

- It would seem to be ruled out by experiment. The ratio of the linear term to the  $T = 0$  term is

$$\Delta = \frac{30\zeta(3)}{\pi^3} aT = 1.16aT, \quad (3.55a)$$

or putting in the numbers ( $300 \text{ K} = (38.7)^{-1} \text{ eV}$ ,  $\hbar c = 197 \text{ MeV fm}$ )

$$\Delta = 0.15 \left( \frac{T}{300 \text{ K}} \right) \left( \frac{a}{1\mu\text{m}} \right), \quad (3.55b)$$

or as Klimchitskaya observed [113], there is a 15% effect at room temperature at a separation of one micron. One would have expected this to have been seen by Lamoreaux [114]; his experiment was reported to be in agreement with the conventional theoretical prediction at the level of 5%. (Lamoreaux [115] is now proposing a new experiment to resolve this issue.)

- Another serious problem is the apparent thermodynamic inconsistency. A linear term in the force implies a linear term in the free energy (per unit area),

$$F = F_0 + \frac{\zeta(3)}{16\pi a^2} T, \quad aT \ll 1, \quad (3.56)$$

which implies a nonzero contribution to the entropy/area at zero temperature:

$$S = - \left( \frac{\partial F}{\partial T} \right)_V = -\frac{\zeta(3)}{16\pi a^2}. \quad (3.57)$$

Taken at face value, this statement appears to be incorrect. We will discuss this problem more closely in section 3.4.3, and will find that although a linear temperature dependence will occur in the free energy at room temperature, the entropy will go to zero as the temperature goes to zero. The point is that the free energy  $F$  for a finite  $\varepsilon$  always will have a zero slope at  $T = 0$ , thus ensuring that  $S = 0$  at  $T = 0$ . The apparent conflict with (3.57) or (3.53) is due to the fact that the curvature of  $F(T)$  near  $T = 0$  becomes infinite when  $\varepsilon \rightarrow \infty$ . So (3.56) and (3.57), corresponding to the modified ideal metal model, describe real metals approximately only for low, but not zero temperature – See the following.

*3.4.1. Lifshitz formula at nonzero temperature* The Casimir surface pressure at finite temperature  $P^T$  between two dielectric plates separated by a distance  $a$  can be obtained from the Lifshitz formula (3.10) by the prescription (3.44b)||

$$P^T = -\frac{1}{\pi\beta} \sum'_{m=0} \int_{\zeta_m}^{\infty} \kappa^2 d\kappa \left[ (A_m^{-1} e^{2\kappa a} - 1)^{-1} + (B_m^{-1} e^{2\kappa a} - 1)^{-1} \right]. \quad (3.58)$$

The relation between  $\kappa$  and the transverse wave vector  $\mathbf{k}_\perp$  is  $\kappa^2 = k_\perp^2 + \zeta_m^2$ , where  $\zeta_m = 2\pi m/\beta$ . Furthermore, the squared reflection coefficients are

$$A_m = \left( \frac{\varepsilon p - s}{\varepsilon p + s} \right)^2, \quad B_m = \left( \frac{s - p}{s + p} \right)^2, \quad (3.59a)$$

$$s^2 = \varepsilon - 1 + p^2, \quad p = \frac{\kappa}{\zeta_m}, \quad (3.59b)$$

with  $\varepsilon(i\zeta_m)$  being the permittivity. Here, the first term in the square brackets in (3.58) corresponds to TM modes, the second to TE modes. Note that whenever  $\varepsilon$  is constant,  $A_m$  and  $B_m$  depend on  $m$  and  $\kappa$  only in the combination  $p$ ,

$$A_m(\kappa) = A(p), \quad B_m(\kappa) = B(p). \quad (3.60)$$

The free energy  $F$  per unit area can be obtained from (3.58) by integration with respect to  $a$  since  $P^T = -\partial F/\partial a$ . We get [117]

$$\beta F = \frac{1}{2\pi} \sum'_{m=0} \int_{\zeta_m}^{\infty} \kappa d\kappa [\ln(1 - \lambda^{\text{TM}}) + \ln(1 - \lambda^{\text{TE}})], \quad (3.61a)$$

where

$$\lambda^{\text{TM}} = A_m e^{-2\kappa a}, \quad \lambda^{\text{TE}} = B_m e^{-2\kappa a}. \quad (3.61b)$$

From thermodynamics the entropy  $S$  and internal energy  $U$  (both per unit area) are related to  $F$  by  $F = U - TS$ , implying

$$S = -\frac{\partial F}{\partial T}, \quad \text{and thus} \quad U = \frac{\partial(\beta F)}{\partial \beta}. \quad (3.62)$$

|| A rederivation of the Casimir force between dissipative metallic mirrors at nonzero temperature has been given by Reynaud, Lambrecht, and Genet [47]. They obtain formulas, generalizing those at zero temperature [116], for the force valid even if the smoothness condition necessary for the derivation of the Lifshitz formula is not satisfied due to the failure of the Poisson summation formula.

As mentioned above the behaviour of  $S$  as  $T \rightarrow 0$  has been disputed, especially for metals where  $\varepsilon \rightarrow \infty$ . We now see the mathematical root of the problem: The quantities  $A_m = B_m \rightarrow 1$  in the  $\varepsilon \rightarrow \infty$  limit except that  $B_0 = 0$  for any finite  $\varepsilon$ . So the question has been whether  $B_0 = 0$  or  $B_0 = 1$  or something in between should be used in this limit as results will differ for finite  $T$ , producing, as we saw above, a difference in the force linear in  $T$ . The corresponding difference in entropy will thus be nonzero. Such a difference would lead to a violation of the third law of thermodynamics, which states that the entropy of a system with a nondegenerate ground state should be zero at  $T = 0$ . Inclusion of the interaction between the plates at different separations cannot change this general property. We will show that this discrepancy vanishes when the limit  $\varepsilon \rightarrow \infty$  is considered carefully.

*3.4.2. Gold as a numerical example* Let us go back to (3.58) for the surface pressure, making use of the best available experimental results for  $\varepsilon(i\zeta)$  as input when calculating the coefficients  $A_m$  and  $B_m$ . We choose gold as an example. Useful information about the real and imaginary parts,  $n'$  and  $n''$ , of the complex permittivity  $n = n' + in''$ , versus the real frequency  $\omega$ , is given in Palik's book [118] and similar sources. The range of photon energies given in Ref. [118] is from 0.1 eV to  $10^4$  eV. (The conversion factor

$$1 \text{ eV} = 1.519 \times 10^{15} \text{ rad/s} \quad (3.63)$$

is useful to have in mind.) When  $n'$  and  $n''$  are known the permittivity  $\varepsilon(i\zeta)$  along the positive imaginary frequency axis, which is a real quantity, can be calculated by means of the Kramers-Kronig relations.

Figure 3 shows how  $\varepsilon(i\zeta)$  varies with  $\zeta$  over seven decades,  $\zeta \in [10^{11}, 10^{18}]$  rad/s. The curve was given in Refs. [77, 119], and is reproduced here for convenience. (We are grateful to A. Lambrecht and S. Reynaud for having given us the results of their accurate calculations.) At low photon energies, below about 1 eV, the data are well described by the Drude model,

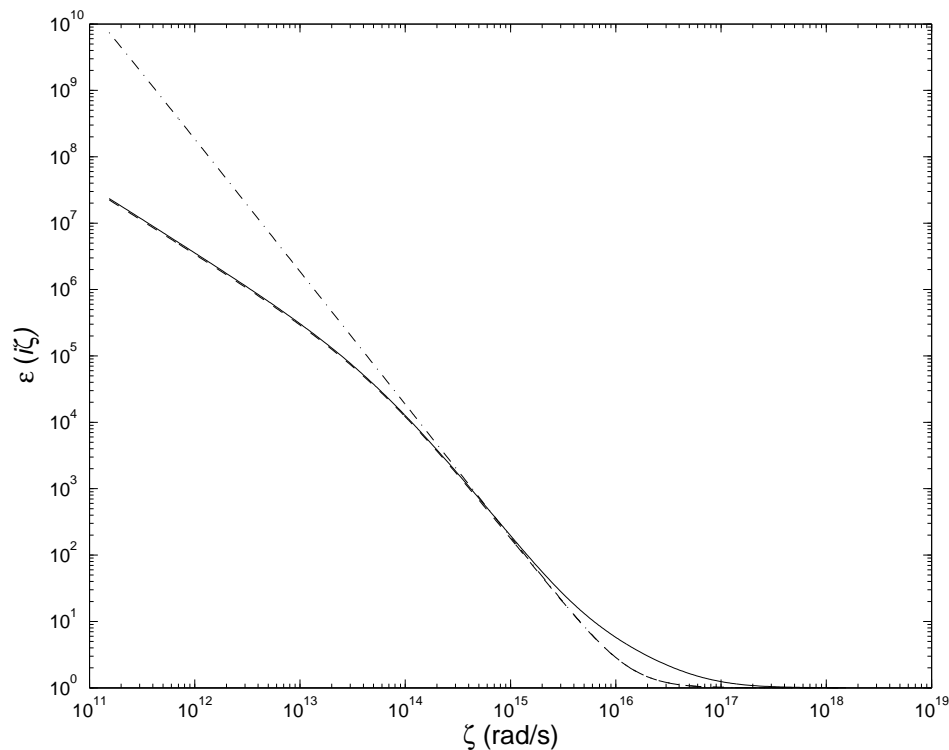
$$\varepsilon(i\zeta) = 1 + \frac{\omega_p^2}{\zeta(\zeta + \nu)}, \quad (3.64)$$

where  $\omega_p$  is the plasma frequency (3.38) and  $\nu$  the relaxation frequency. (Usually,  $\nu$  is taken to be a constant, equal to its room-temperature value, but see below.) The values appropriate for gold at room temperature are [77, 119]

$$\omega_p = 9.0 \text{ eV}, \quad \nu = 35 \text{ meV}. \quad (3.65)$$

The curve in Fig. 3 shows a monotonic decrease of  $\varepsilon(i\zeta)$  with increasing  $\zeta$ , as any permittivity as a function of imaginary frequency has to follow according to thermodynamical requirements. The two dashed curves in the figure show, for comparison, how  $\varepsilon(i\zeta, T)$  varies with frequency if we accept the Drude model for all frequencies, and include the temperature dependence of the relaxation frequency with  $T$  as a parameter. (The latter is given in Fig. 4, according to the Bloch-Grüneisen formula [120], which, however, does not take into account the physical fact that because

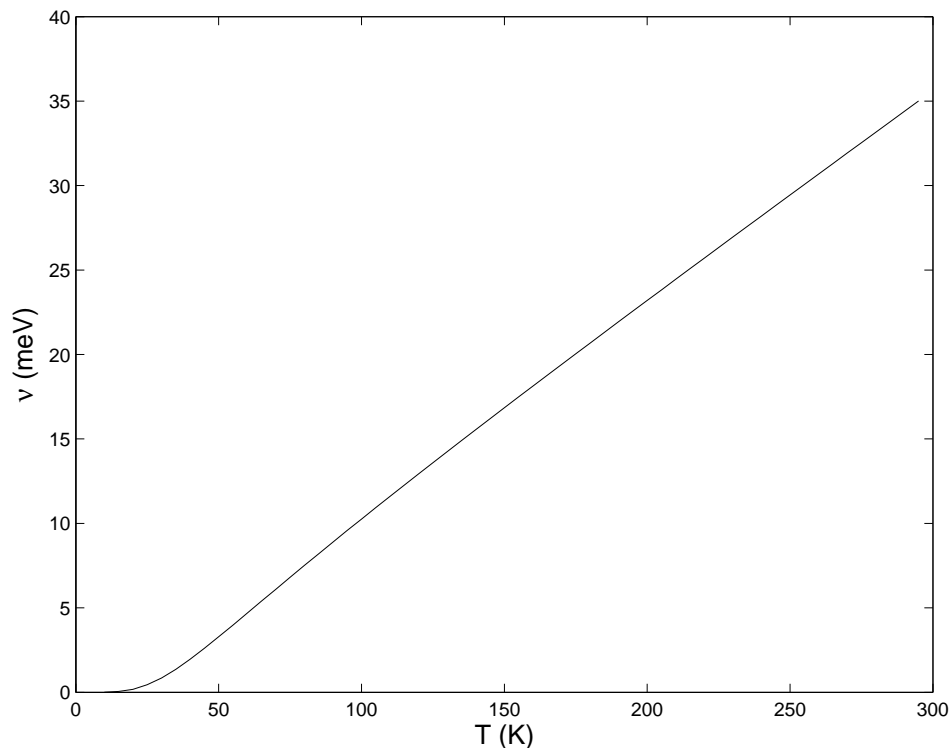




**Figure 3.** Solid line: Permittivity  $\varepsilon(i\zeta)$  as function of imaginary frequency  $\zeta$  for gold. The curve is calculated on the basis of experimental data. Courtesy of Astrid Lambrecht and Serge Reynaud. Dashed lines:  $\varepsilon(i\zeta)$  versus  $\zeta$  with  $T$  as parameter, based upon the temperature dependent Drude model; cf. Appendix D of Ref. [109]. The upper curve is for  $T = 10$  K; the lower is for  $T = 300$  K, which for energies below 1 eV ( $1.5 \times 10^{15}$  rad/s) nicely fits the experimental data. Both curves are below the experimental one for  $\zeta > 2 \times 10^{15}$  rad/s.

of impurities, no actual conductor has zero resistivity at zero temperature [121]. See Appendix D of Ref. [109].) For  $T = 300$  K, the Drude curve is seen to be good for all frequencies up to  $\zeta \sim 2 \times 10^{15}$  rad/s; for higher  $\zeta$  it gives too low values of  $\varepsilon$ . Both Drude curves, for  $T = 10$  K and  $T = 300$  K, are seen to give the same values when  $\zeta \geq 3 \times 10^{14}$  rad/s.

As experiments are usually made at room temperature for various gap widths, we show in Fig. 5 how the surface force density for gold varies with  $a$ , at  $T = 300$  K. The linear slope seen for  $a \geq 4 \mu\text{m}$  is nearly that predicted by (3.54) for high temperatures when the TE zero-mode is excluded (modified ideal metal), which gives a slope of  $2.0 \times 10^{-28}$  Nm<sup>2</sup>/μm. (This is in spite of the fact that  $aT = 0.5$  at  $a = 4 \mu\text{m}$ .) The linear region between 1 and 2 μm corresponds roughly to that in (3.53) (intermediate temperatures). Also shown is the prediction of the temperature dependent Drude model, when  $T = 300$  K. The differences are seen to be very small. Since the Drude values for the permittivity are lower than the empirical ones at high frequencies, as seen in Fig. 3, we expect the predicted Drude forces to be slightly weaker than those based upon the empirical permittivities. This expectation is borne out in Fig. 5; the differences being



**Figure 4.** Temperature dependence of the relaxation frequency for gold based on the Bloch-Grüneisen formula [120].

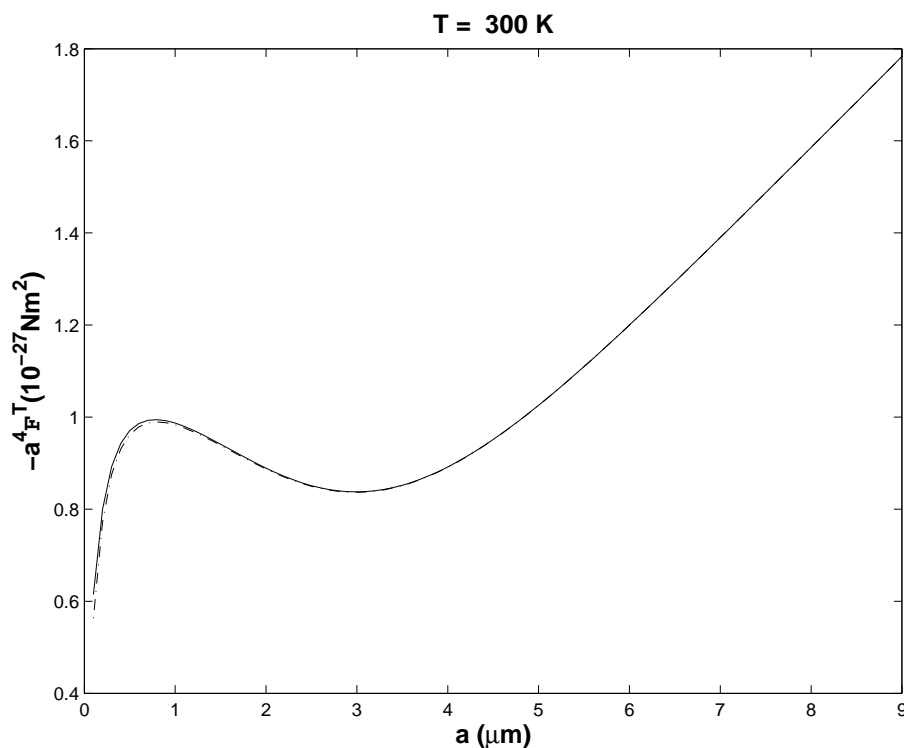
large enough to be slightly visible at short distances, as we would expect since the plasma nature of the material becomes more pronounced for small distances. Note that the temperature dependence of the permittivity is irrelevant here because the temperature is fixed.

It is of interest to check the magnitude of the dispersive effect in these cases. We have therefore made a separate calculation of the expression (3.58) when  $\varepsilon$  is taken to be constant. Figure 6 shows how the force varies with  $aT$  in cases when  $\varepsilon \in \{100, 1000, 10000, \infty\}$  are inserted in the expressions for  $A_m$  and  $B_m$  in (3.59a).

It is seen from the figure that the first three curves asymptotically approach the  $\varepsilon = \infty$  curve, when  $\varepsilon$  increases, as we would expect. Again, we emphasize that the dispersive curve for gold is calculated using the available room-temperature data for  $\varepsilon(i\zeta)$  from Fig. 3. In the nondispersive case, there is of course no permittivity temperature problem since  $\varepsilon$  is taken to be the same for all  $T$ .

There are several points worth noticing from Fig. 6:

- (i) The curves have a horizontal slope at  $T = 0$ . For finite  $\varepsilon$  this property is clearly visible on the curves. This has to be so on physical grounds: If the pressure had a linear dependence on  $T$  for small  $T$  so would the free energy  $F$ , in contradiction with the requirement that the entropy  $S = -\partial F/\partial T$  has to go to zero as  $T \rightarrow 0$ . For the gold data the initial horizontal slope is not resolvable on the scale of this graph, but see the discussion in section 3.4.3.

