TEMPERATURE DEPENDENCE OF THE CASIMIR FORCE FOR METALS

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Starting from the Lifshitz formula for the Casimir force between parallel plates we calculate the difference between the forces at two different settings, one in which the temperature is $T_1 = 350$ K, the other when $T_2 = 300$ K. As material we choose gold, and make use of the Drude dispersion relation. Our results, which are shown graphically, should be directly comparable to experiment. As an analogous calculation based upon the plasma dispersion relation leads to a different theoretical force difference, an experiment of this kind would be a decisive test. We also present an analogous calculation for the case when the two plates are replaced with a sphere-plate system, still with gold as material in both bodies. The sphere is assumed so large that the proximity theorem holds. Discussion of the consistency with the third law of thermodynamics and the validity of the surface impedance approach is provided.

1 Introduction

Recently, there has been something of a heated controversy concerning the temperature dependence of the Casimir effect between parallel metallic plates. The classic result for the pressure on one ideal metal plate separated from another by a distance a is[1] ($\beta = 1/T$; in this paper we set $\hbar = c = k_B = 1$)

$$\mathcal{F}^T = -\frac{\pi^2}{240a^4}\left[1 + \frac{1}{3}\left(\frac{2a}{\beta}\right)^4\right] \quad (aT \ll 1). \tag{1}$$

Recently, however, it has been suggested[2] that the transverse electric zero mode should not contribute, which if the metal is otherwise regarded as ideal (reflection coefficients equal unity), would lead to a presumably observable linear temperature correction to the Casimir force, and a violation of the third law of thermodynamics.[3,4] However, real metals do not possess ideal reflection coefficients, and so are more complicated, and there seems to be no

contradiction with the Nernst heat theorem.[5] It is the purpose of this paper to address the observable and thermodynamical consequences on the Casimir pressure of using the observed permittivity of real metals, particularly gold, which has been used in many recent experiments.

2 A Basic Result Inferred from the Kramers-Kronig Relation

The crucial observation in the proof that the transverse electric zero mode does not contribute at $T = 0$ given in Ref.[5] is the condition $\lim_{\omega \to 0} \omega^2 \varepsilon(\omega) = 0$. This also follows from the Kramers-Kronig (unsubtracted) dispersion relation[6]

$$\chi(\omega) = \frac{\varepsilon(\omega) - 1}{4\pi} = \frac{\omega_p^2}{4\pi} \int_0^\infty d\omega' \frac{p(\omega')}{\omega'^2 - (\omega^2 + i\epsilon)^2}. \tag{2}$$

Here the spectral function is positive, $p(\omega) > 0$, and satisfies the sum rule[a]

$$\int_0^\infty d\omega' \, p(\omega') = 1. \tag{3}$$

The structure of this dispersion relation allows us to make the complex frequency rotation, $\omega \to i\zeta$, so

$$\chi(i\zeta) = \frac{\omega_p^2}{4\pi} \int_0^\infty d\omega' \frac{p(\omega')}{\omega'^2 + \zeta^2}. \tag{4}$$

The proof is now immediate, because

$$\zeta^2 \int_0^\infty d\omega' \frac{p(\omega')}{\omega'^2 + \zeta^2} = 1 - \int_0^\infty d\omega' \frac{\omega'^2 p(\omega')}{\omega'^2 + \zeta^2}, \tag{5}$$

which uses the sum rule. Since the last integral converges to one as $\zeta \to 0$, the desired limit is established. Of course, this behavior is consistent with the Drude model, and not with the plasma model.

