

Dynamical Casimir Effect and Quantum Cosmology

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Abstract

We apply the background field method and the effective action formalism to describe the four-dimensional dynamical Casimir effect. Our picture corresponds to the consideration of quantum cosmology for an expanding FRW universe (the boundary conditions act as a moving mirror) filled by a quantum massless GUT which is conformally invariant. We consider cases in which the static Casimir energy is attractive and repulsive. Inserting the simplest possible inertial term, we find, in the adiabatic (and semiclassical) approximation, the dynamical evolution of the scale factor and the dynamical Casimir stress analytically and numerically (for $SU(2)$ super Yang-Mills theory). Alternative kinetic energy terms are explored in the Appendix.

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1. INTRODUCTION

The Casimir effect [1] can be regarded as the change in the zero-point fluctuations due to nontrivial boundary conditions. Surveys of the effect are given, for instance, by Plunien et al. [2], Mostepanenko and Trunov [3], and Milton [4]. The recent “resource letter” of Lamoreaux [5] contains a wealth of references, although it is admittedly highly incomplete.

In the past, the Casimir effect has been considered as a *static* effect. Growing interest in recent years has been drawn to the *dynamical* variant of the effect, meaning, in essence, that not only the geometrical configurations of the external boundaries (such as plates) but also their velocities play a physical role. Moore [6] is probably the first to have considered the dynamical Casimir effect. Examples of more recent references are [7] and [8].

The recent paper of Nagatani and Shigetomi [9] is an interesting development in this direction. These authors focused attention on the fact that if moving boundaries (mirrors) create radiation, the mirrors have to experience a reaction force. They proposed an effective theory for the back reaction of the dynamical Casimir effect in (1+1) dimensions for a scalar field, this theory being constructed by the background field method in the path integral formalism. In fact, they considered a kind of $2d$ quantum cosmology for describing the dynamical Casimir effect.

In the present paper we show how to apply the effective action formalism, using the background field method, to formulate the dynamical Casimir effect in four dimensions in a convenient and elegant form. We are able to consider an arbitrary matter content (typically a grand unified theory or GUT) and present the dynamical Casimir effect as a kind of quantum cosmological model. Using the background field method, we treat the geometrical configuration of the boundaries classically, but consider the GUT in the interior region as a quantum object.

In the next section we consider a GUT in a three-dimensional space, where the size of the space $a(t)$ is a dynamical variable. Similarly to [9] we make use of the adiabatic approximation. Exploiting the conformal invariance of the theory we calculate the anomaly-induced effective action W . In the simplest case (a torus), W is given by Eq. (2.4). We consider the static Casimir energy in section 3, and show that, for the usual boundary conditions on the torus, the Casimir energy is attractive. In section 4 we start with the effective action, Eq. (4.1), for the dynamical case. Introducing a mass m associated with the scale factor a , with a corresponding kinetic energy in the low-velocity approximation equal to $\frac{1}{2}m\dot{a}^2$ (a phenomenological term), we then consider two cases. If the Casimir energy is attractive, we derive in Eq. (4.15) the time variation of $a(t)$ for large values of t , the last term in the expression being a (special case of the) dynamical correction to the pure quasistatic Casimir result. For the perhaps less realistic repulsive case, the small and large time behavior of the Casimir behavior is extracted in Eqs. (4.24), (4.26). Numerical results in both cases are given in Section 5. The behavior of the scale factor in the two cases is shown in Figs. 1 and 3, while the dynamical stress on the torus is presented in Figs. 2 and 4. In the Appendix we discuss the effects of alternative kinetic energy terms.

2. THE EFFECTIVE ACTION

Let us consider conformally invariant, massless matter in $4d$ -dimensional space-time. The matter may correspond to some GUT (say, $SU(5)$, $SO(10)$, or any other alternative). We are interested first in the study of the static Casimir effect for such a theory when the field is assumed to be bounded in a three-dimensional region. In other words, we are interested in a space having the form $\mathbb{R}^1 \otimes K^3$, where as K^3 one can take any manifold permitting an exact Casimir effect calculation. It can be \mathbb{S}^3 , \mathbb{T}^3 , $\mathbb{S}^1 \otimes \mathbb{S}^2$ or any other compact manifold with a known spectrum of the d'Alembertian operator. We limit ourselves to \mathbb{T}^3 or \mathbb{S}^3 , for the sake of simplicity.

Suppose now that our GUT lives in such a three-dimensional space, where the size of the space is a dynamical variable (moving mirror or moving universe). Hence, we will be interested in the dynamical Casimir effect in a three-dimensional region and the back-reaction from the induced radiation on the moving background geometry. We shall use the adiabatic approximation in this study. A great simplification comes from adopting a physical picture in which the Casimir effect is described as an effective action in curved spacetime (see [10] for an introduction). Here, spacetime is taken to be an expanding universe with topology $\mathbb{R}^1 \otimes \mathbb{S}^3$ or $\mathbb{R}^1 \otimes \mathbb{T}^3$. The corresponding metric is given by

$$ds^2 = dt^2 - a^2(t)ds_3^2, \quad (2.1)$$

where $ds_3^2 = dx^2 + dy^2 + dz^2$ for \mathbb{T}^3 (coordinates are restricted by all radii being equal), or the line element of a three-dimensional sphere \mathbb{S}^3 .