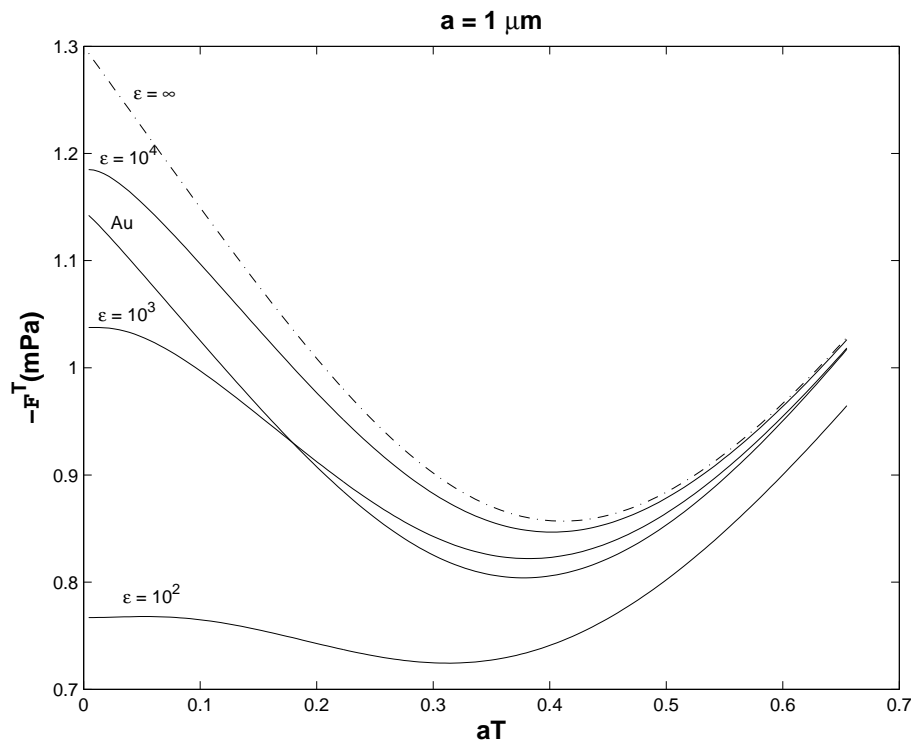


**Figure 5.** Surface pressure for gold, multiplied with  $a^4$ , versus  $a$  when  $T = 300$  K. Input data for  $\varepsilon(i\zeta)$  are taken from Fig. 3.

- (ii) The curves show that the magnitude of the force *diminishes* with increasing  $T$  (for a fixed  $a$ ), in a certain temperature interval up to  $aT \simeq 0.3$ . This perhaps counterintuitive effect is thus clear from the nondispersive curves. This is qualitatively similar to the behavior seen in Fig. 5 for fixed  $T$ , where the minimum occurs for  $aT \sim 0.4$ .
- (iii) It is seen that the curve for  $\varepsilon = \text{const.} = 1000$  gives a reasonably good approximation to the real dispersive curve for gold when  $a = 1 \mu\text{m}$ ; the deviations are less than about 5% except for the lowest values of  $aT$  ( $aT < 0.1$ ). This fact makes our neglect of the temperature dependence of  $\varepsilon(i\zeta)$  appear physically reasonable; the various curves turn out to be rather insensitive with respect to variations in the input values of  $\varepsilon(i\zeta)$ .
- (iv) Also, it can be remarked that  $B_0 = 0$  is required when  $\varepsilon$  is finite. Otherwise the curves in Fig. 6, and thus the free energy, would have a finite slope at  $T = 0$  which again would imply a finite entropy contribution at  $T = 0$  in violation with the third law of thermodynamics.

*3.4.3. Behavior of the Free Energy at Low Temperature* The low temperature correction is dominated by low frequencies,<sup>¶</sup> where the Drude formula is extremely

<sup>¶</sup> This statement is in the context of using of the Euler-Maclaurin summation formula to evaluate (3.58), for example.



**Figure 6.** Nondispersive theory: Surface pressure for  $\varepsilon \in \{100, 1000, 10000, \infty\}$ . For low values of  $aT$  the latter coincides with the expression (3.53). Also shown for comparison is the dispersive result for gold, where experimental input data for  $\varepsilon(i\zeta)$  are taken from Fig. 3. Gap width is  $a = 1 \mu\text{m}$ . The constraint  $a = 1 \mu\text{m}$  applies only to the dispersive case, since otherwise  $a^4 P^T$  is a function of  $aT$  only. Note that room temperature (300 K) corresponds to  $aT = 0.13$ .

accurate. Using this fact, we have performed analytic and numerical calculations which show that the free energy has a quadratic low-temperature dependence, independent of the plate separation:

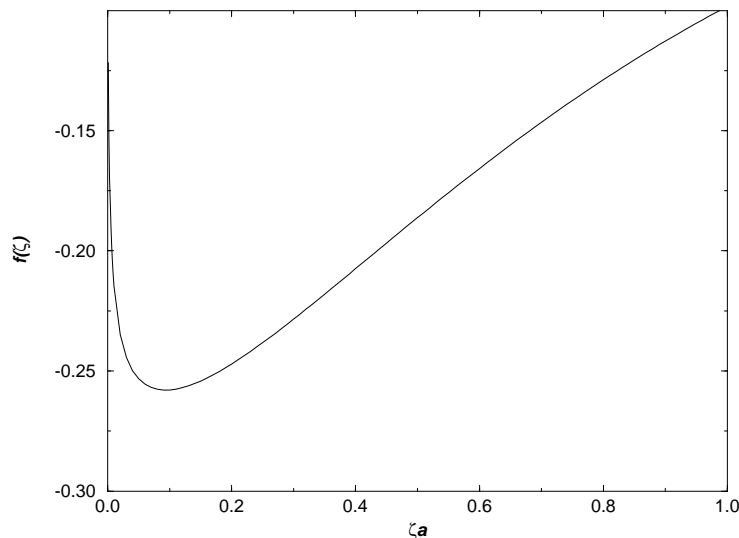
$$F(T) = F_0 + T^2 \frac{\omega_p^2}{48\nu} (2 \ln 2 - 1) = F_0 + T^2 (19 \text{ eV}), \quad T \ll \frac{\nu}{\omega_p^2 a^2} \approx 20 \text{ mK}, \quad (3.66)$$

where we have put in the numbers for gold, (3.65), (the temperature restriction refers to a  $1 \mu\text{m}$  plate separation) rather than the naive extrapolation (3.56)

$$F = F_0 + T \frac{\zeta(3)}{16\pi a^2} = F_0 + \frac{T}{4\pi a^2} 0.30. \quad (3.67)$$

We see from Fig. 7 that the value in (3.67) indeed results if one extrapolates the approximately linear curve there for  $\zeta a > 0.25$  to zero, following the argument given in (3.51). However, we see that the free energy smoothly changes to the quadratic behavior exhibited in (3.66). Of course, the turn-over will be much sharper if we replace the room-temperature relaxation frequency  $\nu(300 \text{ K})$  by the positive value at zero temperature, due to elastic scattering from defects or impurities.

Results consistent with these have been reported by Sernelius and Boström [122]. In particular they show that one cannot ignore the constant value  $\nu(0 \text{ K})$ , so there is



**Figure 7.** The behavior of the free energy for low frequencies, in the Drude model, with parameters suitable for gold, and a plate separation of  $a = 1 \mu\text{m}$ . Here  $F^{\text{TE}} = \frac{T}{2\pi a^2} \sum_{m=0}^{\infty} f(\zeta_m)$ . Here, we have used the room temperature value of the relaxation parameter.

no relevant temperature dependence of the relaxation parameter. Although there is a region of negative entropy, the Nernst heat theorem is not violated, but rather  $S \rightarrow 0$  as  $T \rightarrow 0$  if one goes to sufficiently low temperature, in contradiction to Refs. [123, 45].

*3.4.4. Surface impedance form of reflection coefficient* It has been proposed that the resolution to the temperature problem for the Casimir effect is that the surface impedance form of the reflection coefficients should be used in the Lifshitz formula [124, 125, 126, 127], rather than that based on the bulk permittivity. Here we show that the two approaches are in fact equivalent, and that the former must include transverse momentum dependence.

For the TE modes, the reflection coefficient is given by (3.4a) [75]

$$r^{\text{TE}} = -\frac{k_{1z} - k_{2z}}{k_{1z} + k_{2z}}, \quad (3.68)$$

where

$$k_{az} = \sqrt{\omega^2 \varepsilon - k_{\perp}^2} \rightarrow i\sqrt{\zeta^2 [\varepsilon(i\zeta) - 1] + \kappa^2} = i\kappa_a, \quad (3.69)$$

with  $\kappa^2 = \kappa_2^2 = k_{\perp}^2 + \zeta^2$ , and the subscripts 1 and 2 refer to the metal and the vacuum regions, respectively. Now from Maxwell's equations outside sources we easily derive just inside the metal (the tangential components, designated by  $\perp$ , of  $\mathbf{E}$  and  $\mathbf{B}$  are continuous across the interface)

$$-ik_{1z} \mathbf{k}_{\perp} \cdot \mathbf{B}_{\perp} - i\omega \varepsilon \left(1 - \frac{k_{\perp}^2}{\omega^2 \varepsilon}\right) \mathbf{k}_{\perp} \cdot (\mathbf{n} \times \mathbf{E}_{\perp}) = 0, \quad (3.70a)$$

$$-ik_{1z}\mathbf{k}_\perp \cdot (\mathbf{n} \times \mathbf{E}_\perp) - i\omega\mathbf{k}_\perp \cdot \mathbf{B}_\perp = 0. \quad (3.70b)$$

Here  $\mathbf{n}$  is the normal to the interface. Now the surface impedance is defined by

$$\mathbf{E}_\perp = Z(\omega, \mathbf{k}_\perp)\mathbf{B}_\perp \times \mathbf{n}, \quad \mathbf{n} \times \mathbf{E}_\perp = Z(\omega, \mathbf{k}_\perp)\mathbf{B}_\perp. \quad (3.71)$$

So eliminating  $\mathbf{B}_\perp$  using this definition we find two equations:

$$k_{1z} = -\frac{\omega}{Z}, \quad (3.72)$$

$$k_{1z}^2 = \omega^2\varepsilon - k_\perp^2, \quad (3.73)$$

the latter being the expected dispersion relation (3.69). Substituting this into the expression for the reflection coefficient (3.68) we find

$$r^{\text{TE}} = -\frac{\zeta + Z\kappa}{\zeta - Z\kappa} = -\frac{1 + Zp}{1 - Zp}, \quad p = \frac{\kappa}{\zeta}, \quad (3.74)$$

which apart from (relative) signs (presumably just a different convention choice) coincides with that given in Geyer *et al* [124] or Bezerra *et al* [128]. See also Refs. [129, 130]. The first discussion of the Lifshitz formula in this approach was given in Ref. [102].

However, it is crucial to note that the “surface impedance” so defined depends on the transverse momentum,

$$Z = -\frac{\zeta}{\sqrt{\zeta^2[\varepsilon(i\zeta) - 1] + \kappa^2}}, \quad (3.75)$$

and so  $r^{\text{TE}} \rightarrow 0$  as  $\zeta \rightarrow 0$  just as in the dielectric constant formulation. Of course, we have exactly the same result for the energy as before, since this is nothing but a slight change of notation, as noted in Ref. [131, 36].

It is therefore incorrect to assume that  $Z$  is only a function of frequency, not of transverse momentum, and to use the normal and anomalous skin effect formulas derived for real waves impinging on imperfect conductors.<sup>+</sup> In the above-cited references, this necessary dependence was not included. (For further comments on the insufficiency of the argument in Ref. [124] see Ref. [132].)

How does the usual argument go? The normal component of the wavevector in a conductor is given by

$$k_z = \left[ \omega^2 \left( \varepsilon + i\frac{4\pi\sigma}{\omega} \right) - k_\perp^2 \right]^{1/2} \rightarrow \sqrt{i4\pi\omega\sigma}, \quad \omega \rightarrow 0, \quad (3.76)$$

from which the usual normal skin effect formula follows immediately,

$$Z(\omega) = -(1 - i)\sqrt{\frac{\omega}{8\pi\sigma}}. \quad (3.77)$$

However, the limit in (3.76) here consists in omitting two “small” terms:  $\omega^2\varepsilon$  (which is legitimate) and  $k_\perp^2 \leq \omega^2$ . Here this last is not valid because in going to finite temperature

<sup>+</sup> Of course, in general, the permittivity will be a function both of the frequency and the transverse momentum,  $\varepsilon(\omega, \mathbf{k}_\perp)$ , but we believe the latter dependence is not significant for separations larger than  $\hbar c/\omega_p = 0.02 \mu\text{m}$ .

we have severed the connection between  $\omega \rightarrow i\zeta$  and  $k_\perp$ ; the latter is in no sense ignorable as we take  $\zeta \rightarrow 0$  to determine the low temperature dependence. This is the same error to which we refer in Ref. [109]. (This  $k_\perp$  dependence still seems to be ignored in a recent reanalysis by Torgerson and Lamoreaux [133] (see also Ref. [134]) who argue that low frequencies of order of the inverse transverse size of the plates dominate the low temperature behavior so that a linear term in the temperature does not appear. This seems unlikely since the zero-temperature dependence is extracted by an analytic continuation procedure.)

Not only do Mostepanenko, Klimchitskaya, *et al* [129, 124] ignore transverse momentum dependence, but they apparently do not use the correct values of the frequency in their evaluation of the surface impedance. They use the impedance appropriate to the domain of infrared optics, thereby extrapolating the surface impedance at what they consider a characteristic frequency  $\sim 1/2a$  rather than using the actual zero frequency value [126]. This seems to be a completely *ad hoc* prescription, as opposed to the procedure advocated in Brevik *et al* [109], which uses the actual electrical properties of the materials.

A beginning of a general discussion of nonlocal effects, including the anomalous skin effect, in Casimir phenomena has recently been given by Esquivel and Svetovoy [135]. There they argue that the Leontovich approach [136, 137] advocated by [129, 124] only applies to normal incidence, which is why the surface impedances only depend on frequency. In fact, this is incorrect in general, and if only local functions are used for the permittivity, that is  $\varepsilon = \varepsilon(\omega)$ , the dependence for the TE surface impedance given above is reproduced. For propagating waves the Leontovich approximation is appropriate, but not for the evanescent fields relevant to the Casimir effect, where  $k_\perp/\omega > 1$  occur. They do not calculate temperature effects; the nonlocal anomalous skin effect for  $\omega < \omega_p$  that they compute gives a correction to the Casimir force of order 0.5%, but other nonlocal effects, such as plasmon excitations, could be more significant [138, 139].

### 3.5. Beyond the Proximity Approximation

As we will discuss in the next section, to avoid problems of parallelism, most recent experiments to measure the force between conductors have not been made between parallel plates, but between a plate and a spherical surface, or between crossed cylinders. The Lifshitz and Casimir formulas do not apply to these situations. However, in the 1930s, it was recognized that if the separation between the sphere and the plate is very small compared to the radius of curvature of the sphere, the latter force may be derived from the force for the parallel plate configuration. This result is usually called the Proximity Force Theorem [140], which here says that the attractive force  $\mathcal{F}$  between a sphere of radius  $R$  and a flat surface is simply the circumference of the sphere times the energy per unit area for parallel plates, or, from (2.29),

$$\mathcal{F} = 2\pi R \mathcal{E}(d) = -\frac{\pi^3}{360} \frac{R \hbar c}{d^2}, \quad R \gg d, \quad (3.78)$$

where  $d$  is the distance between the plate and the sphere at the point of closest approach, and  $R$  is the radius of curvature of the sphere at that point. (The exact shape of the “sphere” is not relevant in the strict approximation  $R \gg d$ .) The proof of (3.78) is quite simple. If  $R \gg d$ , each element of the sphere may be regarded as parallel to the plane, so the potential energy of the sphere is

$$V(d) = \int_0^\pi 2\pi R \sin \theta R d\theta \mathcal{E}(d + R(1 - \cos \theta)) = 2\pi R \int_{-R}^R dx \mathcal{E}(d + R - x). \quad (3.79)$$

To obtain the force between the sphere and the plate, we differentiate with respect to  $d$ :

$$\begin{aligned} \mathcal{F} &= -\frac{\partial V}{\partial d} = 2\pi R \int_{-R}^R dx \frac{\partial}{\partial x} \mathcal{E}(d + R - x) \\ &= 2\pi R[\mathcal{E}(d) - \mathcal{E}(d + 2R)] \approx 2\pi R \mathcal{E}(d), \quad d \ll R, \end{aligned} \quad (3.80)$$

provided that  $\mathcal{E}(a)$  falls off with  $a$ . This result was already given in Refs. [141, 142, 143]. The proximity theorem itself dates back to a paper by Derjaguin in 1934 [144, 145].

Let us apply this theorem to the MIM model (3.53) for the force between parallel plates at low temperature. The corresponding free energy is

$$F = -\frac{\pi^2}{720a^3} + \frac{\pi^2}{45}aT^4 - \frac{\zeta(3)}{2\pi}T^3 + \frac{\zeta(3)}{16\pi a^2}T, \quad (3.81)$$

where the term constant in  $a$  is determined by the high-temperature limit (3.54) – see Ref. [29], p. 56. This free energy is to be used in the proximity force theorem, with the result for the force between a sphere and a plate [39, 114, 146, 10]

$$\mathcal{F} = -\frac{\pi^3}{360} \frac{R}{d^3} \left[ 1 - 16(Td)^4 + \frac{360\zeta(3)}{\pi^3}(Td)^3 - \frac{45\zeta(3)}{\pi^3}Td \right]. \quad (3.82)$$

The terms linear in  $T$  would not be present in the IM model. At room temperature, 300 K, and at 1  $\mu\text{m}$  separation, the successive terms correspond to corrections of  $-0.46\%$ ,  $+3.1\%$ , and  $-23\%$ , respectively. This model, of course, does not begin to reflect the true temperature dependence, discussed for parallel plates above. A full discussion of the temperature dependence for the force between a spherical lens and a plate will appear elsewhere.

Emig has recently presented exact results for Casimir forces between periodically deformed surfaces [147, 148, 149]. In the latest paper, the authors calculate the force between a flat plate and one with a rectangular (square) corrugation, of amplitude  $\Delta a$ . This was probed experimentally by Roy and Mohideen [150], with clear deviations from the proximity approximation. (See also Refs. [151, 152] for measurements of the so-called “lateral Casimir effect.”) For short wavelength corrugations for either TE or TM modes one gets

$$P = -\frac{\pi^2}{480} \frac{1}{(a - \Delta a)^4} \approx -\frac{\pi^2}{480a^4} \left( 1 + \frac{4\Delta a}{a} \right), \quad (3.83)$$

while for long wavelength corrugations

$$P = -\frac{\pi^2}{480} \frac{1}{2} \left( \frac{1}{(a - \Delta a)^4} + \frac{1}{(a + \Delta a)^4} \right), \quad (3.84)$$



which is as expected from the proximity approximation. For intermediate wavelength corrugations numerical results are given. The force approaches that given by the proximity approximation for large  $\lambda$  like  $\Delta a/\lambda$ , as compared to  $(\Delta a/\lambda)^2$  for sinusoidal corrugations, due to the sharp edges. These behaviors can be understood from the ray optics approach of Jaffe and Scardicchio [23] discussed in the following subsection. The relative contributions of the TE and TM modes vary with the wavelength and the shape of the corrugation, the ratio of the modes approaching unity as  $a/\Delta a$  tends to 1 or  $\infty$ . Insofar as first approximations to these interactions were extracted through use of the proximity force theorem, these results shed valuable light on how to move beyond that approximation.