3 Temperature Dependent Lifshitz Force

We now recall the Lifshitz expression for the Casimir force between two parallel nonmagnetic plates, characterized by a permittivity $\varepsilon(\omega)$, and separated by a gap a. In the notation of Ref. [5] the force per unit area can be written as

$$\mathcal{F}^T = -\frac{1}{\pi \beta a^3} \sum_{m=0}^\infty {}' \int_{m\gamma}^\infty y^2 dy \left[\frac{A_m e^{-2y}}{1 - A_m e^{-2y}} + \frac{B_m e^{-2y}}{1 - B_m e^{-2y}} \right]. \tag{6}$$

Here, y is a dimensionless quantity, $y = qa$, with

$$q = \sqrt{k_\perp^2 + \zeta_m^2}, \quad \zeta_m = 2\pi m/\beta, \quad \gamma = 2\pi a/\beta, \tag{7}$$

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[a]The normalization of $p(\omega)$ plays no role in our argument. The high-frequency behavior of the susceptibility given by the Drude formula is accepted, for which the spectral function is $p(\omega) = (2/\pi)\gamma/(\omega^2 + \gamma^2)$.

and $\mathbf{k}_\perp$ is the transverse wave vector (i.e., the component of $\mathbf{k}$ parallel to the plates). Further, we have defined the squared reflection coefficients A_m and B_m by

$$A_m = \left(\frac{\varepsilon p - s}{\varepsilon p + s} \right)^2, \quad B_m = \left(\frac{s - p}{s + p} \right)^2, \tag{8}$$

where the Lifshitz variables s and p are given by

$$s = \sqrt{\varepsilon - 1 + p^2}, \quad p = q/\zeta_m. \tag{9}$$

The permittivity $\varepsilon(i\zeta_m)$ is a function of the imaginary Matsubara frequency ζ_m. If the medium is non-dispersive, $A_m(q) = A(p)$ and $B_m(q) = B(p)$. The prime on the sum in Eq. (6) means that the $m = 0$ term is counted with half weight.

The free energy F per unit area follows from the relation $\mathcal{F}^T = -\partial F/\partial a$:

$$F = \frac{1}{2\pi\beta a^2} \sum_{m=0}^{\infty} {}' \int_{m\gamma}^{\infty} y\, dy \left[\ln(1 - A_m e^{-2y}) + \ln(1 - B_m e^{-2y}) \right]. \tag{10}$$

Note that A_m refers to the TM mode, B_m refers to the TE mode.

We are looking for temperature effects in the Casimir force, and shall in the following focus attention on the following two temperatures, both easily accessible in the laboratory:

$$T_1 = 300\,\text{K}, \quad T_2 = 350\,\text{K}. \tag{11}$$

Specifically, we want to calculate the difference in Casimir pressure between the two temperatures:

$$\Delta\mathcal{F} = \mathcal{F}(350\,\text{K}) - \mathcal{F}(300\,\text{K}) = |\mathcal{F}(300\,\text{K})| - |\mathcal{F}(350\,\text{K})| \tag{12}$$

(i.e., for convenience we let $\Delta\mathcal{F}$ mean the difference between the magnitudes). This quantity depends on the values of A_m and B_m, which in turn depend on which dispersion relation is adopted for $\varepsilon(i\zeta_m)$. In this way we calculate a decisive quantity which in principle is directly comparable to experiment.

4 Dispersion Relation

As in our preceding paper,[5] we chose gold as the material of which the plates were composed. For this metal we have access to excellent numerical data for $\varepsilon(i\zeta)$ (courtesy of Astrid Lambrecht and Serge Reynaud). The data are shown graphically in Refs. [7,5]. It turns out that for low and moderate frequencies, at least up to about 1.5×10^{15} rad/s (1 eV), the data are nicely reproduced by the Drude dispersion relation

$$\varepsilon(i\zeta) = 1 + \frac{\omega_p^2}{\zeta(\zeta + \nu)}, \tag{13}$$

where, at room temperature, the plasma frequency ω_p and the relaxation frequency ν are equal to

$$\omega_p = 9.0\,\text{eV}, \quad \nu = 35\,\text{meV}. \tag{14}$$

Strictly speaking, one should take into account also the temperature dependence of ν, such that $\nu \to \nu(i\zeta, T)$:

$$\varepsilon(i\zeta, T) = 1 + \frac{\omega_p^2}{\zeta[\zeta + \nu(T)]}. \tag{15}$$

Here, $\nu(T)$ can be calculated via use of the Bloch-Grüneisen formula, as explained in Appendix C in Ref. [5].