Let us calculate now the effective action for such a GUT. Using the fact that the theory is conformally invariant we may use the anomaly-induced effective action [11]:

$$\begin{aligned} W = & b \int d^4x \sqrt{-\bar{g}} \bar{F} \sigma \\ & + b' \int d^4x \sqrt{-\bar{g}} \left\{ \sigma [2\bar{\square}^2 + 4\bar{R}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{4}{3} \bar{R} \bar{\square} + \frac{2}{3} (\bar{\nabla}^\mu \bar{R})(\bar{\nabla}_\mu)] \sigma + (\bar{G} - \frac{2}{3} \bar{\square} \bar{R}) \sigma \right\} \\ & - \frac{1}{12} \times \frac{2}{3} (b + b') \int d^4x \sqrt{-\bar{g}} [\bar{R} - 6\bar{\square} \sigma - 6(\bar{\nabla}_\mu \sigma)(\bar{\nabla}^\mu \sigma)]^2, \end{aligned} \quad (2.2)$$

where our metric is presented in conformal form. Thus $g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu}$, $\sigma = \ln a(\eta)$, η is the conformal time, F is the square of the Weyl tensor, and G is the square of the Gauss-Bonnet invariant. Overbar quantities indicate that the calculation is made with $\bar{g}_{\mu\nu}$. Further,

$$\begin{aligned} b &= \frac{1}{120(4\pi)^2} (N_0 + 6N_{1/2} + 12N_1), \\ b' &= -\frac{1}{360(4\pi)^2} (N_0 + 11N_{1/2} + 62N_1), \end{aligned} \quad (2.3)$$

where N_0 , $N_{1/2}$, and N_1 are the numbers of scalars, spinors, and vectors. For example, for $\mathcal{N} = 4$ $SU(N)$ super YM one gets [12] $b = -b' = (N^2 - 1) [4(4\pi)^2]^{-1}$. We also adopt the scheme wherein the b' -coefficient of the $\square R$ term in the conformal anomaly is zero. Being ambiguous, it does not influence the dynamics [12].

As the simplest case we consider henceforth a torus. Then

$$W = \int d\eta [2b'\sigma\sigma'''' - 2(b+b')(\sigma'' + \sigma'^2)^2]. \quad (2.4)$$

This is a typical effective action for a GUT in a Friedman-Robertson-Walker (FRW) Universe of a special form.

3. THE STATIC CASIMIR ENERGY

Let us briefly overview the static Casimir effect for a torus of side L (for more detail, see [13]). The Casimir energies associated with massless spin- j fields is

$$\mathcal{E}_{N_j} = \frac{N_j}{2} (-1)^{2j} \sum_{\mathbf{n} \in \mathbb{Z}^3} \omega_{\mathbf{n},j}, \quad (3.1)$$

where we see the appearance of the characteristic minus sign associated with a closed Fermion loop. The frequency of each mode is given by

$$\omega_{\mathbf{n},j}^2 = \left(\frac{2\pi}{L}\right)^2 \sum_{i=1}^3 (n_i + g_i^{(j)})^2, \quad \mathbf{n} = (n_1, n_2, n_3), \quad (3.2)$$

Here $g_i^{(j)} = 0, 1/2$ depending on the field type chosen in $\mathbb{R}^1 \otimes \mathbb{T}^3$.

We use the p -dimensional Epstein zeta function $Z_p \left| \begin{smallmatrix} g_1, \dots, g_p \\ h_1, \dots, h_p \end{smallmatrix} \right| (s)$ defined for $\Re s > 1$ by the formula

$$Z_p \left| \begin{smallmatrix} g_1, \dots, g_p \\ h_1, \dots, h_p \end{smallmatrix} \right| (s) = \sum'_{\mathbf{n} \in \mathbb{Z}^p} [(n_1 + g_1)^2 + \dots + (n_p + g_p)^2]^{-ps/2} \\ \times \exp [2\pi i (n_1 h_1 + \dots + n_p h_p)], \quad (3.3)$$

where g_i and h_i are real numbers, and the prime means omitting the term with $(n_1, \dots, n_p) = (-g_1, \dots, -g_p)$ if all the g_i are integers. For $\Re s < 1$ the Epstein function is understood to be the analytic continuation of the right-hand side of Eq. (3.3). Defined in such a way, the Epstein zeta function obeys the functional equation

$$\pi^{-ps/2} \Gamma\left(\frac{1}{2}ps\right) Z_p \left| \begin{smallmatrix} g_1, \dots, g_p \\ h_1, \dots, h_p \end{smallmatrix} \right| (s) = \pi^{-p(1-s)/2} \Gamma\left(\frac{1}{2}p(1-s)\right) \\ \times \exp [-2\pi i (g_1 h_1 + \dots + g_p h_p)] Z_p \left| \begin{smallmatrix} h_1, \dots, h_p \\ -g_1, \dots, -g_p \end{smallmatrix} \right| (1-s), \quad (3.4)$$

The function (3.3) is an entire function in the complex s plane except for the case when all h_i are integers. In the latter case the function (3.3) has a simple pole at $s = 1$.