*3.5.1. Optical Paths* A very interesting strategy for moving beyond the proximity approximation has been suggested by Jaffe and Scardicchio [23]. This is related to the semiclassical closed orbit approach advocated by Schaden and Spruch [153, 154, 155] and earlier by Gutzweiler [156, 157], and also to that of Balian and Bloch [158, 159, 160]. Fulling has also recently proposed similar ideas [161, 162].

In the simplest context, that of parallel plates, the approach is, of course, exact, and is precisely what we wrote down in (3.6). We simply compute the energy using (2.15) with

$$\mathcal{G}(\mathbf{r}, \mathbf{r}) = \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} g(x, x), \quad (3.85)$$

where  $g(x, x')$  is given by (3.6). Rather than carry out the sum as given there, let us sum the terms with even and odd numbers of reflections separately. The former give, when the zero reflection term is omitted,

$$g_{\text{even}}(x, x) = \frac{1}{2\kappa} 2 [\tilde{r}' r e^{-2\kappa a} + (\tilde{r}' r)^2 e^{-4\kappa a} + \dots] = \frac{1}{2\kappa} (\coth \kappa a - 1), \quad (3.86)$$

where in the last step we have inserted the values for the reflection amplitudes appropriate to Dirichlet boundaries,  $r = \tilde{r}' = -1$ . When this is inserted into the expression for the energy we obtain rather immediately the usual result for the Casimir energy between Dirichlet plates:

$$\mathcal{E} = -\frac{1}{96\pi^2 a^3} \int_0^\infty du \frac{u^3}{e^u - 1} = -\frac{\pi^2}{1440a^3}. \quad (3.87)$$

Keeping only the first term in the sum (2 reflections) gives

$$\mathcal{E}^{(2)} = -\frac{1}{16\pi^2 a^3}, \quad (3.88)$$

which is in magnitude only 7.6% low, while keeping 2 plus 4 reflections give an error of 1.8%:

$$\mathcal{E}^{(2)} + \mathcal{E}^{(4)} = -\frac{1}{16\pi^2 a^3} \left(1 + \frac{1}{16}\right). \quad (3.89)$$

The odd reflections give a term in  $g(x, x)$  which depends on  $x$ :

$$g_{\text{odd}}(x, x) = -\frac{1}{2\kappa} (e^{-2\kappa x} + e^{2\kappa(x-a)}) \frac{1}{1 - e^{-2\kappa a}}. \quad (3.90)$$

when this is integrated over  $x$ , the  $a$  dependence of this term disappears, so this gives rise to an irrelevant constant in the energy. Keeping it and the zero-reflection term gives the expression for the total energy as obtained directly from (3.6)

$$\mathcal{E}_0 + \mathcal{E}_{\text{even}} + \mathcal{E}_{\text{odd}} = -\frac{1}{12\pi^2} \frac{1}{a^3} \int_0^\infty dy y^3 \left( \coth y - \frac{1}{y} \right). \quad (3.91)$$

Jaffe and Scardicchio [23] use this method to estimate the force between a sphere and a plate. The results disagree with the proximity approximation when  $d/R$  is bigger than a few percent, but agrees with an exact numerical calculation [163], described in the following subsection, up to  $d/R \approx 0.1$ , where the proximity theorem fails badly.

*3.5.2. Worldline Approach to the Casimir Energy* Gies, Moyaerts, and Langfeld [163, 164] have developed a numerical technique for extracting Casimir energies in nontrivial geometries, such as between a sphere and a plate. It is based on the string-inspired worldline approach. They consider, like Graham *et al* [165, 61, 30] a scalar field in a smooth background potential like (2.40). The worldline representation of the effective action is obtained by introducing a proper time representation of the functional logarithm with ultraviolet regularization, doing the trace in configuration space, and interpreting the matrix element there as a Feynman path integral over all worldlines  $x(\tau)$ . Field theoretic divergences can thus be handled. Other divergences arise from the potential itself, when it approaches some idealized limit, which may not be removed in a physically meaningful way and may or may not contribute to physical observables. The expectation value is evaluated by the “loop-cloud” method, using techniques from statistical mechanics. Although in the “sharp” and “strong” limits in the sense of Graham *et al* [165, 61, 30] divergences occur in the theory, a finite *force* between rigid bodies can be obtained. The general result for  $\delta$ -function planes, discussed in section 2.2, is reproduced numerically, and then the sphere-plate system is considered. The numerical results, for  $d/R$  from  $10^{-3}$  to 10, agree closely with the *geometric mean*\* of the plate-based and the sphere-based proximity force approximation (deviation from either becomes sizable for  $d/R > 0.02$ ). Note that electromagnetic fluctuations (e.g., TM modes) have not been considered in this approach.

### 3.6. Status of the Experimental Measurements on Casimir Forces

Attempts to measure the Casimir effect between solid bodies date back to the middle 1950s. The early measurements were, not surprisingly, somewhat inconclusive [142, 143, 168, 169, 170, 171, 172, 173, 174, 175]. The Lifshitz theory (3.10), for zero temperature, was, however, confirmed accurately in the experiment of Sabisky and

\* The geometric mean version of the proximity force approximation, which coincides with the semiclassical periodic orbit method of Schaden and Spruch [153, 154, 155], has been found to be the most accurate also for concentric cylindrical shells, the Casimir energy for which was calculated by Mazzitelli *et al* [166, 167].

Anderson in 1973 [18]. So there could be no serious doubt of the reality of zero-point fluctuation forces. For a review of the earlier experiments, see Refs. [15, 176].

New technological developments allowed for dramatic improvements in experimental techniques in recent years, and thereby permitted nearly direct confirmation of the Casimir force between parallel conductors. First, in 1997 Lamoreaux used a electromechanical system based on a torsion pendulum to measure the force between a conducting plate and a sphere [114, 146], as given by the proximity force theorem (3.78). Lamoreaux [114, 146] claimed agreement with this theoretical value at the 5% level, although it seems that finite conductivity was not included correctly, nor were roughness corrections incorporated [177]. Further, Lambrecht and Reynaud [77] analyzed the effect of conductivity and found discrepancies with Lamoreaux [178], and therefore stated that it was too early to claim agreement between theory and experiment. See also Refs. [119, 179].

An improved experimental measurement was reported in 1998 by Mohideen and Roy [97], based on the use of an atomic force microscope. They included finite conductivity, roughness, and conventional temperature corrections, although no evidence for latter has been claimed. Spectacular agreement with theory at the 1% level was attained. Improvements were subsequently reported [98, 99]. (The nontrivial effects of corrugations in the surface were examined in Ref. [150, 151, 152].) Erdeth [180] measured the Casimir forces between crossed cylinders at separations of 20–100 nm. The highest precision was achieved with very smooth, gold-plated surfaces. Rather complete analyses of the roughness, conductivity, and temperature corrections to the Lamoreaux and Mohideen experiments have been published [181, 182, 39].

More recently, a new measurement of the Casimir force (3.78) was presented by a group at Bell Labs [183, 184], using a micromachined torsional device, a microelectromechanical system or MEMS, by which they measured the attraction between a polysilicon plate and a spherical metallic surface. Both surfaces were plated with a 200 nm film of gold. The authors included finite conductivity [77, 185] and surface roughness corrections [186, 187], and obtained agreement with theory at better than 0.5% at the smallest separations of about 75 nm. However, potential corrections of greater than 1% exist, so that limits the level of verification of the theory. Their experimental work, which now continues at Harvard, suggests novel nanoelectromechanical applications.

There is only one experiment with a parallel-plate geometry [188], which is of limited accuracy ( $\sim 15\%$ ) due to the difficulty of maintaining parallelism. It is, however, of considerable interest because the interpretation does not depend on the proximity theorem, corrections to which are problematic [148, 23]; see section 3.5. The importance of improving the accuracy of the parallel-plate configuration has been emphasized by Onofrio [189].

The most precise experiment to date, using a MEMS, makes use of both static and dynamical procedures and yields a claimed accuracy of about 0.25% [100, 190], but this accuracy has been disputed [191], due to difficulty in controlling roughness and the concomitant uncertainty in the ability to determine the separation distance.

It has been asserted [190] that this experiment rules out the temperature dependence claimed in Ref. [109] (see section 3.4.2), but this is problematic at this point, especially as comparison is only made with the MIM model (3.53), rather than with the detailed calculation given there.

Very recently, the Harvard group has performed a very interesting Casimir force measurement between a gold-covered plate and a sphere coated with a hydrogen-switchable mirror [192]. Although the mirror becomes transparent in the visible upon hydrogenation, no effect was observed on the Casimir force when the mirror was switched on and off. This shows that, in contradiction to the claims of Mostepanenko *et al*, for example in Ref. [127], the Casimir force is responsive to a very wide range of frequencies, in accordance with the Lifshitz formula and the general dispersion relation for the permittivity.‡ (See also Ref. [193, 194].) In particular, their results show that wavelengths much larger than the separation between the surfaces play a crucial role.

Because all the recent experiments measure forces between relatively thin films, rather than between bulk metals, significant deviations from the Lifshitz formula ( $\sim 2\%$ ) may be expected [195]. This may also be relevant to the claimed accuracy of the first Mohideen experiment [97], which uses a thin metallic coating, regarded as completely transparent.

This may be an appropriate point to comment on the recent paper of Chen *et al* [95]. This is based on a reanalysis of experimental data obtained four years ago in Ref. [99]. Experimental precision of 1.75% and theoretical accuracy of 1.69% is claimed at the shortest distances, 62 nm. However, their analysis seems flawed. They obtain average experimental forces by averaging many measurements, which is only permissible if the averaging is carried out at exactly the same separation between the surfaces. Of course they have no way of knowing this. Furthermore, they apparently use the mean separation parameter  $d_0$  as a free variable in their fit, which essentially negates the possibility of testing the theory, which is most sensitive at the shortest separations [180]. Iannuzzi asserts that at distance of order 100 nm, errors of a few Ångstroms preclude a 1% measurement. Therefore this analysis cannot be used as a serious constraint for either new forces or for setting limits on temperature corrections.††

A difference force experiment has been proposed by Mohideen and collaborators [196, 197]. The idea is to measure the difference in the force between a lens and a plate at room temperature, before and after both surfaces have been heated 50 K by a laser pulse. The measurements are not yet good enough to distinguish between the plasma and the Drude modes of the permittivity, or between the simplified impedance model versus the measured bulk permittivity approach, as discussed in section 3.4.

A proposal has been made to measure the force between eccentric cylinders, in which the axes are parallel but slightly offset [167]. The net force on the inner cylinder is zero,

‡ Iannuzzi quotes Klimchitskaya as now agreeing with this statement. This, however, is hard to reconcile with statements made in Ref. [126] that one should use the extrapolated surface impedance value at  $\omega_c = 1/2a$  rather than the actual zero-frequency value.

†† I thank Davide Iannuzzi for discussion of these points.

of course, when the cylinders are concentric, but this equilibrium point is unstable. The idea is to look for a shift in the mechanical resonant frequency of the outer cylinder due to the Casimir force exerted by the inner one. The chief difficulty may be in maintaining parallelism.

Another active area of experimental effort involving Casimir measurements is the search for new forces at the submicron level. These are based on looking for a discrepancy between the measured and predicted Casimir forces. The most recent limits are given in Krause, Decca, *et al* [198, 100]. Unfortunately, the limits, for an assumed potential of the form

$$V(r) = -\frac{Gm_1m_2}{r} (1 + \alpha e^{-r/\lambda}), \quad (3.92)$$

for  $\lambda < 10^{-7}$  m are only for absurdly large strengths,  $\alpha \leq 10^{14}$ , and as  $\lambda$  decreases the upper limit on  $\alpha$  increases. The Purdue group has also proposed iso-electronic experiments to look at the force between a sphere and two different plates, composed of material with similar electronic properties (and hence similar Casimir forces) but different nuclear properties (and hence presumably different new forces). See Ref. [199] for a brief description of their experiment and the detection of a small, but probably not significant, residual force.

Very recently, there has been a report [200] of an experiment [201] of dropping ultracold neutrons onto a surface. They are trapped between the mirror and the earth's gravitational potential. These gravitational bound states would be modified by any deviation from Newtonian gravity. No such deviations from Newton's law is found down to the 1–10 nm range. See also Nesvizhevsky and Protasov [202] who obtain limits on non-Newtonian forces inferior to those of Casimir measurements, that is,  $\alpha < 10^{21}$  at  $\lambda = 10^{-7}$  m, although it is relatively better that the Casimir limits in the nanometer range, but the limits are extremely weak there,  $\alpha < 10^{26}$ .

It is clear that as micro engineering comes into its own, Casimir forces will have to be taken into account and utilized. A recent interesting paper by Chumak, Milonni, and Berman [203] suggests that the noncontact friction observed by Stipe *et al* [204] on a cantilever near a surface is due in major part to Casimir forces. The Casimir force is responsible for the frequency shift observed of about 4.5% for a gold sample at a separation of 2 nm.

For another example along these lines, Lin *et al* [205] have shown that Casimir-Polder forces between atoms and the surface can provide fundamental limitations on stability of a Bose-Einstein condensate near a microfabricated silicon chip, a system which holds great promise for technological applications.

The recent intense experimental activity is very encouraging to the development of the field. Coming years, therefore, promise ever increasing experimental input into a field that has been dominated by theory for five decades.

## 4. Self-Stress

### 4.1. Surface and Volume Divergences

It is well known that in general the Casimir energy density diverges in the neighborhood of a surface. For flat surfaces and conformal theories (such as the conformal scalar theory considered in Ref. [57], or electromagnetism) those divergences are not present.‡ We saw hints of this in section 2.4. In particular, Brown and Maclay [209] calculated the local stress tensor for two ideal plates separated by a distance  $a$  along the  $z$  axis, with the result for a conformal scalar

$$\langle T^{\mu\nu} \rangle = -\frac{\pi^2}{1440a^4} [4z^\mu z^\nu - g^{\mu\nu}]. \quad (4.1)$$

This result was given recent rederivations in [210, 57]. Dowker and Kennedy [211] and Deutsch and Candelas [212] considered the local stress tensor between planes inclined at an angle  $\alpha$ , with the result, in cylindrical coordinates  $(t, r, \theta, z)$ ,

$$\langle T^{\mu\nu} \rangle = -\frac{f(\alpha)}{720\pi^2 r^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (4.2)$$

where for a conformal scalar, with Dirichlet boundary conditions,

$$f(\alpha) = \frac{\pi^2}{2\alpha^2} \left( \frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2} \right), \quad (4.3)$$

and for electromagnetism, with perfect conductor boundary conditions,

$$f(\alpha) = \left( \frac{\pi^2}{\alpha^2} + 11 \right) \left( \frac{\pi^2}{\alpha^2} - 1 \right). \quad (4.4)$$

For  $\alpha \rightarrow 0$  we recover the pressures and energies for parallel plates, (2.25*b*), (2.39) and (3.87). (These results were later discussed in Ref. [213].)

Although for perfectly conducting flat surfaces, the energy density is finite, for electromagnetism the individual electric and magnetic fields have divergent RMS values,

$$\langle E^2 \rangle \sim -\langle B^2 \rangle \sim \frac{1}{\epsilon^4}, \quad \epsilon \rightarrow 0, \quad (4.5)$$

a distance  $\epsilon$  above a conducting surface. However, if the surface is a dielectric, characterized by a plasma dispersion relation (3.33), these divergences are softened

$$\langle E^2 \rangle \sim \frac{1}{\epsilon^3}, \quad -\langle B^2 \rangle \sim \frac{1}{\epsilon^2}, \quad \epsilon \rightarrow 0, \quad (4.6)$$

so that the energy density also diverges [214]

$$\langle T^{00} \rangle \sim \frac{1}{\epsilon^3}, \quad \epsilon \rightarrow 0. \quad (4.7)$$

‡ In general, this need not be the case. For example, Romeo and Saharian [206] show that with mixed boundary conditions the surface divergences need not vanish for parallel plates. For additional work on local effects with mixed (Robin) boundary conditions, applied to spheres and cylinders, and corresponding global effects, see Refs. [207, 208, 70]. See also section 2.4 and Ref. [72, 73].

The null energy condition ( $n_\mu n^\mu = 0$ )

$$T^{\mu\nu} n_\mu n_\nu \geq 0 \quad (4.8)$$

is satisfied, so that gravity still focuses light.

Graham [215] examined the general relativistic energy conditions required by causality. In the neighborhood of a smooth domain wall, given by a hyperbolic tangent, the energy is always negative at large enough distances. Thus the weak energy condition is violated, as is the null energy condition (4.8). However, when (4.8) is integrated over a complete geodesic, positivity is satisfied. It is not clear if this last condition, the Averaged Null Energy Condition, is always obeyed in flat space. Certainly it is violated in curved space, but the effects always seem small, so that exotic effects such as time travel are prohibited.

However, as Deutsch and Candelas [212] showed many years ago, in the neighborhood of a curved surface for conformally invariant theories,  $\langle T_{\mu\nu} \rangle$  diverges as  $\epsilon^{-3}$ , where  $\epsilon$  is the distance from the surface, with a coefficient proportional to the sum of the principal curvatures of the surface. In particular they obtain the result, in the vicinity of the surface,

$$\langle T_{\mu\nu} \rangle \sim \epsilon^{-3} T_{\mu\nu}^{(3)} + \epsilon^{-2} T_{\mu\nu}^{(2)} + \epsilon^{-1} T_{\mu\nu}^{(1)}, \quad (4.9)$$

and obtain explicit expressions for the coefficient tensors  $T_{\mu\nu}^{(3)}$  and  $T_{\mu\nu}^{(2)}$  in terms of the extrinsic curvature of the boundary.

For example, for the case of a sphere, the leading surface divergence has the form, for conformal fields, for  $r = a + \epsilon$ ,  $\epsilon \rightarrow 0$

$$\langle T_{\mu\nu} \rangle = \frac{A}{\epsilon^3} \begin{pmatrix} 2/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin \theta \end{pmatrix}, \quad (4.10)$$

in spherical polar coordinates, where the constant is  $A = 1/1440\pi^2$  for a scalar, or  $A = 1/120\pi^2$  for the electromagnetic field. Note that (4.10) is properly traceless. The cubic divergence in the energy density near the surface translates into the quadratic divergence in the energy found for a conducting ball [216]. The corresponding quadratic divergence in the stress corresponds to the absence of the cubic divergence in  $\langle T_{rr} \rangle$ .