At high frequencies, $\zeta > 2 \times 10^{15}$ rad/s, the Drude formula gives values for ε which are too low.

In this context it is of interest to know: What frequency region gives the main contribution to the Casimir force? To analyze this point, it is convenient to go back to the expression (6), from which it is seen that the most important region is when y is of order unity, $y \sim 1$. Assuming that the transverse wave vector $\mathbf{k}_\perp$ does not dominate in the expression (7) for q, we thus get the condition $2\pi m a/\beta \sim 1$, or

$$m \sim \frac{1}{2\pi}\frac{1}{aT}. \tag{16}$$

If $a = 1$ μm, we have $aT = 0.13$ at room temperature, resulting in $m = 1$ as the dominant frequency mode. If $a = 0.5$ μm, we expect a somewhat smeared-out distribution over the lowest integer values for m. If $a = 3$ μm or higher, the $m = 0$ mode should be highly dominant: the problem becomes a high-temperature problem. We have done explicit numerical calculations, reproduced here in Table 1 for convenience,[5] which confirm these expectations in detail. The numbers in the table are the relative percentage of each mode m, i.e., the quantities $(\mathcal{F}_m^T/\mathcal{F}^T) \times 100$, where $\mathcal{F}_m^T$ denotes the mth mode contribution to the force:

$$\mathcal{F}^T = \sum_{m=0}^{\infty} \mathcal{F}_m^T. \tag{17}$$

The numbers in the table are calculated from the empirical results for $\varepsilon(i\zeta_m)$.

The Matsubara frequencies, in view of Eq. (7), become

$$\zeta_m = \begin{cases} 2.47m \times 10^{14}\,\text{rad/s} & \text{for} \quad T = 300\,\text{K}, \\ 2.88m \times 10^{14}\,\text{rad/s} & \text{for} \quad T = 350\,\text{K}. \end{cases} \tag{18}$$

From this it follows that from $m = 1$ up to about $m = 6$, the frequencies stay so low that the Drude formula can be used with confidence. And from Table 1 we see that in particular for large distances, from $a = 1$ μm and upwards, these frequencies encompass the large majority of the mode contributions to the

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Table 1. Contribution from the various Matsubara frequencies for gold. The relative contribution of $\mathcal{F}^T$ is given, in percent, for each mode in the interval $m \in [0,7]$. Room temperature is assumed. Empirical permittivities inserted, for all frequencies.

$a(\mu\mathrm{m})$	$m=0$	1	2	3	4	5	6	7
0.5	10.20	31.24	22.95	15.09	9.18	5.28	2.91	1.55
1	20.07	49.37	20.83	6.97	2.03	0.54	0.14	0.03
2	44.56	49.87	5.17	0.37	0.02			
3	70.95	28.41	0.63	0.01				
4	88.88	11.07	0.05					
5	96.58	3.42						
6	99.06	0.94						
7	99.76	0.24						

force. For small gap widths, $a = 0.5\,\mu\mathrm{m}$ and lower, the situation may be more questionable as indicated by the first line in the table, but it seems that even in these cases we can use the Drude formula with sufficient accuracy to make a meaningful comparison with experiments, given the present experimental accuracy. In the following we will use the Drude formula throughout. This simplifies the calculation significantly.

5   Calculated Results

Fig. 1 shows how the Casimir force itself varies with a, for parallel gold plates. Since the force according to Eq. (6) is negative for attraction, we choose to present the magnitude $|\mathcal{F}^T|$ in the figure, which for convenience is presented in semilog form. The curve is calculated for $T = 300$ K, but a similar curve calculated for $T = 350$ K turns out to be visually indistinguishable from the curve shown.