Using Eq. (3.3) we have

$$\mathcal{E}_{N_j} = \frac{\pi}{L} N_j (-1)^{2j} Z_3 \left| \begin{smallmatrix} g_1^{(j)} & g_2^{(j)} & g_3^{(j)} \\ 0 & 0 & 0 \end{smallmatrix} \right| \left(-\frac{1}{3} \right). \quad (3.5)$$

Taking into account the functional equation (3.4) one gets

$$Z_3 \left| \begin{array}{ccc} g_1^{(j)} & g_2^{(j)} & g_3^{(j)} \\ 0 & 0 & 0 \end{array} \right| \left(-\frac{1}{3} \right) = -\frac{1}{2\pi^3} Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ -g_1^{(j)} & -g_2^{(j)} & -g_3^{(j)} \end{array} \right| \left(\frac{4}{3} \right). \quad (3.6)$$

The Casimir energies (3.1) take the form

$$\mathcal{E}_{N_j} = -\frac{(-1)^{2j}}{2\pi^2 L} N_j Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ -g_1^{(j)} & -g_2^{(j)} & -g_3^{(j)} \end{array} \right| \left(\frac{4}{3} \right). \quad (3.7)$$

Finally the Casimir energy associated with a multiplet of fields characterized by the numbers N_0 , $N_{1/2}$, and N_1 can be written as follows

$$\begin{aligned} \mathcal{E} = \sum_j \mathcal{E}_{N_j} = & -\frac{1}{2\pi^2 L} \left[N_0 Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ -g_1^{(0)} & -g_2^{(0)} & -g_3^{(0)} \end{array} \right| \left(\frac{4}{3} \right) \right. \\ & \left. - N_{1/2} Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ -g_1^{(1/2)} & -g_2^{(1/2)} & -g_3^{(1/2)} \end{array} \right| \left(\frac{4}{3} \right) + N_1 Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ -g_1^{(1)} & -g_2^{(1)} & -g_3^{(1)} \end{array} \right| \left(\frac{4}{3} \right) \right]. \quad (3.8) \end{aligned}$$

Thus, the static Casimir energy for a torus is proportional to c/L , where c is defined by the features of the GUT under consideration. Note that the sign of c is *a priori* unpredictable.

If for all the fields of the theory we take the same boundary conditions (periodic or antiperiodic), the Z_3 's are all equal, and consequently the supersymmetry is not broken, and the Casimir energy is zero. On the other hand, if the different fields satisfy different types of boundary conditions, supersymmetry is broken and there is a static Casimir effect.

For the latter situation, consider, as an illustration, the usual case of bosons satisfying periodic boundary conditions and fermions satisfying antiperiodic boundary conditions on the torus. Then we require only two values. For the bosons, the Casimir energy is proportional to

$$Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right| \left(\frac{4}{3} \right) = 16.5323, \quad (3.9)$$

which value is given explicitly in Ref. [14]. For the fermions, the same reference gives the value

$$Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right| \left(\frac{4}{3} \right) = -3.86316. \quad (3.10)$$

So each term in Eq. (3.8) contributes a negative energy. Thus the net Casimir energy is attractive,

$$\mathcal{E} = -\frac{c}{L}, \quad c = 0.837537(N_0 + N_1) + 0.195710N_{1/2} = 1.033247N_{1/2}, \quad (3.11)$$

since the number of fermions must be equal to the number of bosons.¹

¹Any case with periodic boundary conditions in some directions and antiperiodic ones in others may be given in terms of the values given in Eqs. (3.9), (3.10) and the additional values

$$Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{array} \right| \left(\frac{4}{3} \right) = 0.689223, \quad Z_3 \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right| \left(\frac{4}{3} \right) = -2.156887.$$

We should make the following general remarks concerning the physical interpretation of the calculation sketched here. Imposition of periodic boundary conditions at the boundaries of the field volume is a basic physical ingredient in expressions such as Eqs. (3.1), (3.2) for the Casimir energy. It is analogous to the imposition of perfect conducting boundary conditions, or more generally, electromagnetic boundary conditions, at the walls, when considering ordinary electrodynamics, for example within a spherical volume. The physical outcome of a calculation of this kind is the residual energy remaining when the influence of the local stresses is separated off. (Presumably, such stresses are absorbed in a kind of renormalization of physical parameters.) The field theoretical calculation is able to cope only with the cutoff independent part of the physical stress; the local cutoff dependent parts of the stress are automatically lost in the zeta-function regularization process. This is an important point whenever the result of the field theoretical calculation is to be compared with experiments.

As a typical example of this sort, we may mention the calculation of the Casimir energy of a dilute dielectric ball. One may adopt a field theoretical viewpoint (cf., for instance, [15]), from which the Casimir energy is calculated as a cutoff independent, positive, expression. More detailed considerations, using quantum mechanical perturbation theory [16] (cf. also [17]), or quantum statistical mechanics [18] show however how this expression is to be supplemented with attractive cutoff dependent parts. Such terms are presumably not observable. As for the cutoff independent term, agreement between the methods is found, so the situation is in this respect satisfactory.