This is all completely sensible. However, in their paper Deutsch and Candelas [212] expressed a certain skepticism about the validity of the result of Ref. [24] for the spherical shell case (described in part in section 4.4) where the divergences cancel. That skepticism was reinforced in a later paper by Candelas [217], who criticized the authors of Ref. [24] for omitting  $\delta$  function terms, and constants in the energy. These objections seem utterly without merit. In a later critical paper by the same author [218], it was asserted that errors were made, rather than a conscious removal of unphysical divergences.

Of course, surface curvature divergences are present. As Candelas noted [217, 218], they have the form

$$E = E^S \int dS + E^C \int dS (\kappa_1 + \kappa_2) + E_I^C \int dS (\kappa_1 - \kappa_2)^2 + E_{II}^C \int dS \kappa_1 \kappa_2 + \dots, (4.11)$$

where  $\kappa_1$  and  $\kappa_2$  are the principal curvatures of the surface. The question is to what extent are they observable. After all, as has been shown in Ref. [29, 57] and in section 2.4, we can drastically change the local structure of the vacuum expectation value of the energy-momentum tensor in the neighborhood of flat plates by merely exploiting the ambiguity in the definition of that tensor, yet each yields the same finite, observable (and observed!) energy of interaction between the plates. For curved boundaries, much the same is true. *A priori*, we do not know which energy-momentum tensor to employ, and the local vacuum-fluctuation energy density is to a large extent meaningless. It is the global energy, or the force between distinct bodies, that has an unambiguous value. It is the belief of the author that divergences in the energy which go like a power of the cutoff are probably unobservable, being subsumed in the properties of matter. Moreover, the coefficients of the divergent terms depend on the regularization scheme. Logarithmic divergences, of course, are of another class [59].

Dramatic cancellations of these curvature terms can occur. It might be thought that the reason a finite result was found for the Casimir energy of a perfectly conducting spherical shell [19, 22, 24] is that the term involving the squared difference of curvatures in (4.11) is zero only in that case. However, for reasons not yet apparent to the present author, it has been shown that at least for the case of electromagnetism the corresponding term is not present (or has a vanishing coefficient) for an arbitrary smooth cavity [219], and so the Casimir energy for a perfectly conducting ellipsoid of revolution, for example, is finite. This finiteness of the Casimir energy (usually referred to as the vanishing of the second heat-kernel coefficient [10]) for an ideal smooth closed surface was anticipated already in Ref. [22], but contradicted by Ref. [212]. More specifically, although odd curvature terms cancel inside and outside for any thin shell, it would be anticipated that the squared-curvature term, which is present as a surface divergence in the energy density, would be reflected as an unremovable divergence in the energy. For a closed surface the last term in (4.11) is a topological invariant, so gives an irrelevant constant, while no term of the type of the penultimate term can appear due to the structure of the traced cylinder expansion [70]. It would be extraordinarily interesting if this Casimir energy could be computed for an ellipsoidal boundary, but the calculation appears extremely difficult because the Helmholtz equation is not separable in the exterior region.

#### 4.2. Casimir Forces on Spheres via $\delta$ -Function Potentials

This section is an adaptation and an extension of calculations presented in Ref. [62]. This investigation was carried out in response to the program of the MIT group [165, 60, 61, 71, 30]. They rediscovered irremovable divergences in the Casimir energy for



a circle in 2+1 dimensions first discovered by Sen [220, 221], but then found divergences in the case of a spherical surface, thereby casting doubt on the validity of the Boyer calculation [19]. Some of their results, as we shall see, are spurious, and the rest are well known [59]. However, their work has been valuable in sparking new investigations of the problems of surface energies and divergences.

We now carry out the calculation we presented in section 2 in three spatial dimensions, with a radially symmetric background

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} \frac{\lambda}{a} \delta(r-a) \phi^2(x), \quad (4.12)$$

which would correspond to a Dirichlet shell in the limit  $\lambda \rightarrow \infty$ . The time-Fourier transformed Green's function satisfies the equation ( $\kappa^2 = -\omega^2$ )

$$\left[ -\nabla^2 + \kappa^2 + \frac{\lambda}{a} \delta(r-a) \right] \mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (4.13)$$

We write  $\mathcal{G}$  in terms of a reduced Green's function

$$\mathcal{G}(\mathbf{r}, \mathbf{r}') = \sum_{lm} g_l(r, r') Y_{lm}(\Omega) Y_{lm}^*(\Omega'), \quad (4.14)$$

where  $g_l$  satisfies

$$\left[ -\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{l(l+1)}{r^2} + \kappa^2 + \frac{\lambda}{a} \delta(r-a) \right] g_l(r, r') = \frac{1}{r^2} \delta(r-r'). \quad (4.15)$$

We solve this in terms of modified Bessel functions,  $I_\nu(x)$ ,  $K_\nu(x)$ , where  $\nu = l + 1/2$ , which satisfy the Wronskian condition

$$I'_\nu(x) K_\nu(x) - K'_\nu(x) I_\nu(x) = \frac{1}{x}. \quad (4.16)$$

The solution to (4.15) is obtained by requiring continuity of  $g_l$  at each singularity,  $r'$  and  $a$ , and the appropriate discontinuity of the derivative. Inside the sphere we then find ( $0 < r, r' < a$ )

$$g_l(r, r') = \frac{1}{\kappa r r'} \left[ e_l(\kappa r_>) s_l(\kappa r_<) - \frac{\lambda}{\kappa a} s_l(\kappa r) s_l(\kappa r') \frac{e_l^2(\kappa a)}{1 + \frac{\lambda}{\kappa a} s_l(\kappa a) e_l(\kappa a)} \right]. \quad (4.17)$$

Here we have introduced the modified Riccati-Bessel functions,

$$s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_{l+1/2}(x). \quad (4.18)$$

Note that (4.17) reduces to the expected result, vanishing as  $r \rightarrow a$ , in the limit of strong coupling:

$$\lim_{\lambda \rightarrow \infty} g_l(r, r') = \frac{1}{\kappa r r'} \left[ e_l(\kappa r_>) s_l(\kappa r_<) - \frac{e_l(\kappa a)}{s_l(\kappa a)} s_l(\kappa r) s_l(\kappa r') \right]. \quad (4.19)$$

When both points are outside the sphere,  $r, r' > a$ , we obtain a similar result:

$$g_l(r, r') = \frac{1}{\kappa r r'} \left[ e_l(\kappa r_>) s_l(\kappa r_<) - \frac{\lambda}{\kappa a} e_l(\kappa r) e_l(\kappa r') \frac{s_l^2(\kappa a)}{1 + \frac{\lambda}{\kappa a} s_l(\kappa a) e_l(\kappa a)} \right]. \quad (4.20)$$

which similarly reduces to the expected result as  $\lambda \rightarrow \infty$ .

Now we want to get the radial-radial component of the stress tensor to extract the pressure on the sphere, which is obtained by applying the operator

$$\partial_r \partial_{r'} - \frac{1}{2}(-\partial^0 \partial^0 + \nabla \cdot \nabla') \rightarrow \frac{1}{2} \left[ \partial_r \partial_{r'} - \kappa^2 - \frac{l(l+1)}{r^2} \right] \quad (4.21)$$

to the Green's function, where in the last term we have averaged over the surface of the sphere. In this way we find, from the discontinuity of  $\langle T_{rr} \rangle$  across the  $r = a$  surface, the net stress

$$\mathcal{S} = \frac{\lambda}{2\pi a^2} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx \frac{(e_l(x)s_l(x))' - \frac{2e_l(x)s_l(x)}{x}}{1 + \frac{\lambda e_l(x)s_l(x)}{x}}. \quad (4.22)$$

The same result can be deduced by computing the total energy (2.15). The free Green's function, the first term in (4.17) or (4.20), evidently makes no significant contribution to the energy, for it gives a term independent of the radius of the sphere,  $a$ , so we omit it. The remaining radial integrals are simply

$$\int_0^x dy s_l^2(y) = \frac{1}{2x} [(x^2 + l(l+1)) s_l^2 + x s_l s_l' - x^2 s_l'^2], \quad (4.23a)$$

$$\int_x^{\infty} dy e_l^2(y) = -\frac{1}{2x} [(x^2 + l(l+1)) e_l^2 + x e_l e_l' - x^2 e_l'^2], \quad (4.23b)$$

where all the Bessel functions on the right-hand-sides of these equations are evaluated at  $x$ . Then using the Wronskian, we find that the Casimir energy is

$$E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln [1 + \lambda I_\nu(x) K_\nu(x)]. \quad (4.24)$$

If we differentiate with respect to  $a$ , with  $\lambda/a$  fixed, we immediately recover the force (4.22). This expression, upon integration by parts, coincides with that given by Barton [222], and was first analyzed in detail by Scandurra [223]. For strong coupling, it reduces to the well-known expression for the Casimir energy of a massless scalar field inside and outside a sphere upon which Dirichlet boundary conditions are imposed, that is, that the field must vanish at  $r = a$ :

$$\lim_{\lambda \rightarrow \infty} E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln [I_\nu(x) K_\nu(x)], \quad (4.25)$$

because multiplying the argument of the logarithm by a power of  $x$  is without effect, corresponding to a contact term. Details of the evaluation of Eq. (4.25) are given in Ref. [57], and will be considered in section 4.4 below. (See also Refs. [224, 225, 226].)

The opposite limit is of interest here. The expansion of the logarithm is immediate for small  $\lambda$ . The first term, of order  $\lambda$ , is evidently divergent, but irrelevant, since that may be removed by renormalization of the tadpole graph. In contradistinction to the claim of Refs. [61, 60, 30, 71], the order  $\lambda^2$  term is finite, as established in Ref. [57]. That term is

$$E^{(\lambda^2)} = \frac{\lambda^2}{4\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} [I_{l+1/2}(x) K_{l+1/2}(x)]^2. \quad (4.26)$$

The sum on  $l$  can be carried out using a trick due to Klich [227]: The sum rule

$$\sum_{l=0}^{\infty} (2l+1) e_l(x) s_l(y) P_l(\cos \theta) = \frac{xy}{\rho} e^{-\rho}, \quad (4.27)$$

where  $\rho = \sqrt{x^2 + y^2 - 2xy \cos \theta}$ , is squared, and then integrated over  $\theta$ , according to

$$\int_{-1}^1 d(\cos \theta) P_l(\cos \theta) P_{l'}(\cos \theta) = \delta_{ll'} \frac{2}{2l+1}. \quad (4.28)$$

In this way we learn that

$$\sum_{l=0}^{\infty} (2l+1) e_l^2(x) s_l^2(x) = \frac{x^2}{2} \int_0^{4x} \frac{dw}{w} e^{-w}. \quad (4.29)$$

Although this integral is divergent, because we did not integrate by parts in (4.26), that divergence does not contribute:

$$E^{(\lambda^2)} = \frac{\lambda^2}{4\pi a} \int_0^{\infty} dx \frac{1}{2} x \frac{d}{dx} \int_0^{4x} \frac{dw}{w} e^{-w} = \frac{\lambda^2}{32\pi a}, \quad (4.30)$$

which is exactly the result (4.25) of Ref. [57], which also follows from (2.20) here.

However, before we wax too euphoric, we recognize that the order  $\lambda^3$  term appears logarithmically divergent, just as Refs. [30] and [71] claim. This does not signal a breakdown in perturbation theory, as the divergence (2.21) in the  $D = 1$  calculation did. Suppose we subtract off the two leading terms,

$$E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \left[ \ln(1 + \lambda I_\nu K_\nu) - \lambda I_\nu K_\nu + \frac{\lambda^2}{2} (I_\nu K_\nu)^2 \right] + \frac{\lambda^2}{32\pi a}. \quad (4.31)$$

To study the behavior of the sum for large values of  $l$ , we can use the uniform asymptotic expansion (Debye expansion),

$$\nu \gg 1: \quad I_\nu(x) K_\nu(x) \sim \frac{t}{2\nu} \left[ 1 + \frac{A(t)}{\nu^2} + \frac{B(t)}{\nu^4} + \dots \right]. \quad (4.32)$$

Here  $x = \nu z$ , and  $t = 1/\sqrt{1+z^2}$ . The functions  $A$  and  $B$ , etc., are polynomials in  $t$ . We now insert this into (4.31) and expand not in  $\lambda$  but in  $\nu$ ; the leading term is

$$E^{(\lambda^3)} \sim \frac{\lambda^3}{24\pi a} \sum_{l=0}^{\infty} \frac{1}{\nu} \int_0^{\infty} \frac{dz}{(1+z^2)^{3/2}} = \frac{\lambda^3}{24\pi a} \zeta(1). \quad (4.33)$$

Although the frequency integral is finite, the angular momentum sum is divergent. The appearance here of the divergent  $\zeta(1)$  seems to signal an insuperable barrier to extraction of a finite Casimir energy for finite  $\lambda$ . The situation is different in the limit  $\lambda \rightarrow \infty$  – See section 4.4.

This divergence has been known for many years, and was first calculated explicitly in 1998 by Bordag *et al* [59], where the second heat kernel coefficient gave an equivalent result,

$$E \sim \frac{\lambda^3}{48\pi a} \frac{1}{s}, \quad s \rightarrow 0. \quad (4.34)$$

A possible way of dealing with this divergence was advocated in Ref. [223]. Very recently, Bordag and Vassilevich [228] have reanalyzed such problems from the heat kernel approach. They show that this  $O(\lambda^3)$  divergence corresponds to a surface tension counterterm, an idea proposed by me in 1980 [27, 229] in connection with the zero-point energy contribution to the bag model. Such a surface term corresponds to  $\lambda/a$  fixed, which then necessarily implies a divergence of order  $\lambda^3$ . Bordag argues that it is perfectly appropriate to insert a surface tension counterterm so that this divergence may be rendered finite by renormalization.

#### 4.3. TM Spherical Potential

Of course, the scalar model considered in the previous subsection is merely a toy model, and something analogous to electrodynamics is of far more physical relevance. There are good reasons for believing that cancellations occur in general between TE (Dirichlet) and TM (Robin) modes. Certainly they do occur in the classic Boyer energy of a perfectly conducting spherical shell [19, 22, 24], and the indications are that such cancellations occur even with imperfect boundary conditions [222]. Following the latter reference, let us consider the potential

$$\mathcal{L}_{\text{int}} = \frac{1}{2} \lambda a \frac{1}{r} \frac{\partial}{\partial r} \delta(r-a) \phi^2(x). \quad (4.35)$$

In the limit  $\lambda \rightarrow \infty$  this corresponds to TM boundary conditions. The reduced Green's function is thus taken to satisfy

$$\left[ -\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{l(l+1)}{r^2} + \kappa^2 - \frac{\lambda a}{r} \frac{\partial}{\partial r} \delta(r-a) \right] g_l(r, r') = \frac{1}{r^2} \delta(r-r'). \quad (4.36)$$

At  $r = r'$  we have the usual boundary conditions, that  $g_l$  be continuous, but that its derivative be discontinuous,

$$r^2 \frac{\partial}{\partial r} g_l \Big|_{r=r'-}^{r=r'+} = -1, \quad (4.37)$$

while at the surface of the sphere the derivative is continuous,

$$\frac{\partial}{\partial r} r g_l \Big|_{r=a-}^{r=a+} = 0, \quad (4.38a)$$

while the function is discontinuous,

$$g_l \Big|_{r=a-}^{r=a+} = -\lambda \frac{\partial}{\partial r} r g_l \Big|_{r=a}. \quad (4.38b)$$

Equations (4.38a) and (4.38b) are the analogues of the boundary conditions (2.30a), (2.30b) treated in section 2.3.

It is then easy to find the Green's function. When both points are inside the sphere,

$$r, r' < a : \quad g_l(r, r') = \frac{1}{\kappa r r'} \left[ s_l(\kappa r_{<}) e_l(\kappa r_{>}) - \frac{\lambda \kappa a [e'_l(\kappa a)]^2 s_l(\kappa r) s_l(\kappa r')}{1 + \lambda \kappa a e'_l(\kappa a) s'_l(\kappa a)} \right], \quad (4.39a)$$

and when both points are outside the sphere,

$$r, r' > a : \quad g_l(r, r') = \frac{1}{\kappa r r'} \left[ s_l(\kappa r_{<}) e_l(\kappa r_{>}) - \frac{\lambda \kappa a [s'_l(\kappa a)]^2 e_l(\kappa r) e_l(\kappa r')}{1 + \lambda \kappa a e'_l(\kappa a) s'_l(\kappa a)} \right]. \quad (4.39b)$$

It is immediate that these supply the appropriate Robin boundary conditions in the  $\lambda \rightarrow \infty$  limit:

$$\lim_{\lambda \rightarrow 0} \frac{\partial}{\partial r} r g_l \Big|_{r=a} = 0. \quad (4.40)$$

The Casimir energy may be readily obtained from (2.15), and we find, using the integrals (4.23a), (4.23b)

$$E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln [1 + \lambda x e'_l(x) s'_l(x)]. \quad (4.41)$$

The stress may be obtained from this by applying  $-\partial/\partial a$ , and regarding  $\lambda a$  as constant [see (4.35)], or directly, from the Green's function by applying the operator,

$$t_{rr} = \frac{1}{2i} \left[ \nabla_r \nabla_{r'} - \kappa^2 - \frac{l(l+1)}{r^2} \right] g_l \Big|_{r'=r}, \quad (4.42)$$

which is the same as that in (4.21), except that

$$\nabla_r = \frac{1}{r} \partial_r r, \quad (4.43)$$

appropriate to TM boundary conditions (see Ref. [230], for example). Either way, the total stress on the sphere is

$$\mathcal{S} = -\frac{\lambda}{2\pi a^2} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x^2 \frac{[e'_l(x) s'_l(x)]'}{1 + \lambda x e'_l(x) s'_l(x)}. \quad (4.44)$$

The result for the energy (4.41) is similar, but not identical, to that given by Barton [222].

Suppose we now combine the TE and TM Casimir energies, (4.24) and (4.41):

$$E^{\text{TE}} + E^{\text{TM}} = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln \left[ \left( 1 + \lambda \frac{e_l s_l}{x} \right) (1 + \lambda x e'_l s'_l) \right]. \quad (4.45)$$

In the limit  $\lambda \rightarrow \infty$  this reduces to the familiar expression for the perfectly conducting spherical shell [24]:

$$\lim_{\lambda \rightarrow \infty} E = -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{\infty} dx x \left( \frac{e'_l}{e_l} + \frac{e''_l}{e'_l} + \frac{s'_l}{s_l} + \frac{s''_l}{s'_l} \right). \quad (4.46)$$

Here we have, as appropriate to the electrodynamic situation, omitted the  $l=0$  mode. This expression yields a finite Casimir energy, as we will see in section 4.4. What about finite  $\lambda$ ? In general, it appears that there is no chance that the divergence found in the previous section in order  $\lambda^3$  can be cancelled. But suppose the coupling for the TE and TM modes are different. If  $\lambda^{\text{TE}} \lambda^{\text{TM}} = 4$ , a cancellation appears possible.