Fig. 2 shows the force difference $\Delta\mathcal{F}$, calculated according to Eq. (12). Typical orders of magnitude for $\Delta\mathcal{F}$ are in the millipascal range, when a is less than about $0.5\,\mu\mathrm{m}$. It is notable that it is the low-temperature term, $T = 300$ K, which yields the strongest force at low and intermediate distances. (When $a > 2.8\,\mu\mathrm{m}$, $\Delta\mathcal{F}$ changes sign.) We note that the influence from temperature in the formula (6) is rather complex since T occurs at three different places: (i) in the prefactor; (ii) in the lower limit of the integral, and (iii) in the dependence of A_m and B_m on T via the permittivity $\varepsilon = \varepsilon(i2\pi mT)$. (We will discuss the temperature dependence of the relaxation frequence ν below.) Actually we showed the essentials of the temperature dependence of the force already in Fig. 5 in Ref. [5]: Assuming the Drude formula, and ignoring the temperature dependence of the relaxation frequency, we found the force $|\mathcal{F}^T|$ to *diminish* with increasing T up to $aT \simeq 0.35$. For higher values of aT, the force was found to increase again. In view of this, indeed we expect
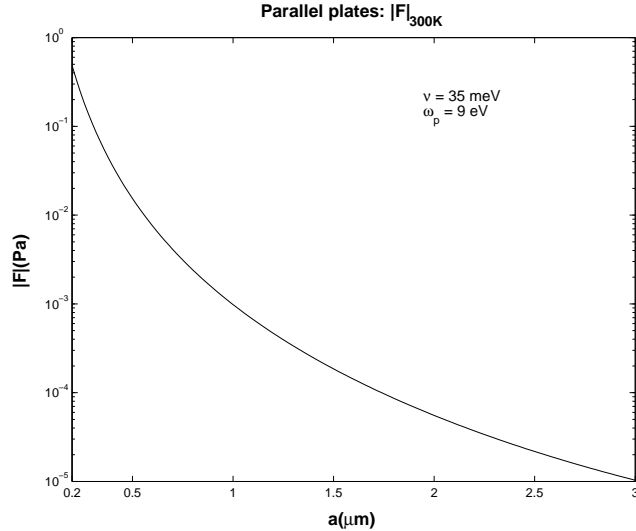
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Figure 1. Magnitude of Casimir pressure between parallel plates, versus gap width a, when $T = 300$ K. The curve for $T = 350$ K overlaps the one shown.

that the lower-temperature term dominates in Fig. 2 for low and intermediate distances. (Note also that the change in sign for $a = 2.8\,\mu$m, mentioned above, corresponds nicely to the expected transitional region since in this case $aT \simeq 0.37$.)

So far, we have assumed a constant relaxation frequency, $\nu = 35$ meV. Will our results be changed significantly if we take into account the temperature dependence of ν; cf. Eq. (15)? The answer turns out to be no. We have made an explicit calculation of this, based upon the Bloch-Grüneisen formula, yielding $\nu = 35.6$ meV for $T = 300$ K, and $\nu = 41.8$ meV for $T = 350$ K. The results were visually indistinguishable from those given in Fig. 2, so that our conclusion is that the assumed constancy of ν is justified for practical purposes.

Fig. 3 shows how the corresponding difference in free energy,

$$\Delta F = F(350\,\text{K}) - F(300\,\text{K}) = |F(300\,\text{K})| - |F(350\,\text{K})|, \qquad (19)$$

varies versus a for parallel plates. As expected, it is again the case $T = 300$ K which is the dominant one for small and moderate gap widths.