4. DYNAMICAL PROPERTIES

Now let us turn to a simplified discussion of the dynamical Casimir effect. We here take into account that we have a dynamical radius $a(t)L$, $a(t)$ being a dimensionless scale factor. Then, the total effective action is given as

$$\Gamma = W - L \int d\eta a(\eta) \mathcal{E}, \quad (4.1)$$

where W is given by Eq. (2.4) and $\mathcal{E} = -c/(aL)$ as displayed in Eq. (3.8). Because the action is dimensionless, the length L disappears from the calculation, and we have

$$\Gamma = \int d\eta [2b'\sigma\sigma'''' - 2(b+b')(\sigma'' + \sigma'^2)^2 + c]. \quad (4.2)$$

This is a typical effective action to describe a quantum FRW Universe.

In order to consider dynamical properties we add to the above effective action a phenomenological term, which has the form of a kinetic energy. We associate a mass m with the scale factor a , and take the corresponding kinetic energy to be given by $\frac{1}{2}m\dot{a}^2$. Our essential idea is that the geometrical configuration of the space is treated classically and that the GUT field is a quantum object which induces the Casimir effect. One might in principle introduce other expressions for the kinetic energy, but this expression is clearly the simplest choice that one can make. The Newtonian form is moreover in correspondence with our use of the adiabatic approximation, meaning that $|\dot{a}(t)| \ll 1$; cf. also the analogous argument

in [9] in connection with the (1+1) dimensional case.² Introducing the physical time t via $d\eta/dt = 1/a$, we now write Γ as

$$\Gamma = \int dt \left[\frac{1}{2} m \dot{a}^2 + 2b' \ln a (\ddot{a} a^2 + 3 \ddot{a} \dot{a} a + \ddot{a}^2 a + \ddot{a} \dot{a}^2) - 2(b + b') \frac{(\dot{a}^2 + a \ddot{a})^2}{a} + \frac{c}{a} \right], \quad (4.3)$$

where $\dot{a} = da/dt$.

From the variational equation $\delta\Gamma/\delta a = 0$ we obtain, after some algebra,³

$$m\ddot{a} - 2b' \left(\ddot{a}^2 + 2 \frac{d^2}{dt^2} (a\ddot{a}) \right) - 2(b + b') \left(2a \ddot{a} \ddot{a} + 4\dot{a} \ddot{a} + 3\dot{a}^2 - 12 \frac{\dot{a}^2 \ddot{a}}{a} + 3 \frac{\dot{a}^4}{a^2} \right) + \frac{c}{a^2} = 0. \quad (4.4)$$

It is remarkable that the the logarithm is absent in Eq. (4.4); there seems to be no reason *a priori* why this should be so.

We limit ourselves to the $\mathcal{N} = 4$ $SU(N)$ super YM theory for which, as mentioned, $(b + b') = 0$. Then Eq. (4.4) simplifies to

$$m\ddot{a} + 2b(2 \ddot{a} a + 4 \ddot{a} \dot{a} + 3\dot{a}^2) + \frac{c}{a^2} = 0. \quad (4.5)$$

Both the terms involving b and c are dynamical, quantum mechanical, effects, which in dimensional terms are proportional to \hbar . However, we will see that it is sensible (if $c \neq 0$) to regard the b term as a small correction to the Casimir-determined geometry. We denote the $b = 0$ solution by $a_0(t)$; it satisfies the equation

$$m\ddot{a}_0 + \frac{c}{a_0^2} = 0, \quad (4.6)$$

implying

$$\frac{1}{2} m \dot{a}_0^2 = \frac{c}{a_0} + \text{const.} \quad (4.7)$$

²We consider other possibilities for the kinetic energy term in the Appendix.

³This and subsequent equations are dimensionally consistent if we restore dimensions:

$$[a] = [t] = [m^{-1}] = \text{Length.}$$

4.1 Attractive Casimir energy, $c > 0$

If the Casimir energy is attractive, as actually realized in our illustrative calculation given in Sec. 3, see Eq. (3.11), we will assume, as boundary conditions, that $a_0(t \rightarrow \infty) = \infty$, $\dot{a}_0(t \rightarrow \infty) = 0$. Then, the constant in Eq. (4.7) becomes equal to zero, and we get the Casimir solution

$$a_0(t) = At^{\frac{2}{3}}, \quad \text{with} \quad A = \left(\frac{9c}{2m} \right)^{\frac{1}{3}}. \quad (4.8)$$

It is worth noticing here that the proportionality of $a_0(t)$ to $t^{2/3}$ is precisely the behaviour shown by the scale factor in the Einstein-de Sitter universe. This may be surprising at first sight, but does not seem to be so unreasonable after all, since the Einstein-de Sitter universe is flat, thus in correspondence with our neglect of Riemannian curvature terms in the formalism above.