Let us illustrate this by retaining only the leading terms in the uniform asymptotic expansions: ( $x = \nu z$ )

$$\frac{e_l(x) s_l(x)}{x} \sim \frac{t}{2\nu}, \quad x e'_l(x) s'_l(x) \sim -\frac{\nu}{2t}, \quad \nu \rightarrow \infty. \quad (4.47)$$

Then the logarithm appearing in the integral for the energy (4.45) is approximately

$$\ln \sim \ln \left( -\frac{\lambda^{\text{TM}} \nu}{2t} \right) + \ln \left( 1 + \frac{\lambda^{\text{TE}} t}{2\nu} \right) + \ln \left( 1 - \frac{2t}{\lambda^{\text{TM}} \nu} \right). \quad (4.48)$$

The first term here presumably gives no contribution to the energy, because it is independent of  $\lambda$  upon differentiation, and further we may interpret  $\sum_{l=0}^{\infty} \nu^2 = 0$  [see (4.52)]. Now if we make the above identification of the couplings,

$$\hat{\lambda} = \frac{\lambda^{\text{TE}}}{2} = \frac{2}{\lambda^{\text{TM}}}, \quad (4.49)$$

all the odd powers of  $\nu$  cancel out, and

$$E \sim -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln \left( 1 - \frac{\hat{\lambda}^2 t^2}{\nu^2} \right). \quad (4.50)$$

The divergence encountered for the TE mode is thus removed, and the power series is simply twice the sum of the even terms there. This will be finite. Presumably, the same is true if the subleading terms in the uniform asymptotic expansion are retained.

It is interesting to approximately evaluate (4.50). The integral over  $z$  may be easily evaluated as a contour integral, leaving

$$E \sim -\frac{1}{a} \sum_{l=0}^{\infty} \nu^2 \left( 1 - \sqrt{1 - \frac{\hat{\lambda}^2}{\nu^2}} \right). \quad (4.51)$$

This  $l$  sum appears to be divergent, an artifact of the asymptotic expansion, since we know the  $\lambda^2$  term is finite. However, if we expand the square root for small  $\hat{\lambda}^2/\nu^2$ , we see that the  $O(\hat{\lambda}^2)$  term vanishes if we interpret the sum as

$$\sum_{l=0}^{\infty} \nu^{-s} = (2^s - 1)\zeta(s), \quad (4.52)$$

in terms of the Riemann zeta function. The leading term is  $O(\hat{\lambda}^4)$ :

$$E \sim -\frac{\hat{\lambda}^4}{8a} \sum_{l=0}^{\infty} \frac{1}{\nu^2} = -\frac{\hat{\lambda}^4 \pi^2}{16a}. \quad (4.53)$$

To recover the correct leading  $\lambda$  behavior in (4.30) requires the inclusion of the subleading  $\nu^{-2n}$  terms displayed in (4.32).

Much faster convergence is achieved if we consider the results with the  $l=0$  term removed, as appropriate for electromagnetic modes. Let's illustrate this for the order  $\lambda^2$  TE mode (now, for simplicity, write  $\lambda = \lambda^{\text{TE}}$ ) Then, in place of the energy (4.30) we have

$$\tilde{E}^{(\lambda^2)} = \frac{\lambda^2}{32\pi a} + \frac{\lambda^2}{4\pi a} \int_0^{\infty} \frac{dx}{x^2} \sinh^2 x e^{-2x} = \frac{\lambda^2}{a} \left( \frac{1}{32\pi} + \frac{\ln 2}{4\pi} \right) = \frac{\lambda^2}{a} (0.0651061). \quad (4.54)$$

Now the leading term in the uniform asymptotic expansion is no longer zero:

$$\begin{aligned} E^{(0)} &= -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \left( -\frac{\lambda^2 t^2}{8\nu^2} \right) \\ &= \frac{\lambda^2}{8\pi a} \sum_{l=1}^{\infty} \nu^0 \left( -\frac{\pi}{2} \right) = \frac{\lambda^2}{16a} = \frac{\lambda^2}{a} (0.0625), \end{aligned} \quad (4.55)$$

which is 4% lower than the exact answer (4.54). The next term in the uniform asymptotic expansion is

$$\begin{aligned} E^{(2)} &= -\frac{\lambda^2}{4\pi a} [3\zeta(2) - 4] \int_0^\infty dz t^2 \frac{t^2 - 6t^4 + 5t^6}{8} \\ &= \frac{\lambda^2}{a} \left( \frac{3\pi^2}{2048} - \frac{3}{256} \right) = \frac{\lambda^2}{a} (0.002\,736\,8), \end{aligned} \quad (4.56)$$

which reduces the estimate to

$$E^{(0)} + E^{(2)} = \frac{\lambda^2}{a} (0.065\,236\,8), \quad (4.57)$$

which is now 0.2% high. Going out one more term gives

$$\begin{aligned} E^{(4)} &= -\frac{\lambda^2}{8\pi a} [15\zeta(4) - 16] \int_0^\infty dz t^2 \frac{t^4}{16} (7 - 148t^2 + 554t^4 - 708t^6 + 295t^8) \\ &= -\frac{\lambda^2}{a} \left( \frac{59\pi^4}{524288} - \frac{177}{16328} \right) = -\frac{\lambda^2}{a} (0.000\,158\,570), \end{aligned} \quad (4.58)$$

and the estimate for the energy is now only 0.04% low:

$$E^{(0)} + E^{(2)} + E^{(4)} = \frac{\lambda^2}{a} (0.065\,078\,23). \quad (4.59)$$

We could also make similar remarks about the TM contributions.

#### 4.4. Perfectly Conducting Spherical Shell

Now we consider a massless scalar in three space dimensions, with a spherical boundary on which the field vanishes. This corresponds to the TE modes for the electrodynamic situation first solved by Boyer [19, 22, 24]. The purpose of this section (adapted from Ref. [57]) is to emphasize anew that, contrary to the implication of Ref. [60, 61, 30, 71], the corresponding Casimir energy is also finite for this configuration.

The general calculation in  $D$  spatial dimensions was given in Ref. [224]; the pressure is given by the formula

$$P = - \sum_{l=0}^{\infty} \frac{(2l + D - 2)\Gamma(l + D - 2)}{l! 2^D \pi^{(D+1)/2} \Gamma(\frac{D-1}{2}) a^{D+1}} \int_0^\infty dx x \frac{d}{dx} \ln [I_\nu(x) K_\nu(x) x^{2-D}]. \quad (4.60)$$

Here  $\nu = l - 1 + D/2$ . For  $D = 3$  this expression reduces to

$$P = -\frac{1}{8\pi^2 a^4} \sum_{l=0}^{\infty} (2l + 1) \int_0^\infty dx x \frac{d}{dx} \ln [I_{l+1/2}(x) K_{l+1/2}(x) / x]. \quad (4.61)$$

This precisely corresponds to the strong limit  $\lambda \rightarrow \infty$  given in (4.25), if we recall the comment made about contact terms there. In Ref. [224] we evaluated expression (4.60) by continuing in  $D$  from a region where both the sum and integrals existed. In that way, a completely finite result was found for all positive  $D$  not equal to an even integer.

Here we will adopt a perhaps more physical approach, that of allowing the time-coordinates in the underlying Green's function to approach each other, as described in

Ref. [24]. That is, we recognize that the  $x$  integration above is actually a (dimensionless) imaginary frequency integral, and therefore we should replace

$$\int_0^\infty dx f(x) = \frac{1}{2} \int_{-\infty}^\infty dy e^{iy\delta} f(|y|), \quad (4.62)$$

where at the end we are to take  $\delta \rightarrow 0$ . Immediately, we can replace the  $x^{-1}$  inside the logarithm in (4.61) by  $x$ , which makes the integrals converge, because the difference is proportional to a delta function in the time separation, a contact term without physical significance.

To proceed, we use the uniform asymptotic expansions for the modified Bessel functions, (4.32). This is an expansion in inverse powers of  $\nu = l + 1/2$ , low terms in which turn out to be remarkably accurate even for modest  $l$ . The leading terms in this expansion are

$$\ln [xI_{l+1/2}(x)K_{l+1/2}(x)] \sim \ln \frac{zt}{2} + \frac{1}{\nu^2}g(t) + \frac{1}{\nu^4}h(t) + \dots, \quad (4.63)$$

where  $x = \nu z$  and  $t = (1 + z^2)^{-1/2}$ . Here

$$g(t) = \frac{1}{8}(t^2 - 6t^4 + 5t^6), \quad (4.64a)$$

$$h(t) = \frac{1}{64}(13t^4 - 284t^6 + 1062t^8 - 1356t^{10} + 565t^{12}). \quad (4.64b)$$

The leading term in the pressure is therefore

$$P_0 = -\frac{1}{8\pi^2 a^4} \sum_{l=0}^{\infty} (2l+1)\nu \int_0^\infty dz t^2 = -\frac{1}{8\pi a^4} \sum_{l=0}^{\infty} \nu^2 = \frac{3}{32\pi a^4} \zeta(-2) = 0. \quad (4.65)$$

where in the last step we have used the formal zeta function evaluation (4.52).§ Here the rigorous way to argue is to recall the presence of the point-splitting factor  $e^{i\nu z\delta}$  and to carry out the sum on  $l$  using

$$\sum_{l=0}^{\infty} e^{i\nu z\delta} = -\frac{1}{2i} \frac{1}{\sin z\delta/2}, \quad (4.66)$$

so

$$\sum_{l=0}^{\infty} \nu^2 e^{i\nu z\delta} = -\frac{d^2}{d(z\delta)^2} \frac{i}{2 \sin z\delta/2} = \frac{i}{8} \left( -\frac{2}{\sin^3 z\delta/2} + \frac{1}{\sin z\delta/2} \right). \quad (4.67)$$

Then  $P_0$  is given by the divergent expression

$$P_0 = \frac{i}{\pi^2 a^4 \delta^3} \int_{-\infty}^{\infty} \frac{dz}{z^3} \frac{1}{1+z^2}, \quad (4.68)$$

which we argue is zero because the integrand is odd, as justified by averaging over contours passing above and below the pole at  $z = 0$ .

The next term in the uniform asymptotic expansion (4.63), that involving  $g$ , likewise gives zero pressure, as intimated by the formal zeta function identity (4.52), which

§ Note that the corresponding TE contribution for the electromagnetic Casimir pressure would not be zero, for there the sum starts from  $l = 1$ .



vanishes at  $s = 0$ . The same conclusion follows from point splitting, using (4.66) and arguing that the resulting integrand  $\sim z^2 t^3 g'(t)/z\delta$  is odd in  $z$ . Again, this cancellation does not occur in the electromagnetic case because there the sum starts at  $l = 1$ .

So here the leading term which survives is that of order  $\nu^{-4}$  in (4.63), namely

$$P_2 = \frac{1}{4\pi^2 a^4} \sum_{l=0}^{\infty} \frac{1}{\nu^2} \int_0^{\infty} dz h(t), \quad (4.69)$$

where we have now dropped the point-splitting factor because this expression is completely convergent. The integral over  $z$  is

$$\int_0^{\infty} dz h(t) = \frac{35\pi}{32768} \quad (4.70)$$

and the sum over  $l$  is  $3\zeta(2) = \pi^2/2$ , so the leading contribution to the stress on the sphere is

$$\mathcal{S}_2 = 4\pi a^2 P_2 = \frac{35\pi^2}{65536a^2} = \frac{0.00527094}{a^2}. \quad (4.71)$$

Numerically this is a terrible approximation.

What we must do now is return to the full expression and add and subtract the leading asymptotic terms. This gives

$$\mathcal{S} = \mathcal{S}_2 - \frac{1}{2\pi a^2} \sum_{l=0}^{\infty} (2l+1) R_l, \quad (4.72)$$

where

$$R_l = Q_l + \int_0^{\infty} dx \left[ \ln zt + \frac{1}{\nu^2} g(t) + \frac{1}{\nu^4} h(t) \right], \quad (4.73)$$

where the integral

$$Q_l = - \int_0^{\infty} dx \ln[2x I_\nu(x) K_\nu(x)] \quad (4.74)$$

was given the asymptotic form in Ref. [224, 29] ( $l \gg 1$ ):

$$Q_l \sim \frac{\nu\pi}{2} + \frac{\pi}{128\nu} - \frac{35\pi}{32768\nu^3} + \frac{565\pi}{1048577\nu^5} - \frac{1208767\pi}{2147483648\nu^7} + \frac{138008357\pi}{137438953472\nu^9}. \quad (4.75)$$

The first two terms in (4.75) cancel the second and third terms in (4.73), of course. The third term in (4.75) corresponds to  $h(t)$ , so the last three terms displayed in (4.75) give the asymptotic behavior of the remainder, which we call  $w(\nu)$ . Then we have, approximately,

$$\mathcal{S} \approx \mathcal{S}_2 - \frac{1}{\pi a^2} \sum_{l=0}^n \nu R_l - \frac{1}{\pi a^2} \sum_{l=n+1}^{\infty} \nu w(\nu). \quad (4.76)$$

For  $n = 1$  this gives  $\mathcal{S} \approx 0.00285278/a^2$ , and for larger  $n$  this rapidly approaches the value first given in Ref. [224], and rederived in [225, 226, 231]

$$\mathcal{S}^{\text{TE}} = 0.002817/a^2, \quad (4.77)$$

a value much smaller than the famous electromagnetic result [19, 21, 24, 22],

$$\mathcal{S}^{\text{EM}} = \frac{0.04618}{a^2}, \quad (4.78)$$

because of the cancellation of the leading terms noted above. Indeed, the TM contribution was calculated separately in Ref. [230], with the result

$$\mathcal{S}^{\text{TM}} = -0.02204 \frac{1}{a^2}, \quad (4.79)$$

and then subtracting the  $l = 0$  modes from both contributions we obtain (4.78)

$$\mathcal{S}^{\text{EM}} = \mathcal{S}^{\text{TE}} + \mathcal{S}^{\text{TM}} + \frac{\pi}{48a^2} = \frac{0.0462}{a^2}. \quad (4.80)$$

#### 4.5. Dielectric Spheres

The Casimir self-stress on a uniform dielectric sphere was first worked out in 1979 [216]. It was generalized to the case when both electric permittivity and magnetic permeability are present in 1997 [232]. Since this calculation is summarized in my monograph [29], we content ourselves here with simply stating the result for the pressure, ( $x = \sqrt{\varepsilon\mu}|y|$ ,  $x' = \sqrt{\varepsilon'\mu'}|y|$  where  $\varepsilon', \mu'$  are the interior, and  $\varepsilon, \mu$  are the exterior, values of the permittivity and the permeability)

$$P = -\frac{1}{2a^4} \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{iy\delta} \sum_{l=1}^{\infty} \frac{2l+1}{4\pi} \left\{ x \frac{d}{dx} \ln D_l + 2x'[s'_l(x')e'_l(x') - e_l(x')s''_l(x')] - 2x[s'_l(x)e'_l(x) - e_l(x)s''_l(x)] \right\}, \quad (4.81)$$

where the ‘‘bulk’’ pressure has been subtracted, and

$$D_l = [s_l(x')e'_l(x) - s'_l(x')e_l(x)]^2 - \xi^2[s_l(x')e'_l(x) + s'_l(x')e_l(x)]^2, \quad (4.82)$$

with the parameter  $\xi$  being

$$\xi = \frac{\sqrt{\frac{\varepsilon'}{\varepsilon} \frac{\mu}{\mu'}} - 1}{\sqrt{\frac{\varepsilon'}{\varepsilon} \frac{\mu}{\mu'}} + 1}, \quad (4.83)$$

and  $\delta$  is the temporal regulator introduced in (4.62). This result is obtained either by computing the radial-radial component of the stress tensor, or from the total energy.

In general, this result is divergent. However, consider the special case  $\sqrt{\varepsilon\mu} = \sqrt{\varepsilon'\mu'}$ , that is, when the speed of light is the same in both media. Then  $x = x'$  and the Casimir energy derived from (4.81) reduces to

$$E = 4\pi a^3 P = -\frac{1}{4\pi a} \int_{-\infty}^{\infty} dy e^{iy\delta} \sum_{l=1}^{\infty} (2l+1) x \frac{d}{dx} \ln[1 - \xi^2((s_l e_l)')^2], \quad (4.84)$$

where

$$\xi = \frac{\mu - \mu'}{\mu + \mu'} = -\frac{\varepsilon - \varepsilon'}{\varepsilon + \varepsilon'}. \quad (4.85)$$

If  $\xi = 1$  we recover the case of a perfectly conducting spherical shell, treated in section 4.4 [cf. (4.46)], for which  $E$  is finite. In fact (4.84) is finite for all  $\xi$ .

Of particular interest is the dilute limit, where [227]

$$E \approx \frac{5\xi^2}{32\pi a} = \frac{0.0994718\xi^2}{2a}, \quad \xi \ll 1. \quad (4.86)$$

[This evaluation is carried out in the same manner as that of (4.26).] It is remarkable that the value for a spherical conducting shell (4.78), for which  $\xi = 1$ , is only 7% lower, which as Klich remarks, is accounted for nearly entirely by the next term in the small  $\xi$  expansion.

There is another dilute limit which is also quite surprising. For a purely dielectric sphere ( $\mu = 1$ ) the leading term in an expansion in powers of  $\varepsilon - 1$  is finite [233, 234, 59, 235]:

$$E = \frac{23}{1536\pi} \frac{(\varepsilon - 1)^2}{a} = (\varepsilon - 1)^2 \frac{0.004767}{a}. \quad (4.87)$$

This result coincides with the sum of van der Waals energies of the material making up the ball [80]. The term of order  $(\varepsilon - 1)^3$  is divergent [59]. The establishment of the result (4.87) was the death knell for the Casimir energy explanation of sonoluminescence [236] – See section 6.

The temperature correction to this result was first worked out by Nesterenko, Lambiase, and Scapetta [237, 238]. See also Ref. [239].

#### 4.6. Cylinders

It is much more difficult to carry out Casimir calculations for cylindrical geometries. We restrict our attention here to cylinders of circular cross section and infinite length. Although calculations have been carried out for parallelepiped geometries [240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251], the effects included refer only to the interior modes of oscillation. This is because the wave equation is not separable outside a cube or a rectangular solid. As a result, divergences occur which cannot be legitimately removed, which nevertheless are artificially removed by zeta-function methods. It is the view of the author that such finite results are without meaning.