Finally, we consider the case which is probably the one of principal experimental interest, namely a sphere of radius R situated above a plane surface. Let the minimum distance between the spherical surface and the plane be a. Assuming the sphere large enough for the proximity theorem to be valid,[1] we
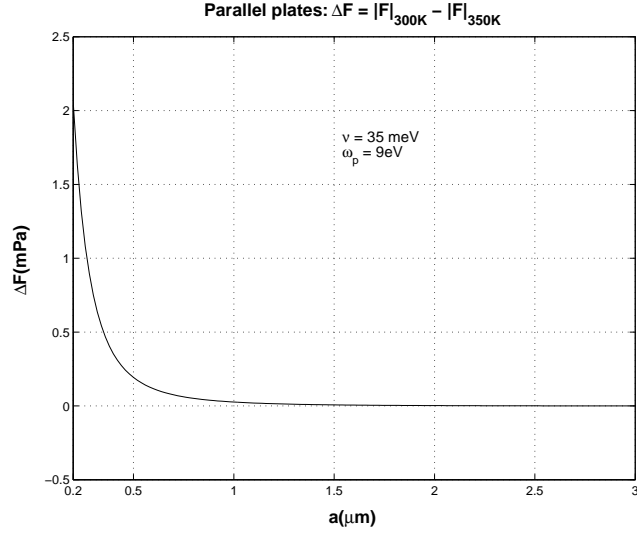
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Figure 2. Force difference between parallel plates, Eq. (12), versus gap width.
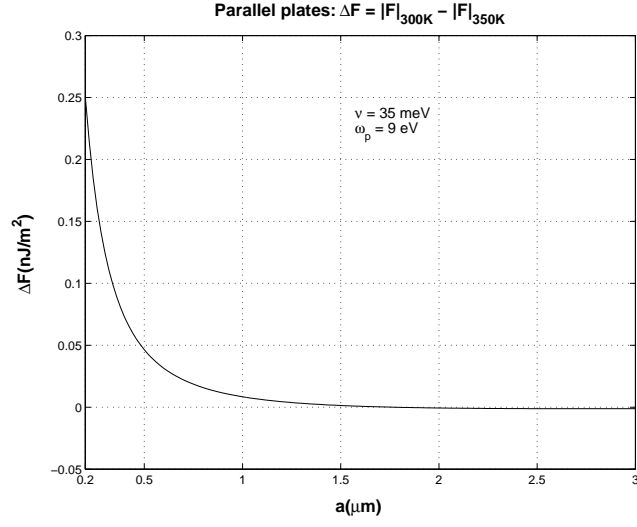

Figure 3. Free energy difference for parallel plates, Eq. (19), versus gap width.

have for the force $\mathcal{F}_{\mathrm{ps}}^{T}$ on the sphere:

$$\mathcal{F}_{\mathrm{ps}}^{T} = 2\pi R F(a), \tag{20}$$

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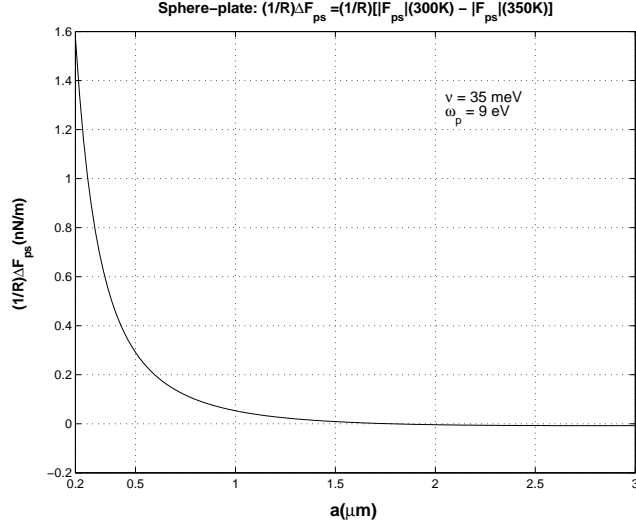**Sphere–plate: $(1/R)\Delta F_{ps} = (1/R)[|F_{ps}|(300K) - |F_{ps}|(350K)]$**

Figure 4. Force difference between sphere and plate, Eq. (21), versus gap width a.

where $F(a)$ is the free energy for parallel plates as given by Eq. (10). Fig. 4 shows how the difference

$$\frac{1}{R}\Delta\mathcal{F}_{\mathrm{ps}} = \frac{1}{R}\left[\mathcal{F}_{ps}(350\,\mathrm{K}) - \mathcal{F}_{ps}(300\,\mathrm{K})\right] \tag{21}$$

varies with a. Again, it is the lower-temperature term that is the dominant one.