Now we turn to the solution $a(t)$, taking into account the b correction. We shall limit ourselves to giving a perturbative solution, implying an expansion of $a(t)$ around $a_0(t)$ assuming b to be small:

$$a(t) = a_0(t) + ba_1(t). \quad (4.9)$$

We consider only times for which the correction term is small:

$$ba_1/a_0 \ll 1. \quad (4.10)$$

Thus, we may expand the Casimir term in Eq. (4.5) as $c/a^2 = (c/a_0^2)(1 - 2ba_1/a_0)$. A first order expansion of the other terms in Eq. (4.5) then yields, when we take into account the Casimir solution (4.8), the inhomogeneous equation

$$\ddot{a}_1 - \frac{4}{9t^2}a_1 = \frac{8A^2}{9m}t^{-\frac{8}{3}}. \quad (4.11)$$

The homogeneous version of Eq. (4.11) has solutions of the form t^α , with $\alpha = 4/3$ and $\alpha = -1/3$. We write the independent solutions as

$$f(t) = t^{\frac{4}{3}}, \quad g(t) = t^{-\frac{1}{3}}. \quad (4.12)$$

The Wronskian Δ between f and g is simple; $\Delta = f\dot{g} - g\dot{f} = -5/3$. Writing for brevity the right hand side of Eq. (4.11) as r we then get, as the solution of the inhomogeneous equation,

$$\begin{aligned} a_1(t) &= f(t) \left(C_1 - \frac{1}{\Delta} \int rg \, dt \right) + g(t) \left(C_2 + \frac{1}{\Delta} \int rf \, dt \right) \\ &= t^{\frac{4}{3}} \left(C_1 - \frac{4A^2}{15m} t^{-2} \right) + t^{-\frac{1}{3}} \left(C_2 + \frac{8A^2}{5m} t^{-\frac{1}{3}} \right) \\ &= C_1 t^{4/3} + C_2 t^{-1/3} + \frac{4}{3} \frac{A^2}{m} t^{-2/3}, \end{aligned} \quad (4.13)$$

with C_1 and C_2 being constants. As for the values of these constants, we have first to observe our restriction (4.10), which implies that

$$\frac{b}{A} \left(C_1 t^{\frac{2}{3}} + C_2 t^{-1} + \frac{4A^2}{3m} t^{-\frac{4}{3}} \right) \ll 1. \quad (4.14)$$

If we require the perturbative approximation to be valid for large times, we must have $C_1 = 0$. If we also set $C_2 = 0$, our perturbative solution becomes

$$a(t) = At^{\frac{2}{3}} + \frac{4A^2}{3m} bt^{-\frac{2}{3}}, \quad (4.15)$$

which is only valid for large enough t , i.e., for

$$\frac{bA}{m} t^{-\frac{4}{3}} \ll 1. \quad (4.16)$$

The static Casimir force is

$$F_{\text{Cas}} = -\frac{\partial}{\partial a} \left(-\frac{c}{a} \right) = -\frac{c}{a^2}, \quad (4.17)$$

whereas the dynamical force is

$$F_{\text{dyn}} = m\ddot{a}. \quad (4.18)$$

Substituting Eq. (4.15) into Eqs. (4.17) and (4.18), and observing the relation between c and A in Eq. (4.8), we get

$$F_{\text{dyn}} = F_{\text{Cas}} \left(1 - \frac{4bA}{m} t^{-4/3} \right), \quad (4.19)$$

which shows that the dynamical force is the Casimir force modified by a small dynamical correction when the perturbative approximation is valid.

Before we turn to a numerical solution of Eq. (4.5), we discuss the repulsive case.

4.2 Repulsive Casimir energy, $c < 0$

Now let us consider the case when $c < 0$, a repulsive Casimir energy. In this case we must take $\dot{a}_0|_{t \rightarrow \infty} \neq 0$. Let us write the $b = 0$ equation (4.7) in the form

$$\dot{a}_0 = \pm \sqrt{\frac{c_1}{a_0} + c_2}, \quad (4.20)$$

$$c_1 = \frac{2c}{m}, \quad c_2 = v_\infty^2, \quad v_\infty = \dot{a}_0|_{t \rightarrow \infty}. \quad (4.21)$$

From (4.20) we obtain

$$\frac{1}{c_2} \left[a_0 \sqrt{\frac{c_1}{a_0} + c_2} - \frac{c_1}{\sqrt{c_2}} \ln \left(2\sqrt{c_2 a_0} + 2\sqrt{c_2 a_0 + c_1} \right) \right] = \pm t + c_3, \quad (4.22)$$

where c_3 is a further integration constant. We see from Eq. (4.20) that

$$a_0 \geq \frac{-2c}{mv_\infty^2}. \quad (4.23)$$

For long times, the solution behaves as

$$a_0(t) \sim \sqrt{c_2}t, \quad t \gg 1. \quad (4.24)$$

For short times, suppose a_0 approaches the minimum value (4.23); then

$$c_3 = -\frac{c_1}{c_2^{3/2}} \ln 2\sqrt{-c_1}, \quad (4.25)$$

and

$$a_0(t) \sim -\frac{c_1}{c_2} - \frac{c_2^2}{4c_1}t^2, \quad t \ll 1. \quad (4.26)$$

5. NUMERICAL SOLUTION AND DISCUSSION

5.1 $c > 0$

Let us consider numerical solutions of dynamical equations (4.5) for $SU(2)$ super Yang-Mills theory, for the attractive case. We suppose that the initial behavior of $a(t)$ is given by the perturbative form Eq. (4.15). We may always set $m = 1$ since that amounts to using dimensionless variables for a and t . Let us take as an illustration

$$N = 2, \quad c = 1 \quad (A = 1.65096), \quad t_0 = 0.5, \quad (5.1)$$

For later times we integrate the exact equations numerically, starting with the initial conditions at t_0 :

$$a_0 = 1.06744, \quad \dot{a}_0 = 1.35019, \quad \ddot{a}_0 = -0.802706, \quad \dddot{a}_0 = 1.81582. \quad (5.2)$$