But even though circular-cylinder calculations are possible, they are considerably more complex than the corresponding spherical calculations. This is not merely because spherical Bessel functions are simpler than cylinder functions. The fundamental difficulty in these geometries is that there is in general no decoupling between TE and TM modes [252]. Progress in understanding has therefore been much slower in this regime. It was only in 1981 that it was found that the electromagnetic Casimir energy of a perfectly conducting cylinder was attractive, the energy per unit length being [253]

$$\mathcal{E}_{\text{em,cyl}} = -\frac{0.01356}{a^2}, \quad (4.88)$$

for a circular cylinder of radius  $a$ . The corresponding result for a scalar field satisfying Dirichlet boundary conditions of the cylinder is repulsive [254],

$$\mathcal{E}_{\text{D,cyl}} = \frac{0.000606}{a^2}. \quad (4.89)$$

These ideal limits are finite, but, as with the spherical geometry, less ideal configurations have unremovable divergences. For example, a cylindrical  $\delta$ -shell potential, as described earlier, has divergences (in third order) [255]. And it is expected that a dielectric cylinder will have a divergent Casimir energy, although the coefficient of  $(\varepsilon - 1)^2$  will be finite for a dilute dielectric cylinder [256], corresponding to a finite van der Waals energy between the molecules that make up the material [257]. Recent progress in understanding these points will be described below.

*4.6.1. Dielectric cylinders* The following calculation represents work in progress with Ines Cavero-Pelaez. Although the calculation remains incomplete, we offer it here as a detailed example of how a complicated electromagnetic calculation is formulated in the Green's function approach. We start from the equations satisfied by the Green's dyadics for Maxwell's equations in a medium characterized by a permittivity  $\varepsilon$  and a permeability  $\mu$  (see Ref. [216]):

$$\nabla \times \mathbf{\Gamma}' - i\omega\mu\mathbf{\Phi} = \frac{1}{\varepsilon}\nabla \times \mathbf{1}, \quad (4.90a)$$

$$-\nabla \times \mathbf{\Phi} - i\omega\varepsilon\mathbf{\Gamma}' = \mathbf{0}, \quad (4.90b)$$

where

$$\mathbf{\Gamma}'(\mathbf{r}, \mathbf{r}', \omega) = \mathbf{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) + \frac{\mathbf{1}}{\varepsilon(\omega)}, \quad (4.91)$$

and where the unit dyadic  $\mathbf{1}$  includes a three-dimensional  $\delta$  function,

$$\mathbf{1} = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}'). \quad (4.92)$$

The two dyadics are solenoidal,

$$\nabla \cdot \mathbf{\Phi} = \mathbf{0}, \quad (4.93a)$$

$$\nabla \cdot \mathbf{\Gamma}' = \mathbf{0}. \quad (4.93b)$$

The corresponding second-order equations are

$$(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{\Gamma}' = -\frac{1}{\varepsilon}\nabla \times (\nabla \times \mathbf{1}), \quad (4.94a)$$

$$(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{\Phi} = i\omega\nabla \times \mathbf{1}. \quad (4.94b)$$

Classically, these Green's dyadic equations are equivalent to Maxwell's equations, and give the solution thereto when a polarization source  $\mathbf{P}$  is present,

$$\mathbf{E}(x) = \int (dx')\mathbf{\Gamma}(x, x') \cdot \mathbf{P}(x'). \quad (4.95)$$

Quantum mechanically, they give the one-loop vacuum expectation values of the product of fields (at a given frequency  $\omega$ )

$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}') \rangle = \frac{\hbar}{i}\Gamma(\mathbf{r}, \mathbf{r}'), \quad (4.96a)$$

$$\langle \mathbf{H}(\mathbf{r})\mathbf{H}(\mathbf{r}') \rangle = -\frac{\hbar}{i}\frac{1}{\omega^2\mu^2}\nabla \times \Gamma(\mathbf{r}, \mathbf{r}') \times \overleftarrow{\nabla}'. \quad (4.96b)$$

Thus, from knowledge of the classical Green's dyadics, we can calculate the one-loop vacuum energy or stress.

We now introduce the appropriate partial wave decomposition for a cylinder, a slight modification of that given for a conducting cylindrical shell [253]||:

$$\begin{aligned} \Gamma'(\mathbf{r}, \mathbf{r}'; \omega) &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ (\nabla \times \hat{z})f_m(r; k, \omega)\chi_{mk}(\theta, z) \right. \\ &\quad \left. + \frac{i}{\omega\varepsilon}\nabla \times (\nabla \times \hat{z})g_m(r; k, \omega)\chi_{mk}(\theta, z) \right\}, \end{aligned} \quad (4.97a)$$

$$\begin{aligned} \Phi(\mathbf{r}, \mathbf{r}'; \omega) &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ (\nabla \times \hat{z})\tilde{g}_m(r; k, \omega)\chi_{mk}(\theta, z) \right. \\ &\quad \left. - \frac{i\varepsilon}{\omega\mu}\nabla \times (\nabla \times \hat{z})\tilde{f}_m(r; k, \omega)\chi_{mk}(\theta, z) \right\}, \end{aligned} \quad (4.97b)$$

where the cylindrical harmonics are

$$\chi_{mk}(\theta, z) = \frac{1}{\sqrt{2\pi}}e^{im\theta}e^{ikz}, \quad (4.98)$$

and the dependence of  $f_m$  etc. on  $\mathbf{r}'$  is implicit (they are further vectors in the second tensor index). Because of the presence of these harmonics, we have

$$\nabla \times \hat{z} \rightarrow \hat{r}\frac{im}{r} - \hat{\theta}\frac{\partial}{\partial r} \equiv \mathcal{M}, \quad (4.99a)$$

$$\nabla \times (\nabla \times \hat{z}) \rightarrow \hat{r}ik\frac{\partial}{\partial r} - \hat{\theta}\frac{mk}{r} - \hat{z}d_m \equiv \mathcal{N}, \quad (4.99b)$$

in terms of the cylinder operator

$$d_m = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r^2}. \quad (4.100)$$

Now if we use the Maxwell equation (4.90b) we conclude¶

$$\tilde{g}_m = g_m, \quad (4.101a)$$

$$(d_m - k^2)\tilde{f}_m = -\omega^2\mu f_m. \quad (4.101b)$$

From the other Maxwell equation (4.90a) we deduce (we now make the second, previously suppressed, position arguments explicit) (the prime on the differential

|| It might be thought that we could immediately use the general waveguide decomposition of modes into those of TE and TM type, for example as given in Ref. [258]. However, this is here impossible because the TE and TM modes do not separate. See Ref. [252].

¶ The ambiguity in solving for these equations is absorbed in the definition of subsequent constants of integration.

operator signifies action on the second, primed argument)

$$d_m \mathcal{D}_m \tilde{\mathbf{f}}_m(r; r', \theta', z') = \frac{\omega^2 \mu}{\varepsilon} \chi_{mk}^*(\theta', z') \mathcal{M}^{*'} \frac{1}{r} \delta(r - r'), \quad (4.102a)$$

$$d_m \mathcal{D}_m \mathbf{g}_m(r; r', \theta', z') = -i\omega \chi_{mk}^*(\theta', z') \mathcal{N}^{*'} \frac{1}{r} \delta(r - r'), \quad (4.102b)$$

where the Bessel operator appears,

$$\mathcal{D}_m = d_m + \lambda^2, \quad \lambda^2 = \omega^2 \varepsilon \mu - k^2. \quad (4.103)$$

Now we do the separation of variables on the second argument,

$$\tilde{\mathbf{f}}_m(r, \mathbf{r}') = \left[ \mathcal{M}^{*'} F_m(r, r'; k, \omega) + \mathcal{N}^{*'} \tilde{F}_m(r, r'; k, \omega) \right] \chi_{mk}^*(\theta', z'), \quad (4.104a)$$

$$\mathbf{g}_m(r, \mathbf{r}') = -\frac{i}{\omega} \left[ \mathcal{N}^{*'} G_m(r, r'; k, \omega) + \mathcal{M}^{*'} \tilde{G}_m(r, r'; k, \omega) \right] \chi_{mk}^*(\theta', z'), \quad (4.104b)$$

where we have now introduced the two scalar Green's functions  $F_m$ ,  $G_m$ , which satisfy

$$d_m \mathcal{D}_m F_m(r, r') = \frac{\omega^2 \mu}{\varepsilon} \frac{1}{r} \delta(r - r'), \quad (4.105a)$$

$$d_m \mathcal{D}_m G_m(r, r') = \omega^2 \frac{1}{r} \delta(r - r'), \quad (4.105b)$$

while  $\tilde{F}_m$  and  $\tilde{G}_m$  are annihilated by the operator  $d_m \mathcal{D}_m$ .

In the following we will apply these equations to a dielectric-diamagnetic cylinder of radius  $a$ , where the interior of the cylinder is characterized by a permittivity  $\varepsilon$  and a permeability  $\mu$ , while the outside is vacuum, so  $\varepsilon = \mu = 1$  there. Let us compute the Green's dyadics for the case that the source point is outside,  $r' > a$ . If the field point is also outside,  $r, r' > a$ , the Green's dyadics have the form ( $\mu = \varepsilon = 1$ )

$$\begin{aligned} \mathbf{\Gamma}' = & \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \mathcal{M} \left( -\frac{d_m - k^2}{\omega^2} \right) \left[ \mathcal{M}^{*'} F_m(r, r'; k, \omega) + \mathcal{N}^{*'} \tilde{F}_m(r, r'; k, \omega) \right] \right. \\ & \left. + \frac{1}{\omega^2} \mathcal{N} \left[ \mathcal{N}^{*'} G_m(r, r'; k, \omega) + \mathcal{M}^{*'} \tilde{G}_m(r, r'; k, \omega) \right] \right\} \chi_{mk}(\theta, z) \chi_{mk}^*(\theta', z'), \quad (4.106a) \end{aligned}$$

$$\begin{aligned} \mathbf{\Phi} = & \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{i}{\omega} \mathcal{M} \left[ \mathcal{N}^{*'} G_m(r, r'; k, \omega) + \mathcal{M}^{*'} \tilde{G}_m(r, r'; k, \omega) \right] \right. \\ & \left. - \frac{i}{\omega} \mathcal{N} \left[ \mathcal{M}^{*'} F_m(r, r'; k, \omega) + \mathcal{N}^{*'} \tilde{F}_m(r, r'; k, \omega) \right] \right\} \chi_{mk}(\theta, z) \chi_{mk}^*(\theta', z'). \quad (4.106b) \end{aligned}$$

From the differential equation (4.105a) we see that the Green's function  $F$  has the form ( $m \neq 0$ )

$$\begin{aligned} F_m = & -\frac{\omega^2}{\lambda^2} \left[ \frac{1}{2|m|} \left( \frac{r_{<}}{r_{>}} \right)^{|m|} + \frac{\pi}{2i} J_m(kr_{<}) H_m(kr_{>}) \right] \\ & + a_m H_m(\lambda r) H_m(\lambda r') + b_m r^{-|m|} H_m(\lambda r') + c_m r'^{-|m|} H_m(\lambda r) + d_m r^{-|m|} r'^{-|m|}, \quad (4.107) \end{aligned}$$

while  $G_m$  has the same form with the constants  $a_m$ ,  $b_m$ ,  $c_m$ ,  $d_m$  replaced by  $a'_m$ ,  $b'_m$ ,  $c'_m$ ,  $d'_m$ , respectively. The homogeneous functions have the form

$$\tilde{F}_m = \tilde{a}_m H_m(\lambda r) H_m(\lambda r') + \tilde{b}_m r^{-|m|} H_m(\lambda r') + \tilde{c}_m r'^{-|m|} H_m(\lambda r) + \tilde{d}_m r^{-|m|} r'^{-|m|}, \quad (4.108)$$

and  $\tilde{G}_m$  replaces  $\tilde{a} \rightarrow \tilde{a}'$ , etc.

When the source point is outside and the field point is inside, there are only homogeneous solutions of the equations, so we may write for  $r < a, r' > a$

$$F_m = e_m r^{|m|} r'^{-|m|} + f_m r^{|m|} H_m(\lambda r') + g_m J_m(\lambda' r) r'^{-|m|} + h_m J_m(\lambda' r) H_m(\lambda r'), \quad (4.109)$$

and similarly for  $G_m, \tilde{F}_m, \tilde{G}_m$ , with the constants denoted by  $e'_m, \tilde{e}_m$ , and  $\tilde{e}'_m$ , respectively. Here the outside and inside forms of  $\lambda$  are given by

$$\lambda^2 = \omega^2 - k^2, \quad \lambda'^2 = \omega^2 \mu \varepsilon - k^2. \quad (4.110)$$

The various constants are to be determined, as far as possible, by the boundary conditions at  $r = a$ . The boundary conditions at the surface of the dielectric cylinder are the continuity of the tangential components of the electric field, of the normal component of the electric displacement, of the normal component of the magnetic induction, and of the tangential components of the magnetic field:

$$\begin{aligned} \mathbf{E}_t & \text{ is continuous,} & \varepsilon E_n & \text{ is continuous,} \\ \mathbf{H}_t & \text{ is continuous,} & \mu H_n & \text{ is continuous.} \end{aligned} \quad (4.111)$$

These conditions are redundant, but we will impose all of them as a check of consistency. In terms of the Green's dyadics, the conditions read

$$\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\Gamma}' \Big|_{r=a} \text{ is continuous,} \quad (4.112a)$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\Gamma}' \Big|_{r=a} \text{ is continuous,} \quad (4.112b)$$

$$\hat{\boldsymbol{r}} \cdot \varepsilon \boldsymbol{\Gamma}' \Big|_{r=a} \text{ is continuous,} \quad (4.112c)$$

$$\hat{\boldsymbol{r}} \cdot \mu \boldsymbol{\Phi}' \Big|_{r=a} \text{ is continuous,} \quad (4.112d)$$

$$\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\Phi}' \Big|_{r=a} \text{ is continuous,} \quad (4.112e)$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\Phi}' \Big|_{r=a} \text{ is continuous.} \quad (4.112f)$$

A fairly elaborate system of linear equations for the various constants results. However, they are not quite sufficient to determine all the relevant physical combinations. We also need to impose one of the Helmholtz equations, say (4.94b). From that equation we learn

$$b' - k \operatorname{sgn} m \tilde{b} = 0, \quad (4.113a)$$

$$b - \frac{\operatorname{sgn} m}{k} \tilde{b}' = 0, \quad (4.113b)$$

$$\hat{d} + \hat{d}' = 0, \quad (4.113c)$$

$$f + \frac{\mu \operatorname{sgn} m}{\varepsilon k} \tilde{f}' = 0, \quad (4.113d)$$

$$f' + \frac{\varepsilon}{\mu} k \operatorname{sgn} m \tilde{f} = 0, \quad (4.113e)$$

$$\hat{e} - \frac{\mu}{\varepsilon} \hat{e}' = 0, \quad (4.113f)$$

where we have introduced the abbreviations for any constant  $K$

$$\hat{K} = K - k \operatorname{sgn} m \tilde{K}, \quad \hat{K}' = K' - \frac{\operatorname{sgn} m}{k} \tilde{K}'. \quad (4.114)$$

Then from the boundary conditions we can solve for the remaining constants: First,

$$\hat{c} = \hat{c}' = 0, \quad (4.115a)$$

$$\hat{g} = \hat{g}' = 0, \quad (4.115b)$$

and

$$\tilde{h}'_m = -\frac{\varepsilon^2}{\mu} (1 - \varepsilon\mu) \frac{\omega^2 m k}{\lambda \lambda' D} h_m H_m(\lambda a) J_m(\lambda' a), \quad (4.116a)$$

$$\tilde{a}'_m = \frac{\lambda'^2}{\lambda^2} \frac{\varepsilon}{\mu} (1 - \varepsilon\mu) \frac{\omega^2 m k}{\lambda \lambda' D} h_m J_m(\lambda' a)^2, \quad (4.116b)$$

$$a_m = \frac{\omega^2}{\lambda^2} \frac{\pi}{2i} \frac{J_m(\lambda a)}{H_m(\lambda a)} + \frac{\lambda'^2 \varepsilon}{\lambda^2 \mu} h_m \frac{J_m(\lambda' a)}{H_m(\lambda a)}, \quad (4.116c)$$

all in terms of

$$h_m = \frac{\mu}{\varepsilon} \omega^2 \frac{\lambda \lambda' D}{\Xi}, \quad (4.116d)$$

where the denominators occurring here are

$$D = \varepsilon \lambda a J'_m(\lambda' a) H_m(\lambda a) - \lambda' a J_m(\lambda' a) H'_m(\lambda a), \quad (4.117a)$$

$$\tilde{D} = \mu \lambda a J'_m(\lambda' a) H_m(\lambda a) - \lambda' a J_m(\lambda' a) H'_m(\lambda a), \quad (4.117b)$$

$$\Xi = (\lambda \lambda')^2 D \tilde{D} - (\varepsilon \mu - 1)^2 k^2 m^2 \omega^2 H_m^2(\lambda a) J_m^2(\lambda' a). \quad (4.117c)$$

The second set of constants is

$$\tilde{h}_m = -\frac{\mu}{\varepsilon^2} (1 - \varepsilon\mu) \frac{m k}{\lambda \lambda' \tilde{D}} h'_m H_m(\lambda a) J_m(\lambda' a), \quad (4.118a)$$

$$\tilde{a}_m = -\frac{\lambda'^2}{\lambda^2} \frac{(1 - \varepsilon\mu)}{\varepsilon} \frac{m k}{\lambda \lambda' \tilde{D}} h'_m J_m(\lambda' a)^2, \quad (4.118b)$$

$$a'_m = \frac{\omega^2}{\lambda^2} \frac{\pi}{2i} \frac{J_m(\lambda a)}{H_m(\lambda a)} + \frac{\lambda'^2}{\lambda^2 \varepsilon} h'_m \frac{J_m(\lambda' a)}{H_m(\lambda a)}, \quad (4.118c)$$

in terms of

$$h'_m = \varepsilon \omega^2 \frac{\lambda \lambda' \tilde{D}}{\Xi}. \quad (4.118d)$$

It might be thought that  $m = 0$  is a special case, and indeed

$$\frac{1}{2|m|} \left( \frac{r_{<}}{r_{>}} \right)^{|m|} \rightarrow \frac{1}{2} \ln \frac{r_{<}}{r_{>}}, \quad (4.119)$$

but just as the latter is correctly interpreted as the limit as  $|m| \rightarrow 0$ , so the coefficients in the Green's functions turn out to be just the  $m = 0$  limits of those given above, so the  $m = 0$  case is properly incorporated.

It is now easy to check that, as a result of the conditions (4.113a), (4.113b), (4.113c), (4.113d), (4.113e), (4.113f), (4.115a) and (4.115b), the terms in the Green's functions that involve powers of  $r$  or  $r'$  do not contribute to the electric or magnetic fields. As



we might well have anticipated, only the pure Bessel function terms contribute. (This observation was not noted in Ref. [253].)