A dedicated experiment to look for the temperature dependence we have proposed is probably essential to settle this issue. The recent experiment by Decca et al.[8] is claimed to be in disagreement with our prediction. However, that comparison is in fact not based on our detailed calculations, and the experiment is subject to large, uncontrolled errors.[9]

6   Behavior of the Free Energy at Low Temperature

The low temperature correction is dominated by low frequencies,[b] where the Drude formula is extremely accurate. Using this fact, we have performed analytic and numerical calculations which show the free energy has a quadratic low-temperature dependence, independent of the plate separation:

$$F(T) = F_0 + T^2\frac{\omega_p^2}{48\nu}(2\ln 2 - 1) = F_0 + T^2(19\ \mathrm{eV}), \tag{22}$$

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[b]This statement is in the context of using of the Euler-Maclaurin summation formula to evaluate Eq. (6), for example.
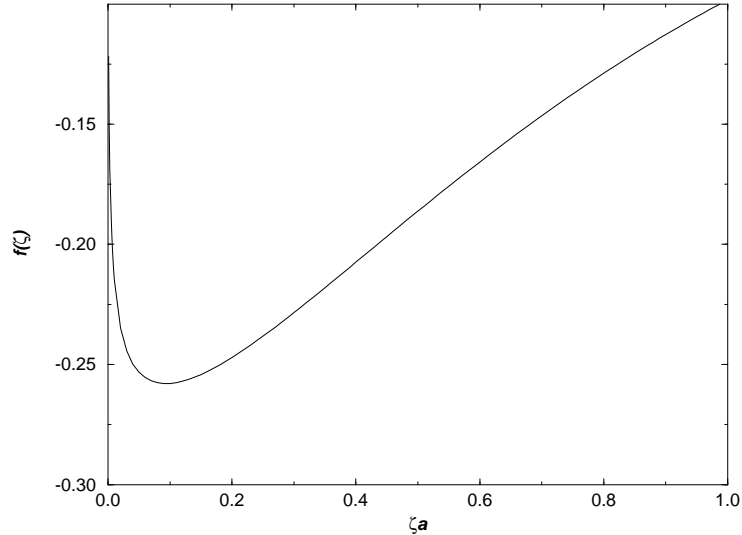
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Figure 5. The behavior of the free energy for low frequencies, in the Drude model, with parameters suitable for gold, and a plate separation of $a = 1$ μm. Here $F^{\mathrm{TE}} = \frac{T}{2\pi a^2} \sum_{m=0}^{\infty} {}' f(\zeta_m)$.

putting in the numbers for gold, rather than the naive extrapolation

$$F = F_0 + T\frac{\zeta(3)}{16\pi a^2} = F_0 + \frac{T}{4\pi a^2}0.30 \qquad (23)$$

We see from Fig. 5 that this value indeed results if one extrapolates the approximately linear curve there for $\zeta a > 0.25$ to zero, following the argument given in Eq. (2.8) of Ref. [5]. However, we see that the free energy smoothly changes to the quadratic behavior exhibited in Eq. (22).

Results consistent with these have been reported by Sernelius and Boström.[10]

7 Surface impedance form of reflection coefficient

It has been proposed that the resolution to the temperature problem for the Casimir effect is that the surface impedance form of the reflection coefficients should be used in the Lifshitz formula,[4] rather than that based on the bulk permittivity. Here we show that the two approaches are in fact equivalent, and that the former must include transverse momentum dependence.

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For the TE modes, the reflection coefficient is given by[6]

$$r^{\text{TE}} = -\frac{k_{1z} - k_{2z}}{k_{1z} + k_{2z}}, \tag{24}$$

where

$$k_z = \sqrt{\omega^2 \varepsilon - k_\perp^2} \rightarrow i\sqrt{\zeta^2[\varepsilon(i\zeta) - 1] + q^2}, \tag{25}$$

with $q^2 = k_\perp^2 + \zeta^2$, and the subscripts 1 and 2 refer to the metal and the vacuum regions, respectively. Now from Maxwell's equations outside sources we easily derive just inside the metal (the tangential components of $\mathbf{E}$ and $\mathbf{B}$ are continuous across the interface)

$$-ik_{1z}\mathbf{k}_\perp \cdot \mathbf{B}_\perp - i\omega\varepsilon\left(1 - \frac{k_\perp^2}{\omega^2\varepsilon}\right)\mathbf{k}_\perp \cdot (\mathbf{n} \times \mathbf{E}_\perp) = 0, \tag{26}$$

$$-ik_{1z}\mathbf{k}_\perp \cdot (\mathbf{n} \times \mathbf{E}_\perp) - i\omega\mathbf{k}_\perp \cdot \mathbf{B}_\perp = 0. \tag{27}$$