For those conditions we have a numerical solution for $a(t)$ as shown in Fig. 1. For comparison we also show in the figure the unperturbed solution (4.8) due to the static Casimir force. It will be noticed that for large t there are significant deviations from the unperturbed solution, which must be due to $C_1 \neq 0$ in the perturbative solution (4.13). In fact, for the entire range of $t = 0.1$ –80, the exact solution shown in Fig. 1 is roughly reproduced by Eq. (4.13) with $C_1 = C_2 = -5$. For the Casimir force we have the behavior as shown in Fig. 2. The exact solution has oscillations, but overall is close to the unperturbed solution. Not surprisingly, the Casimir energy dominates the force. Note also that for another choice of initial conditions one will find somewhat different behaviour. The essential property of the approximation under discussion is that there are always dynamical oscillations around the static Casimir force.

5.2 $c < 0$

Next we consider numerical solutions of dynamical equations (4.5) for $SU(2)$ super Yang-Mills theory when $c < 0$. We suppose that the initial behavior of $a(t)$ is given by form Eq. (4.22). Let us take the illustrative values

$$N = 2, \quad c_1 = -1, \quad c_2 = 1, \quad a_0|_{t=0} = 1, \quad t_0 = 0.1, \quad (5.3)$$

For later times we integrate the exact equations numerically, starting from the initial conditions at t_0 :

$$a_0 = 1.00241, \quad \dot{a}_0 = 0.0490327, \quad \ddot{a}_0 = 0.497511, \quad \dddot{a}_0 = -0.049793. \quad (5.4)$$

Note that the perturbative values of these parameters, given from Eq. (4.26) are close to these:

$$a_0 = 1.0025, \quad \dot{a}_0 = 0.05, \quad \ddot{a}_0 = 0.5, \quad \dddot{a}_0 = 0. \quad (5.5)$$

For those conditions we have a numerical solution for $a(t)$ as shown in Fig. 3. For the Casimir force we have the behavior as shown in Fig. 4. For both cases the exact solution has very small oscillations, but overall is close to the Casimir solution, and is accurately described by the limits of that solution, Eqs. (4.26) and (4.24).

Thus, we have presented a formalism to describe the dynamical Casimir effect in the adiabatic approximation. It may be applied to an arbitrary GUT. Without any technical problems one can generalize the present consideration to any specific four-dimensional background (we limited ourselves to a discussion of a toroidal FRW universe as providing the moving boundary conditions). But the limitations of our approach must be stressed: It would be extremely interesting to suggest new formulations of the dynamical Casimir effect beyond the adiabatic approximation.

ACKNOWLEDGMENTS

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APPENDIX A: ALTERNATIVE KINETIC ENERGY TERMS

In the text, we introduced an *ad hoc* kinetic energy term into the action, referring to the change of scale with physical time,

$$\int dt \frac{1}{2} m \dot{a}^2. \quad (A1)$$

This is rather natural in the adiabatic context, where $|\dot{a}| \ll 1$, for then simple scaling properties obtain, as evidenced by the dimensional consistency of the resulting equations of motion when m has dimensions of mass. But we can not offer very strong arguments in

its favor, in the absence of dynamical information. So, in this appendix we consider two alternatives, which provide somewhat different models for the dynamical evolution of the world.

In the first, we suppose that the same kinetic energy should be integrated over conformal time,⁴

$$\int d\eta \frac{1}{2} m \dot{a}^2 = m \int dt \frac{1}{2} \frac{\dot{a}^2}{a}, \quad (\text{A2})$$

so that the Casimir evolution equation is, in place of Eq. (4.7),

$$\frac{1}{2} \frac{m \dot{a}^2}{a} = \frac{c}{a} + k, \quad (\text{A3})$$

where we have dropped the subscript 0 for simplicity, and written the constant of integration as k . The solution of this equation is very simple,

$$a = a_0 + t \sqrt{\frac{2}{m}} \sqrt{c + k a_0} + \frac{k}{2m} t^2, \quad a_0 = a(0). \quad (\text{A4})$$

If $c > 0$, we can set $k = 0$ and obtain instead of the behavior exhibited in Eq. (4.8), a linear growth of the scale,

$$a = a_0 + \sqrt{\frac{2c}{m}} t. \quad (\text{A5})$$

If $c < 0$, as before we cannot set the integration constant equal to zero; if we again choose the initial velocity to be zero, or $a_0 = -c/k$, we get a result very like Eq. (4.26):

$$a = -\frac{c}{k} + \frac{k}{2m} t^2, \quad (\text{A6})$$

but now valid for all times.