We are now in a position to calculate the pressure on the surface of the sphere from the radial-radial component of the stress tensor,

$$T_{rr} = \frac{1}{2} \left[ \varepsilon(E_\theta^2 + E_z^2 - E_r^2) + \mu(H_\theta^2 + H_z^2 - H_r^2) \right], \quad (4.120)$$

so as a result of the boundary conditions (4.111), the pressure on the cylindrical walls are given by the expectation value of the squares of field components just outside the cylinder:

$$\begin{aligned} T_{rr} \Big|_{r=a-} - T_{rr} \Big|_{r=a+} &= \frac{\varepsilon - 1}{2} \left( E_\theta^2 + E_z^2 + \frac{1}{\varepsilon} E_r^2 \right) \Big|_{r=a+} \\ &+ \frac{\mu - 1}{2} \left( H_\theta^2 + H_z^2 + \frac{1}{\mu} H_r^2 \right) \Big|_{r=a+}. \end{aligned} \quad (4.121)$$

These expectation values are given by (4.96a) and (4.96b), where the latter may also be written as

$$\langle \mathbf{H}(\mathbf{r}) \mathbf{H}(\mathbf{r}') \rangle = -\frac{1}{\omega\mu} \Phi(\mathbf{r}, \mathbf{r}') \times \overleftarrow{\nabla}'. \quad (4.122)$$

It is quite straightforward to compute the vacuum expectation values in terms of the coefficients given above. Further details will be supplied in a forthcoming publication. The resulting expression for the pressure may then, in a standard manner, be expressed after a Euclidean rotation,

$$\omega \rightarrow i\zeta, \quad \lambda \rightarrow i\kappa, \quad (4.123)$$

so that the Bessel functions are replaced by the modified Bessel functions,

$$J_m(x') H_m(x) \rightarrow \frac{2}{\pi i} I_m(y') K_m(y), \quad (4.124)$$

where  $y = \kappa a$ ,  $y' = \kappa' a$ , as

$$\begin{aligned} P &= \frac{\varepsilon - 1}{16\pi^3 a^4} \int d\zeta a \, dk a \sum_{m=-\infty}^{\infty} \frac{1}{\tilde{\Xi}} \left\{ \frac{k^2 a^2 - \zeta^2 a^2 \mu}{y^2} I_m(y') y' I_m'(y') [y K_m'(y)]^2 \right. \\ &- \frac{\mu}{y^2} (k^2 a^2 - \zeta^2 a^2 \varepsilon) [y' I_m'(y')]^2 y K_m'(y) K_m(y) \\ &- \left[ \mu y^2 + \frac{m^2}{y^2} (\mu k^2 a^2 - \zeta^2 a^2) \right] I_m(y') y' I_m'(y') [K_m(y)]^2 \\ &+ \left( -\frac{\varepsilon\mu - 1}{\varepsilon} \frac{m^2 k^2 a^2 \zeta^2 a^2}{y^4} \left[ \frac{\varepsilon\mu - 1}{y'^2} + 2(\varepsilon + 1) \right] \right. \\ &\left. \left. + y'^2 \left[ 1 + \frac{m^2}{y^4} \left( k^2 a^2 - \frac{\zeta^2 a^2}{\varepsilon} \right) \right] \right) [I_m(y')]^2 y K_m'(y) K_m(y) \right\}, \end{aligned} \quad (4.125)$$

where

$$\tilde{\Xi} = \Delta \tilde{\Delta} + (\varepsilon\mu - 1)^2 \frac{m^2 k^2 a^2 \zeta^2 a^2}{y^2 y'^2} I_m^2(y') K_m^2(y), \quad (4.126a)$$

$$\Delta = \varepsilon y I_m'(y') K_m(y) - y' K_m'(y) I_m(y'), \quad (4.126b)$$

$$\tilde{\Delta} = \mu y I_m'(y') K_m(y) - y' K_m'(y) I_m(y'). \quad (4.126c)$$

This result reduces to the well-known expression for the Casimir pressure when the speed of light is the same inside and outside the cylinder, that is, when  $\varepsilon\mu = 1$ . Then, it is easy to see that the denominator reduces to

$$\tilde{\Xi} = \Delta\tilde{\Delta} = \frac{(\varepsilon + 1)^2}{4\varepsilon} [1 - \xi^2 y^2 [(I_m K_m)']^2], \quad (4.127)$$

where  $\xi = (\varepsilon - 1)/(\varepsilon + 1)$ . In the numerator introduce polar coordinates,

$$y^2 = k^2 a^2 + \zeta^2 a^2, \quad ka = y \sin \theta, \quad \zeta a = y \cos \theta, \quad (4.128)$$

and carry out the trivial integral over  $\theta$ . The result is

$$P = -\frac{1}{8\pi^2 a^4} \int_0^\infty dy y^2 \sum_{m=-\infty}^\infty \frac{d}{dy} \ln \left( 1 - \xi^2 [y(K_m I_m)']^2 \right), \quad (4.129)$$

which is exactly the finite result derived in Ref. [257], and analyzed in a number of papers [259, 260, 261]. For  $\xi = 1$  this is the formal result for a perfectly conducting cylindrical shell first analyzed in Ref. [253]. On the other hand, if  $\xi$  is regarded as small, and (4.129) is expanded in powers of  $\xi^2$ , then the term of order  $\xi^2$  turns out to vanish, for reasons not yet understood [257, 261, 29]. Recall that the corresponding coefficient for a dilute dielectric-diamagnetic sphere (4.86) is not zero.

*4.6.2. Bulk Casimir Stress* The above expression is incomplete. It contains an unobservable “bulk” energy contribution, which the formalism would give if either medium, that of the interior with dielectric constant  $\varepsilon$  and permeability  $\mu$ , or that of the exterior with dielectric constant and permeability unity, fills all space. The corresponding stresses are computed from the free Green’s functions,

$$F_m^{(0)}(r, r') = \frac{\mu}{\varepsilon} G_m^{(0)}(r, r') = -\frac{\omega^2 \mu}{\lambda'^2 \varepsilon} \left[ \frac{1}{2|m|} \left( \frac{r_{<}}{r_{>}} \right)^{|m|} + \frac{\pi}{2i} J_m(\lambda' r_{<}) H_m(\lambda' r_{>}) \right]. \quad (4.130)$$

It should be noted that such a Green’s function does not satisfy the appropriate boundary conditions, and therefore we cannot use (4.121), but rather one must compute the interior and exterior stresses individually. Because the two scalar Green’s functions differ only by a factor of  $\mu/\varepsilon$  in this case, these are

$$T_{rr}^{(0)}(a-) = \frac{1}{2\pi i} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty \frac{d\omega}{2\pi} \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{\lambda'^2}{\omega^2} \left[ \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G_m^{(0)} + \left( \lambda'^2 - \frac{m^2}{a^2} \right) G_m^{(0)} \right] \Big|_{r=r'=a-}, \quad (4.131)$$

while the outside bulk stress is given by the same expression with  $\lambda' \rightarrow \lambda$ . When we substitute the appropriate interior and exterior Green’s functions given in (4.130), and perform the Euclidean rotation,  $\omega \rightarrow i\zeta$  we obtain the following rather simple formula for the bulk contribution to the pressure:

$$\begin{aligned} P^b &= T_{rr}^{(0)}(a-) - T_{rr}^{(0)}(a+) \\ &= \frac{1}{8\pi^3 a^2} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty dk \int_{-\infty}^\infty d\zeta [y'^2 I'_m(y') K'_m(y') - (y'^2 + m^2) I_m(y') K_m(y') \\ &\quad - y^2 I'_m(y) K'_m(y) - (y^2 + m^2) I_m(y) K_m(y)]. \end{aligned} \quad (4.132)$$

This term must be *subtracted* from the pressure given in (4.125). Note that this term is the direct analog of that found in the case of a dielectric sphere in Ref. [216] – See (4.81). Note also that  $P^b = 0$  in the special case  $\varepsilon\mu = 1$ .

In the following, we will be interested in dilute dielectric media, where  $\mu = 1$  and  $\varepsilon - 1 \ll 1$ . We easily find that when the integrand in (4.132) is expanded in powers of  $\varepsilon - 1$  the leading terms yield

$$\begin{aligned}
 P^b &\approx -\frac{1}{4\pi^2 a^4} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dy y \int_0^{2\pi} \frac{d\theta}{2\pi} \left[ (\varepsilon - 1) \zeta^2 a^2 I_m(y) K_m(y) \right. \\
 &\quad \left. + \frac{1}{4} (\varepsilon - 1)^2 \frac{(\zeta a)^4}{y} [I_m(y) K_m(y)]' + O(\varepsilon - 1)^3 \right] \\
 &= -\frac{\varepsilon - 1}{8\pi^2 a^4} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dy y^3 \left[ I_m(y) K_m(y) + \frac{3(\varepsilon - 1)}{16} y [I_m(y) K_m(y)]' + O((\varepsilon - 1)^3) \right],
 \end{aligned} \tag{4.133}$$

where we have introduced polar coordinates as in (4.128).

*4.6.3. Dilute Dielectric Cylinder* We now turn to the case of a dilute dielectric medium filling the cylinder, that is, set  $\mu = 1$  and consider  $\varepsilon - 1$  as small. The leading term in the pressure,  $O[(\varepsilon - 1)^1]$ , is obtained from (4.125) by setting  $\mu = \varepsilon = 1$  everywhere in the integrand. The denominator  $\tilde{\Xi}$  is then unity, and we get

$$P \approx -\frac{\varepsilon - 1}{8\pi^2 a^4} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dy y^3 I_m(y) K_m(y), \tag{4.134}$$

which is exactly what is obtained to leading order from the bulk stress (4.133):

$$P - P^b = O[(\varepsilon - 1)^2], \tag{4.135}$$

which is consistent with the interpretation of the Casimir energy as arising from the pairwise interaction of dilutely distributed molecules. In fact, from Ref. [257, 262], we know that the van der Waals energy vanishes even in order  $(\varepsilon - 1)^2$ , so we expect the same to occur with the Casimir energy, although the latter should diverge in  $O[(\varepsilon - 1)^3]$  [256].

We now obtain the expression for the  $O[(\varepsilon - 1)^2]$  term. Because the general expression (4.125) is proportional to  $\varepsilon - 1$ , we need only expand the integrand to first order in this quantity. Let us write it as

$$P = \frac{\varepsilon - 1}{16\pi^3 a^4} \int_{-\infty}^{\infty} d\zeta a \int_{-\infty}^{\infty} dk a \sum_{m=-\infty}^{\infty} \frac{N}{\Delta \tilde{\Delta}}, \tag{4.136}$$

where we have noted that the  $(\varepsilon - 1)^2$  in  $\tilde{\Xi}$  (4.126a) can be dropped. Then introducing polar coordinates as in (4.128), and expanding the numerator and denominator according to

$$N = N^{(0)} + (\varepsilon - 1)N^{(1)} + \dots, \quad \Delta \tilde{\Delta} = 1 + (\varepsilon - 1)\Delta^{(1)} + \dots, \tag{4.137}$$

the second-order term in the unsubtracted Casimir pressure is given by

$$P^{(2)} = \frac{(\varepsilon - 1)^2}{16\pi^3 a^4} \int_0^\infty dy y \int_0^{2\pi} d\theta (N^{(1)} - \Delta^{(1)} N^{(0)}). \quad (4.138)$$

Here the correction to the denominator is

$$\begin{aligned} \Delta^{(1)} = & y I'_m(y) K_m(y) - y \sin^2 \theta [I_m(y) K_m(y)]' + \sin^2 \theta (m^2 + y^2) I'_m(y) K'_m(y) \\ & - y^2 \sin^2 \theta I'_m(y) K'_m(y), \end{aligned} \quad (4.139)$$

and the first two term in the numerator expansion are

$$N^{(0)} = - [y^2 + m^2(1 - 2 \sin^2 \theta)] I_m(y) K_m(y) - y^2(1 - 2 \sin^2 \theta) I'_m(y) K'_m(y), \quad (4.140a)$$

$$\begin{aligned} N^{(1)} = & -\frac{1}{2} (m^2 + y^2) [y^2 + m^2(1 - 2 \sin^2 \theta)] K_m^2(y) I_m^2(y) \\ & + y \sin^2 \theta [(m^2 + y^2) + m^2(1 - 2 \sin^2 \theta) - 4m^2 \cos^2 \theta] I_m^2(y) K_m(y) K'_m(y) \\ & + \frac{1}{2} y^2 \sin^2 \theta (1 - 2 \sin^2 \theta) (m^2 + y^2) I_m^2(y) K_m'^2(y) \\ & - \frac{1}{2} y^2 \sin^2 \theta [y^2 + m^2(1 - 2 \sin^2 \theta)] K_m^2(y) I_m'^2(y) \\ & + y^4 [\sin^2 \theta - \sin^2 \theta (1 - 2 \sin^2 \theta)] I_m(y) I'_m(y) K_m(y) K'_m(y) \\ & + y^3 [\sin^2 \theta + \sin^2 \theta (1 - 2 \sin^2 \theta)] I_m'^2(y) K_m(y) K'_m(y) \\ & + \frac{1}{2} y^4 \sin^2 \theta (1 - 2 \sin^2 \theta) I_m'^2 K_m'^2. \end{aligned} \quad (4.140b)$$

The angular integrals are trivially

$$\int_0^{2\pi} d\theta \sin^2 \theta = \pi, \quad \int_0^{2\pi} d\theta \sin^4 \theta = \frac{3}{4}\pi, \quad (4.141)$$

and then the straightforward reduction of (4.138) is

$$\begin{aligned} P^{(2)} = & \frac{(\varepsilon - 1)^2}{64\pi^2 a^4} \sum_{m=-\infty}^{\infty} \int_0^\infty dy \left\{ y(y^2 + m^2)(2y^2 - m^2) I_m^2(y) K_m^2(y) \right. \\ & + 2y^2(2y^2 + m^2) K_m^2(y) I_m(y) I'_m(y) \\ & - y^3(y^2 + m^2) I_m^2(y) K_m'^2(y) - y^3(2y^2 - m^2) K_m^2(y) I_m'^2(y) \\ & \left. + 4y^4 K_m(y) K'_m(y) I_m'^2(y) + 2y^4 I_m(y) I'_m(y) K_m'^2(y) + y^5 [I'_m(y) K'_m(y)]^2 \right\}. \end{aligned} \quad (4.142)$$

Our challenge now is to evaluate this quantity.

#### 4.7. Perspective

I have been working on this problem, on and off, since 1998, when I learned of Romeo's proof [262] that the renormalized van der Waals energy for a dilute dielectric cylinder was zero. Unfortunately, I had labored under a misconception concerning the form of the Green's dyadic, which was not in a sufficiently general form until I started re-examining this problem with my graduate student Ines Cavero-Pelaez this past year. We now have a consistent formal result, which only requires some delicate analysis to extract the answer. The results, and further details, will follow in a paper to appear later this year.

This promises to add another bit of understanding to our knowledge of Casimir forces, knowledge that seems to grow only incrementally based on specific calculations, since a general understanding is still not at hand.

## 5. Casimir Effects for Solitons

Our discussion throughout this article so far has been confined to idealized boundaries, although we alluded to a dynamical basis in the sections referring to the delta-function potentials. Of course, from the beginning of the subject, it has always been the goal to describe the interactions due to real interfaces, be they constituted of atoms and molecules, or due to solutions of the quantum field equations themselves. The most natural thing to consider is a solitonic background, where the soliton is a classical field configuration which minimizes the energy, and then consider the effect of quantum fluctuations around this background field. Perhaps the first physical ideas along this direction were presented in the context of the bag model [263, 264, 265, 266, 68, 267, 268]. The bag is supposed to represent, semiclassically, the notions of confinement, in which within the bag particles carrying color charge (quarks and gluons) are free to move subject to perturbative QCD interactions, whereas outside the bag, no colored objects can exist. Such a bag picture has never actually been derived from QCD, but it forms an enormously fruitful phenomenological framework. Similar pictures can be derived from truncated models [269, 270, 271, 272, 273, 274, 275].

Casimir energies have been discussed in connection with the bag model since 1980 [27, 229, 26]. (Actually, a zero-point energy parameter was put in the model from the beginning.) Unfortunately, no reliable result could be derived because interior contributions alone are inherently divergent. Efforts, however, more or less successful, were made to extract finite parts [276], and a summary of some of the phenomenological results can be found in Ref. [29]. Progress toward understanding the divergences promise to lead to more reliable predictions in the near future.

However, for kink and soliton solutions, reliable Casimir effects have been found in a number of cases. The reviews given at QFEXT03 [49] by van Nieuwenhuizen, Bordag, and Jaffe are a useful starting point. For example, Refs. [277, 278, 279] show that quantum corrections to the mass and central charge of supersymmetric solitons are nonvanishing even though zero-point energies of bosons and fermions seem to cancel. The Bogomolnyi bound is saturated because there is only one fermionic zero mode.

Very interesting methods have been presented in the past few years by the group led by Jaffe [280, 165, 281], based on subtraction from the local spectral density (related to derivatives of the phase shift) the first few Born terms, which correspond to low-order Feynman diagrams, which may be renormalized in the standard manner. For any smooth background a finite renormalized vacuum energy is obtained. These methods have been used, as noted in section 2 and 4, to critically discuss energies and self energies of idealized boundaries. In solitonic physics Fahri *et al* [282] have used these methods to compute quantum fluctuations around static classical solitons in Euclidean

electroweak theory, which are unstable, in an attempt to find stable quantum solitons. (See also Ref. [283].) No solutions have yet been found, yet some promising nonspherical candidates exist.

Bordag [284] considers the vacuum energy of a fermion in the background of a Nielsen-Olesen vortex (string) [285]. The vacuum energy is defined by zeta-function regularization, and is expressed in terms of the Jost function, evaluated by using the Abel-Plana formula. The quantum correction found in this way, however, is very small. It may be that in other cases, such as electroweak strings, the quantum vacuum energy might have more physical relevance, even leading to the stability of the string.

We should also mention that Casimir energies play an important role in lattice simulations of QCD. For its role in QCD string formation see, for example, Juge, Kuti, and Morningstar [286, 287] and Lüscher and Weisz [288].

## 6. Dynamical Casimir Effects

Everything discussed above referred to static configurations. In such a case the concept of energy is well-defined, but even then, as we have seen, it is not easy or noncontroversial to extract a physically observable effect. When the boundaries are moving, the situation is far more difficult.

In one dimension, the problem seems tractable. We can consider a point undergoing harmonic oscillations, and ask what are the consequences for a scalar field which must vanish at that point. We expect that the result is the production of real quanta of the field. This is the dynamical Casimir effect. However the only reliable results seem to be for motions which can be treated perturbatively, or in the opposite extreme, where the adiabatic (instantaneous) approximation applies for very rapid changes.

In three dimensions, the situation is still more challenging. Here we should mention the suggestion of Schwinger [289, 290], followed up by Eberlein [291, 292], Chodos [293, 294], Carlson [295, 296], Visser [297, 298, 299, 300, 301, 301], and others, that the copious production of light in sonoluminescence [302, 236] was due to the dynamical Casimir effect, due to the rapid expansion and contraction of a micron-sized air bubbles in water. The original estimation that there was sufficient energy available for this mechanism was based on a naive use of the cutoff value of  $\frac{1}{2} \sum \hbar\omega$ . An actual calculation showed that the energy was insufficient by 10 orders of magnitude [232]. Dynamically, photons indeed should be produced by QED by a rapidly oscillating bubble, but to produce the requisite number ( $10^6$  per flash) necessitated, if not superluminal velocities at least macroscopic collapse time scales of order  $10^{-15}$  s, rather than the observed  $10^{-11}$  s scale [80].