Here $\mathbf{n}$ is the normal to the interface. Now the surface impedance is defined by

$$\mathbf{E}_\perp = Z(\omega, \mathbf{k}_\perp)\mathbf{B}_\perp \times \mathbf{n}, \quad \mathbf{n} \times \mathbf{E}_\perp = Z(\omega, \mathbf{k}_\perp)\mathbf{B}_\perp. \tag{28}$$

So eliminating $\mathbf{B}_\perp$ using this definition we find two equations:

$$k_{1z} = -\frac{\omega}{Z}, \tag{29}$$

$$k_{1z}^2 = \omega^2\varepsilon - k_\perp^2, \tag{30}$$

the latter being the expected dispersion relation (25). Substituting this into the expression for the reflection coefficient we find

$$r^{\text{TE}} = -\frac{\zeta + Zq}{\zeta - Zq} = -\frac{1 + Zp}{1 - Zp}, \quad p = \frac{q}{\zeta}, \tag{31}$$

which apart from (relative) signs (presumably just a different convention choice) coincides with that given in Geyer et al.[4] or Bezerra et al.[11].

7.1 Dependence on Transverse Momentum

However, it is crucial to note that the "surface impedance" so defined depends on the transverse momentum,

$$Z = -\frac{\zeta}{\sqrt{\zeta^2[\varepsilon(i\zeta) - 1] + q^2}}, \tag{32}$$

and so $r^{\text{TE}} \rightarrow 0$ as $\zeta \rightarrow 0$ just as in the dielectric constant formulation. Of course, we have exactly the same result for the energy as before, since this is nothing but a slight change of notation.

It is therefore incorrect to assume that Z is only a function of frequency, not of transverse momentum, and to use the normal and anomalous skin effect formulas derived for real waves impinging on imperfect conductors.[c]

How does the usual argument go? The normal component of the wavevector in a conductor is given by

$$k_z = \left[\omega^2\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right) - k_\perp^2\right]^{1/2} \rightarrow \sqrt{i4\pi\omega\sigma},\tag{33}$$

from which the usual normal skin effect formula follows immediately,

$$Z(\omega) = -(1-i)\sqrt{\frac{\omega}{8\pi\sigma}}.\tag{34}$$

However, the last step here consists in omitting two "small" terms: ε (okay) and $k_\perp^2 \le \omega^2$. Here this last is not valid because in going to finite temperature we have severed the connection between $\omega \rightarrow i\zeta$ and $k_\perp$; the latter is in no sense ignorable as we take $\zeta \rightarrow 0$ to determine the low temperature dependence. This is the same error to which we refer in our published paper.[5]

These considerations are consistent with those of Esquivel et al.[12]

8 Conclusions

Our results of main interest are probably those shown in Figs. 2 and 4. These force curves show a dependence upon a that reflect our underlying choice of the Drude dispersion relation. We can compare our results with those recently obtained by Chen et al.[13] They make use of the plasma dispersion relation instead of the Drude relation, and obtain results for the Casimir forces that differ from ours even in *sign*. The force curves thus give rise to a very useful critical test, in principle. It would be quite interesting if the experimentalists could measure these force curves directly. We have also commented on the purported violation of basic principles of thermodynamics and on the claimed necessity to use surface impedances at ~ 1 μm plate separations and find no merit in these objections.

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[c]Of course, in general, the permittivity will be a function both of the frequency and the transverse momentum, $\varepsilon(\omega, \mathbf{k}_\perp)$, but we believe the latter dependence is not significant for separations larger that $\hbar c/\omega_p = 0.02$ μm.

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