Perhaps a more natural possibility is to use the conformal time everywhere in the kinetic energy term,⁵

$$\int d\eta \frac{m}{2} \left(\frac{da}{d\eta} \right)^2 = \frac{m}{2} \int dt a \dot{a}^2. \quad (\text{A7})$$

The solution to the purely Casimir dynamical equation is

$$\frac{m}{2} a \dot{a}^2 = \frac{c}{a} + k, \quad (\text{A8})$$

which is integrated to

$$t = \frac{1}{k^2} \sqrt{\frac{m}{2}} \left\{ \frac{2}{3} [(ka + c)^{3/2} - (ka_0 + c)^{3/2}] - 2c [(ka + c)^{1/2} - (ka_0 + c)^{1/2}] \right\}. \quad (\text{A9})$$

⁴In this case the parameter m is dimensionless.

⁵In this case the parameter m has dimension 1/Length².

When $c > 0$ again we can take $k \rightarrow 0$, which leads to the $k = 0$ result

$$a^2 = a_0^2 + 2\sqrt{\frac{2c}{m}}t. \quad (\text{A10})$$

If $c < 0$ and we choose again $ka_0 + c = 0$, we obtain for short times

$$a = -\frac{c}{k} + \frac{k^3}{2mc^2}t^2, \quad t \ll 1, \quad (\text{A11})$$

again very similar to Eq. (4.26).

In Figs. 5 and 6 we show the effect of the inclusion of the dynamical b term for these kinetic energy structures. Qualitatively, the results do not depend much on whether the Casimir term is positive or negative. The example given for the first alternative kinetic energy is similar to the simple model result for $c < 0$ shown in Fig. 3 except that the growth in t is quadratic rather than linear. The evolution for the second form of kinetic energy resembles the simple model result for $c > 0$ shown in Fig. 1.

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FIGURES

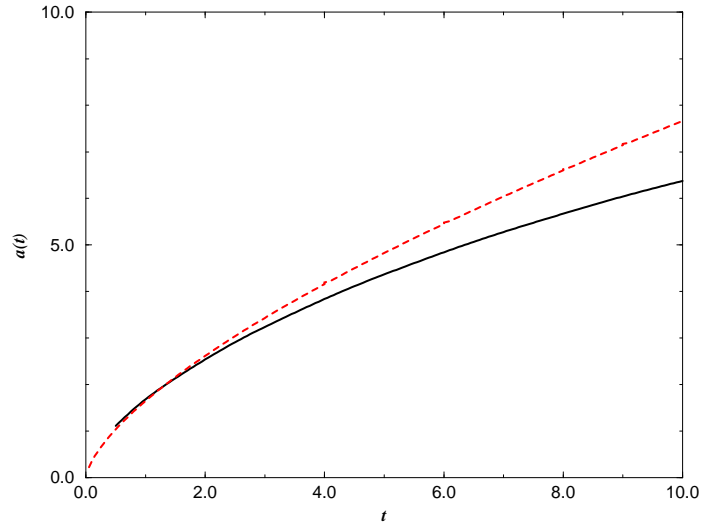


FIG. 1: Casimir (dashed line) and dynamical behavior for $a(t)$ for $c > 0$.

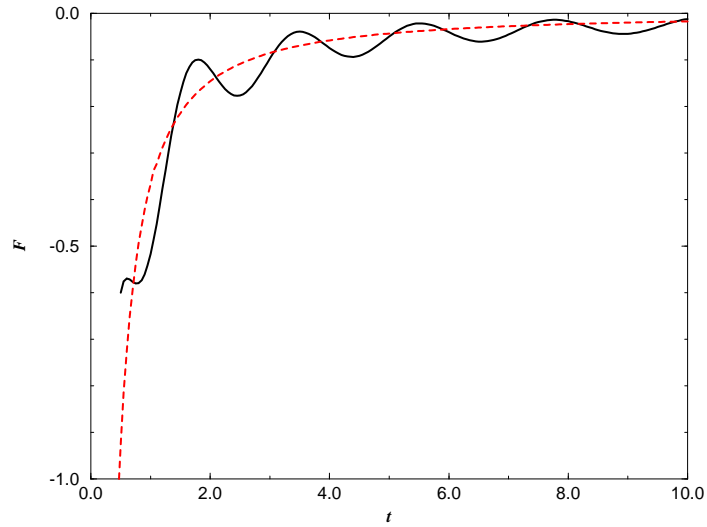


FIG. 2: Casimir (dashed line) and dynamical behavior of F for $c > 0$.

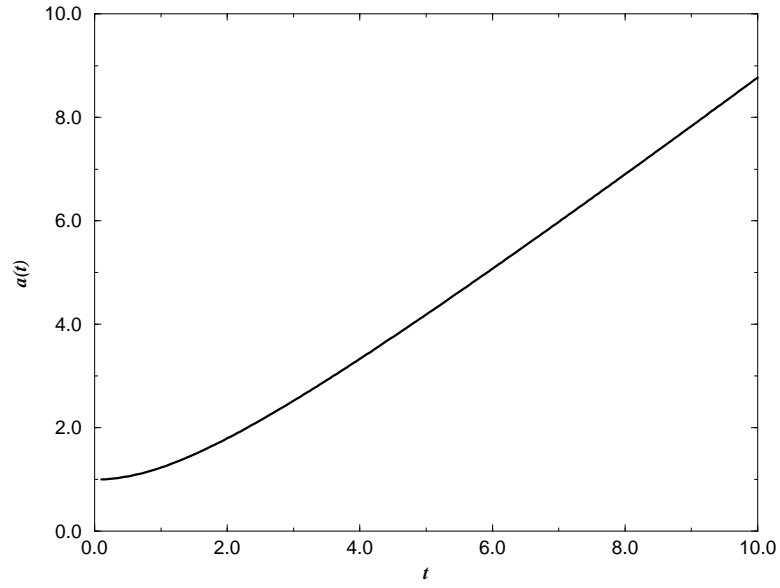


FIG. 3: Dynamical behavior of $a(t)$ for $c < 0$.