### 6.1. Fulling/Unruh/Hawking Radiation

One regime where definitive results exist for quantum particle production is in the general relativistic context. The Moore-Fulling-Davies Effect is the production of

photons by a mirror undergoing uniform acceleration [303, 304, 305]. The photon spectrum is thermal, with the temperature proportional to the acceleration of the mirror. The Unruh effect is very similar [306]. If the free equations of quantum field theory are examined in the frame of an accelerated observer, with acceleration  $a$ , it is found that such an observer sees a heat-bath of photons, again with  $T = a/2\pi$ . (For a precise description of these phenomena see the classic book by Birrell and Davies [307].)

These phenomena naturally are mirrored in gravitational phenomena. The celebrated Hawking radiation [308] is the production of quanta by a black hole. Energy is extracted from the black hole by particle-antiparticle production outside the horizon. One particle escapes, while the other falls into the black hole. The resulting thermal radiation has a temperature, in accordance with the expectation from the above, proportional to the surface gravity of the black hole, or inversely proportional to its mass  $M$ :

$$T = 1.2 \times 10^{26} \text{K} \left( \frac{1 \text{ gm}}{M} \right). \quad (6.1)$$

(Again, see Ref. [307].)

Scully and collaborators [309] have proposed an experiment to measure the Unruh effect by injecting atoms into a microwave or optical cavity, which atoms then undergo acceleration. Hu *et al* [86, 310] persuasively argue that this experiment will not detect the Unruh effect, because the latter does not refer to the radiation produced by the accelerated detector (which is nil), Lorentz invariance, crucial to the Unruh effect, is broken by the cavity, the thermal distribution of photons in the cavity is not that of the Unruh effect, and finally that the injection mechanism will produce cavity excitation so that acceleration no longer plays a crucial role.

For recent work on moving charges, detectors, and mirrors by Hu's group, see Ref. [311].

## 6.2. Terrestrial Applications

Most of the calculations of the dynamical Casimir effect have considered scalar fields. For example, Crocce *et al* [312] consider a cavity bifurcated by a semiconducting film whose properties can be changed in time by laser pulses, modeled by a time-dependent potential

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}V(t)\delta(x)\phi^2. \quad (6.2)$$

The inhomogeneous wave equation is solved with the time-dependence given as a first-order perturbation, with the result that if the film is driven parametrically, that is, in resonance with one of the modes of the cavity, particle (photon) production grows exponentially. This is in line with the expectations from the 1 + 1 dimensional cavity, where if the length undergoes periodic oscillations at a multiple of the fundamental frequency  $\omega_0 = \pi/L$  of the cavity

$$L(t) = L + \Delta L \sin k\omega_0 t, \quad (6.3)$$

for large times the energy produced is [313, 314, 315, 316, 10]

$$E(t) = -\frac{k^2\omega_0}{24} + \frac{(k^2 - 1)\omega_0}{24} \cosh k\omega_0 \frac{\Delta L}{L} t, \quad k = 1, 2, 3, \dots \quad (6.4)$$

The  $k = 0$  value is the Casimir energy corresponding to (2.14). A numerical simulation method for calculating particle production in both cosmological and terrestrial settings is given in Ref. [317].

One of the few treatments of the  $3 + 1$  dimensional situation for electromagnetic fields is that of Uhlmann *et al* [318], who consider a rectangular cavity of length  $L$  with perfectly conducting walls, with a narrow dielectric slab of width  $a$  at one end possessing a time-dependent permittivity  $\varepsilon(t)$ . The time dependence is still treated perturbatively. Only TM modes are effective in producing photons, the number of which increase exponentially on resonance, just as in Ref. [312]:

$$\langle N \rangle(t) = \sinh^2 \left( \frac{k_{\perp}^2 \chi a}{\omega \varepsilon_0 L} t \right), \quad (6.5)$$

where  $k_{\perp}^2$  is the square of the transverse wave vector,  $\omega$  is the resonant frequency, and  $\chi$  is the amplitude of the sinusoidal time-varying permittivity,

$$\frac{\varepsilon_0}{\varepsilon(t)} = \text{constant} + \chi \sin \omega t. \quad (6.6)$$

The challenge will be to devise a practical experiment where this effect can be observed in the microwave regime.

## 7. Casimir Effect and the Cosmological Constant

### 7.1. Cosmological Constant Problem and Recent Observations

It has been appreciated for many years that there is an apparently fundamental conflict between quantum field theory and the smallness of the cosmological constant [5, 6, 319, 320]. This is because the zero-point energy of the quantum fields (including gravity) in the universe should give rise to an observable cosmological vacuum energy density,

$$u_{\text{cosmo}} \sim \frac{1}{L_{\text{Pl}}^4}, \quad (7.1)$$

where the Planck length is

$$L_{\text{Pl}} = \sqrt{G_N} = 1.6 \times 10^{-33} \text{ cm}. \quad (7.2)$$

(We use natural units with  $\hbar = c = 1$ . The conversion factor is  $\hbar c \simeq 2 \times 10^{-14} \text{ GeV cm}$ .) This means that the cosmic vacuum energy density would be

$$u_{\text{cosmo}} \sim 10^{118} \text{ GeV cm}^{-3}, \quad (7.3)$$

which is 123 orders of magnitude larger than the critical mass density required to close the universe:

$$\rho_c = \frac{3H_0^2}{8\pi G_N} = 1.05 \times 10^{-5} h_0^2 \text{ GeV cm}^{-3}, \quad (7.4)$$



in terms of the dimensionless Hubble constant,  $h_0 = H_0/100 \text{ km s}^{-1}\text{Mpc}^{-1} \approx 0.7$ . From relativistic covariance the cosmological vacuum energy density must be the 00 component of the expectation value of the energy-momentum tensor, which we can identify with the cosmological constant:

$$\langle T^{\mu\nu} \rangle = -u g^{\mu\nu} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}. \quad (7.5)$$

[We use the metric with signature  $(-1, 1, 1, 1)$ .] Of course this is absurd with  $u$  given by Eq. (7.3), which would have caused the universe to expand to zero density long ago.

For most of the past century, it was the prejudice of theoreticians that the cosmological constant was exactly zero, although no one could give a convincing argument. In the last few years, however, with the new data gathered on the brightness-redshift relation for very distant type Ia supernovæ [321, 322, 323, 324, 325], corroborated by observations of the anisotropy in the cosmic microwave background [326], observations of large-scale structure [327], and of the Sachs-Wolf effect [328]. Thus, it seems clear that the cosmological constant is near the critical value, and in fact makes up the majority of the energy in the universe,

$$\Omega_\Lambda = \Lambda/8\pi G\rho_c \simeq 0.75. \quad (7.6)$$

Dark matter makes up most of the rest. Data are consistent with the value for the ratio of pressure to energy predicted by the cosmological constant interpretation,  $w = p/\rho = -1$  [329, 330]. For reviews of the observational situation, see Ref. [331, 332]. It is very hard to understand how the cosmological constant can be nonzero but small. (For a recent example of how difficult this problem is to solve, see Dolgov [333].)

## 7.2. Quantum Fluctuations

In Ref. [334, 335] we have presented a plausible scenario for understanding this puzzle. It seems quite clear that vacuum fluctuations in the gravitational and matter fields in flat Minkowski space give a zero cosmological constant. On the other hand, since the work of Kaluza and Klein [336, 337, 338] it has been an exciting possibility that there exist extra dimensions beyond those of Minkowski space-time. Why do we not experience those dimensions? The simplest possibility seems to be that those extra dimensions are curled up in a space  $\mathcal{S}$  of size  $a$ , smaller than some observable limit.

Of course, in recent years, the idea of extra dimensions has become much more compelling. Superstring theory requires at least 10 dimensions, six of which must be compactified, and the putative M theory, supergravity, is an 11 dimensional theory. Perhaps, if only gravity experiences the extra dimensions, they could be of macroscopic size. Various scenarios have been suggested [339, 340, 341].

Macroscopic extra dimensions imply deviations from Newton's law at such a scale. Five years ago, millimeter scale deviations seemed plausible, and many theorists hoped that the higher-dimensional world was on the brink of discovery. Experiments were initiated [342, 343]. Recently, the results of definitive Cavendish-type experiments have appeared [344, 345, 346, 347], which indicate no deviation from Newton's law

down to 100  $\mu\text{m}$ . (The experimental constraints on non-Newtonian gravity discussed in section 3.6 are so weak as to be useless in this connection.)

This poses an extremely serious constraint for model-builders.

Earlier we had proposed [334] that a very tight constraint indeed emerges if we recognize that compact dimensions of size  $a$  necessarily possess a quantum vacuum or Casimir energy of order  $u(z) \sim a^{-4}$ . These can be calculated in simple cases. Appelquist and Chodos [348, 349] found that the Casimir energy for the case of scalar field on a circle,  $\mathcal{S} = S^1$ , was

$$u_C = -\frac{3\zeta(5)}{64\pi^6 a^4} = -\frac{5.056 \times 10^{-5}}{a^4}, \quad (7.7)$$

which needs only to be multiplied by 5 for graviton fluctuations. The general case of scalars on  $\mathcal{S} = S^N$ ,  $N$  odd, was considered by Candelas and Weinberg [350], who found that the Casimir energy was positive for  $3 \leq N \leq 19$ , with a maximum at  $N = 13$  of  $u_C = 1.374 \times 10^{-3}/a^4$ . The even dimensional case was much more subtle, because it was divergent. Kantowski and Milton [63] showed that the coefficient of the logarithmic divergence was unique, and adopting the Planck length as the natural cutoff, found

$$S^N, N \text{ even} : \quad u_C^N = \frac{\alpha_N}{a^4} \ln \frac{a}{L_{\text{Pl}}}, \quad (7.8)$$

but  $\alpha_N$  was always negative for scalars. In a second paper [351] we extended the analysis to vectors, tensors, fermions, and to massive particles, among which cases positive values of the (divergent) Casimir energy could be found. In an unsuccessful attempt to find stable configurations, the analysis was extended to cases where the internal space was the product of spheres [352].

It is important to recognize that these Casimir energies correspond to a cosmological constant in our 3 + 1 dimensional world, not in the extra compactified dimensions or “bulk.” They constitute an effective source term in the 4-dimensional Einstein equations. Note that because the scale  $a$  makes no reference to four-dimensional space, the total free energy of the universe (of volume  $V$ ) arising from this source is  $F = V u_C$ , so as required for dark energy or a cosmological constant,

$$p = -\frac{\partial}{\partial V} F = -u_C \quad T^{\mu\nu} = -u_C g^{\mu\nu}, \quad \text{i.e.,} \quad w = -1. \quad (7.9)$$

The goal, of course, in all these investigations was to include graviton fluctuations. However, it immediately became apparent that the results were gauge- and reparameterization-dependent unless the DeWitt-Vilkovisky formalism was adopted [353, 354, 355, 356]. This was an extraordinarily difficult task. Among the earlier papers in which the unique effective action is given in simple cases we cite Ref. [357]; see references therein. Only in 2000 did the general analysis for gravity appear, with results for a few special geometries [358]. Cho and Kantowski obtain the unique divergent part of the effective action for  $\mathcal{S} = S^2$ ,  $S^4$ , and  $S^6$ , as polynomials in  $\Lambda a^2$ . (Unfortunately, once again, they are unable to find any stable configurations.)

The results for the coefficient  $\alpha_N$  in (7.8) are  $\alpha_2 = 1.70 \times 10^{-2}$ ,  $\alpha_4 = -0.489$ , and  $\alpha_6 = 5.10$ , for  $\Lambda a^2 \sim G/a^2 \ll 1$ . Graviton fluctuations dominate matter fluctuations,

except in the case of a large number of matter fields in a small number of dimensions. Of course, it would be very interesting to know the graviton fluctuation results for odd-dimensional spaces, but that seems to be a more difficult calculation; it is far easier to compute the divergent part, which appears as a heat kernel coefficient, than the finite part, which is all there is in odd-dimensional spaces.

These generic results may be applied to recent popular scenarios. For example, in the ADD scheme only gravity propagates in the bulk, while the RS approach has other bulk fields in a single extra dimension.

Let us now perform some simple estimates of the cosmological constant in these models. The data require a positive cosmological constant, so we can exclude those cases where the Casimir energy is negative. If we use the divergent results for even dimensions, merely requiring that this be less than the critical density  $\rho_c$  implies the inequality ( $\alpha > 0$ )

$$a > [\alpha \ln(a/L_{\text{Pl}})]^{1/4} 80 \mu\text{m}, \quad (7.10)$$

where we can approximate  $(\ln a/L_{\text{Pl}})^{1/4} \approx 2.9$ . The absence of deviations from Newton's law above  $100 \mu\text{m}$  rules out all but one of the gravity cases ( $S^2$ ) given by Cho and Kantowski [358]. For matter fluctuations only, excluded are  $N > 14$  for a single vector field and  $N > 6$  for a single tensor field. (Fermions always have a negative Casimir energy in even dimensions.) Of course, it is possible to achieve cancellations by including various matter fields and gravity. In general the Casimir energy is obtained by summing over the species of field which propagate in the extra dimensions,

$$u_{\text{tot}} = \frac{1}{a^4} \sum_i [\alpha_i \ln(a/L_{\text{Pl}}) + \beta_i] \approx \frac{\beta_{\text{eff}}}{a^4}, \quad (7.11)$$

which leads to a lower limit analogous to (7.10). Presumably, if exact supersymmetry held in the extra dimensions (including supersymmetric boundary conditions), the Casimir energy would vanish, but this would seem to be difficult to achieve with *large* extra dimensions ( $1 \text{ mm}$  corresponds to  $2 \times 10^{-4} \text{ eV}$ ). (See, for example, Ref. [333].)

That there is a correlation between the currently favored value of the cosmological constant and submillimeter-sized extra dimensions has been noted qualitatively before [359, 360, 361, 362]. A first attempt to calculate the cosmological constant in terms of Casimir energies in the context of deconstructed extra dimensions is given by Bauer, Lindner, and Seidl [363].

In summary, we have proposed the following scenario to explain the predominance of dark energy in the universe.

- (i) Quantum fluctuations of gravity/matter fields in extra dimensions give rise to a dark energy, or cosmological constant,  $\propto 1/a^4$  where  $a$  is the size of the extra dimensions.
- (ii) The dark energy will be too large unless  $a > 10 - 300 \mu\text{m}$ .
- (iii) Laboratory (Cavendish) tests of Newton's law require  $a < 100 \mu\text{m}$ .

- (iv) Thus, extra dimensions may be on the verge of discovery. If serious limits on the validity of Newtonian gravity can be extended down to  $10 \mu\text{m}$ , then we would have to conclude that
- (v) Extra dimensions probably do not exist, and dark energy has another origin, for example quintessence [364]. However, the fact that the rapidly improving data favor the cosmological constant interpretation of dark energy [329, 330], makes alternatives disfavored, since they would generally exhibit time-dependence.

## 8. Future Prospects and Perspectives

In this review we have attempted to present a personal perspective on the progress in understanding quantum vacuum energy and its physical implication in the past four years. The primary stimulus for the development of the subject has been the tremendous progress on the experimental front. This has brought to the fore issues that were regarded as arcane, such as the temperature dependence of the Casimir forces between metal plates, the meaning of infinities encountered in calculations of quantum vacuum energy, and the source of the cosmological dark energy, which it is hard to believe does not have something to do with quantum fluctuations, yet is remarkably small. At this point, no definitive resolution of any of these issues has been given; yet, progress is rapid, and I hope that this status report may help sharpen issues and contribute in some small way to the solution of outstanding problems.

The reader will have noted that this document is far from even-handed. I have continued to focus on the use of Green's function techniques as expounded in my earlier monograph [29]. I do not mean to disparage in any way the valuable progress made using other techniques, including use of zeta-function methods, Jost functions, worldline approaches, and scattering phase-shift formalisms; although I do continue to believe that the Green's function approach is the most physical. I also have focused on topics that are of personal interest, so if I have slighted important subjects and researchers, I beg forbearance.

I want to close this review by briefly mentioning a few topics that do not seem to fit in the sections above. For example, there has been important work in the subject of the Casimir effect in critical systems by Krech and collaborators [365, 51], in which they consider massless excitations caused by critical fluctuations of the order parameter of a condensed-matter system about the critical temperature  $1/\beta_c$ . For  $d$  transverse dimensions, that force is

$$\beta_c \mathcal{F} = (d-1) \frac{\Delta}{a}, \tag{8.1}$$

where  $\Delta$  is universal. For an application to thinning of superfluid helium films, see Ref. [366]. Williams [367] shows that vortex excitations are the source of the critical fluctuations that give rise to the critical Casimir force in this situation.

An acoustic analog of the Casimir effect has been discussed [368, 369, 370].

There have been many extremely interesting contributions to cosmological and brane-world models. For example, Dowker [371] considers the Casimir effect in nontrivial cosmological topologies. A condensate of the metric tensor may stabilize Euclidean Einstein gravity in a manner not unrelated to the Casimir effect [372]. And Brevik has questioned the meaning of the Cardy-Verlinde formula expressing a bound on the entropy [373]. (See also Ref. [374].) Mazur and Mottola [375, 376] have suggested that dark energy is quantum vacuum energy due to a causal boundary effect at the cosmological horizon – namely, that instead of a black hole, there are three regions due to a quantum phase transition (perhaps due to the trace anomaly) in spacetime itself: exterior (Schwarzschild) where  $\rho = p = 0$ ; interior (de Sitter) where  $\rho = -p$ ; and a thin boundary shell where  $\rho = p$ . Details of this proposal are still vague; without a detailed calculation one cannot tell whether even the vacuum energy will emerge correctly.

There have been many contributions on Casimir effects in brane-world scenarios, for example, Refs. [377, 378].

Finally, we note that everything we have considered in this review has been at the one-loop level. Radiative corrections to the Casimir effect have, in fact, been considered by several authors. Most of the calculations have been in situations in which there is only one significant direction: For QED, see Ref. [379, 380], and for  $\lambda\phi^4$  theory, see Ref. [381, 382] and references therein. There is an impressive calculation of the radiative correction to the Casimir effect with a spherical shell boundary, perfectly conducting as far as electromagnetism is concerned, but transparent to electrons by Bordag and Lindig [383].

So we leave the subject of Casimir phenomena as a work in progress. It is clear that quantum fluctuation forces are vitally important both in the very large and the very small domains, and that they will play increasingly central roles in engineering applications. Thus, the subject is an exciting interdisciplinary topic, with both fundamental and technological spinoffs. Thus the uncertainty principle is not just about atomic and subatomic physics, but it may control our future, in many senses.

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