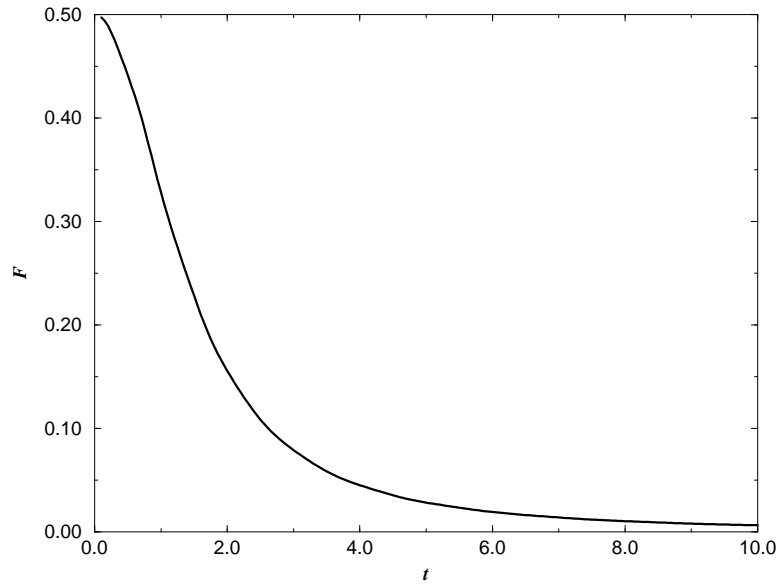


FIG. 4: Dynamical behavior of F for $c < 0$.

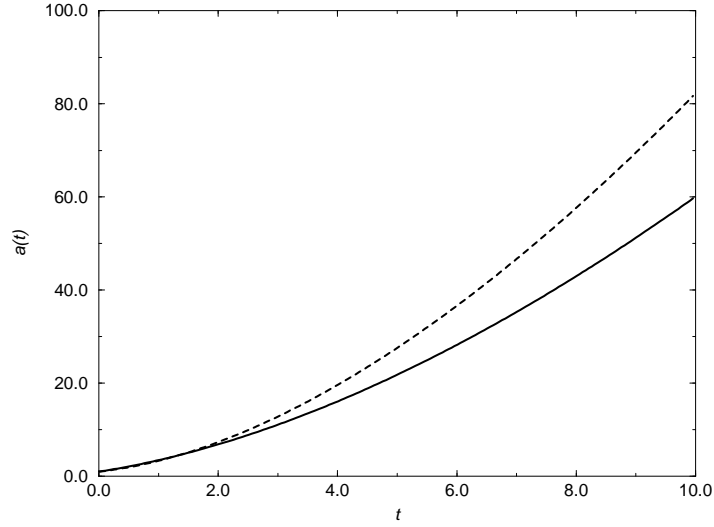


FIG. 5: Dynamical behavior for $a(t)$ for the first alternative kinetic energy term, Eq. (A2). Shown are the behaviors with $m = 1$, and initial condition $a(0) = 1$, evolving initially until $t = 0.01$ according to Eq. (A4), with $c = 1$, $k = 1$ (solid line); and with $c = -1$, $k = 2$ (dashed line).

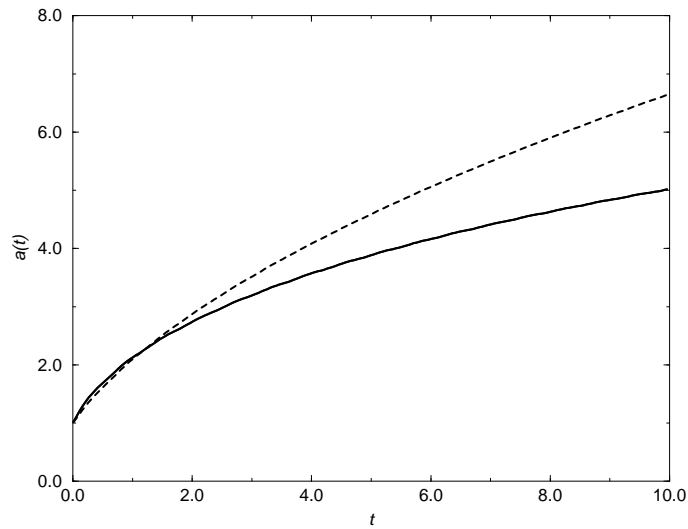


FIG. 6: Dynamical behavior for $a(t)$ for the second alternative kinetic energy term, Eq. (A7). Shown are the behaviors with $m = 1$, and initial condition $a(0) = 1$, evolving initially until $t = 0.01$ according to Eq. (A9), with $c = 1$, $k = 1$ (solid line); and with $c = -1$, $k = 2$ (dashed line).