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Nonperturbative calculation of symmetry breaking in quantum field theory

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Abstract

A new version of the delta expansion is presented, which, unlike the conventional delta expansion, can be used to do nonperturbative calculations in a self-interacting scalar quantum field theory having broken symmetry. We calculate the expectation value of the scalar field to first order in δ, where δ is a measure of the degree of nonlinearity in the interaction term.

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In a private discussion with D. Bessis we learned about the remarkable non-Hermitian Hamiltonian

$$H = p^2/2 + ix^3. \tag{1}$$

The eigenvalues of this Hamiltonian are all real and positive! (We have no proof of this property, but there is ample numerical evidence.) This Hamiltonian is intriguing because it suggests a novel way to apply the *delta expansion* to quantum field theories having a broken symmetry. In this paper we introduce our new version of the delta expansion and use it to calculate the (nonzero) value of $\langle\phi\rangle$ in a scalar quantum field theory [see Eq. (31)].

The delta expansion is a Taylor series in powers of δ, where δ measures the degree of *nonlinearity* of an interaction term [1]. For a scalar quantum field theory, the conventional approach [2] has been to introduce the parameter δ into the Euclidean Lagrangian density by

$$\mathcal{L} = (\nabla\phi)^2/2 + m^2\phi^2/2 + g(\phi^2)^{1+\delta}. \tag{2}$$

The advantage of the delta expansion is that it is *nonperturbative* in the coupling constant g and mass m and has a nonzero radius of convergence. The conventional delta expansion has been used to study renormalization [3], supersymmetry [4], local gauge invariance [5], stochastic quantization [6], and finite-temperature field theory [7].

A drawback of the Lagrangian density in Eq. (2) is that it becomes a $g|\phi|^3$ theory rather than a $g\phi^3$ theory when $\delta = \frac{1}{2}$. The $g|\phi|^3$ theory is symmetric under $\phi \to -\phi$ and cannot exhibit symmetry breaking. In general, the conventional delta expansion is unsuitable for studying theories with symmetry breaking.

In the past it was assumed that to satisfy the physical requirement that the Hamiltonian be bounded below for all δ, the parameter δ should appear in the Lagrangian density (2) as the exponent of ϕ^2 and *not* of ϕ. This assumption appears to be even more reasonable if one expands the interaction term in Eq. (2) in powers of δ:

$$g(\phi^2)^{1+\delta} = g\phi^2 \sum_{k=0}^{\infty} \frac{\delta^k}{k!}(\log \phi^2)^k. \tag{3}$$

One would think that the argument of the logarithm in Eq. (3) should be positive to avoid complex numbers.

However, H in Eq. (1) describes a theory having a positive spectrum. This suggests a surprising new version of the delta expansion in which we replace Eq. (2) by

$$\mathcal{L} = (\nabla\phi)^2/2 + m^2\phi^2/2 - g(i\phi)^{2+\delta}. \tag{4}$$

As with Eq. (2), $\delta = 0$ gives free field theory. At $\delta = 1$ we recover (when the dimension of space-time is one and the bare mass vanishes) the theory described by Eq. (1).

We believe that for $\mathrm{Re}\,\delta > -2$ the quantum-mechanical theory has a positive real spectrum. Here, we verify this to first order in δ. The shift in the nth energy level of $H = p^2/2 - (ix)^{2+\delta}$ is the expectation value of $\delta x^2 \log(ix)$ in the unperturbed harmonic-oscillator basis $|n\rangle$:

$$\begin{aligned}
\Delta E_n &= \langle n|\delta x^2 \log(ix)|n\rangle \\
&= \delta \int_{-\infty}^{\infty} dx\, \psi_n^2(x) x^2 \log(ix) \\
&= \delta \left(\int_{-\infty}^{0} + \int_0^{\infty} \right) dx\, \psi_n^2(x) x^2 \log(ix) \\
&= \delta \int_0^{\infty} dx\, \psi_n^2(x) x^2 [\log(-ix) + \log(ix)] \\
&= 2\delta \int_0^{\infty} dx\, \psi_n^2(x) x^2 \log(x) \\
&= \delta 2^{-5/2} [a_n - (2n+1)\gamma - (5n + 5/2)\log 2].
\end{aligned} \tag{5}$$

The sequence $a_n = 2,\, 8,\, 16,\, 74/3, \ldots$ is given by

$$a_n = 4n + 4 + 4[n/2] + (4n+2) \sum_{k=0}^{[n/2+1/2]} \frac{1}{2k-1},$$

where $[\alpha]$ is the greatest integer in α. (In the above calculation we choose the branch of the logarithm to lie along the negative axis and choose the sheet of the Riemann surface for which $\log 1 = 0$.) Although the Hamiltonian is not self-adjoint, the $\mathrm{O}(\delta)$ shift in the energy levels is *real*! This property persists to all orders in δ [8].

We will now show that in any Euclidean dimension and for all $\delta > -2$ (apart from a special discrete set of negative values of δ), Eq. (4) exhibits symmetry breaking. This result is especially interesting for the case $\delta = 2$ because, although the resulting ϕ^4 theory appears symmetric under $\phi \to -\phi$, symmetry breaking *persists*!

We begin by considering a massless *zero-dimensional* scalar quantum field theory, whose vacuum-persistence amplitude Z and N-point Green's function G_N are

$$Z = \int_{-\infty}^{\infty} dx\, e^{(ix)^{2+\delta}}, \quad G_N = \frac{1}{Z}\int_{-\infty}^{\infty} dx\, e^{(ix)^{2+\delta}} x^N. \tag{6}$$

To evaluate G_N we break the integral into two pieces:

$$\begin{aligned} ZG_N &= \int_{-\infty}^{0} dx\, e^{(ix)^{2+\delta}} x^N + \int_{0}^{\infty} dx\, e^{(ix)^{2+\delta}} x^N \\ &= 2\begin{pmatrix} \mathrm{Re}\ (\text{if } N \text{ even}) \\ i\,\mathrm{Im}\ (\text{if } N \text{ odd}) \end{pmatrix} \int_0^{\infty} dx\, e^{(ix)^{2+\delta}} x^N. \end{aligned} \tag{7}$$

This integral exists if $-1 < \mathrm{Re}\,\delta < 1$. We let $x^{2+\delta} = s$ and use standard Fresnel contour integral methods to get

$$ZG_N = -(-i)^N \frac{2\pi}{N+1} \Big/ \Gamma\left(-\frac{1+N}{2+\delta}\right). \tag{8}$$

Using this result we calculate G_N:

$$G_N = \frac{(-i)^N}{N+1}\Gamma\left(-\frac{1}{2+\delta}\right) \Big/ \Gamma\left(-\frac{1+N}{2+\delta}\right). \tag{9}$$

The nonvanishing of G_{2N+1} in Eq. (9) confirms that there is symmetry breaking.

Let us examine the case $\delta = 2$ (ϕ^4 theory) in depth. Since the one-point Green's function G_1 for this model is nonvanishing, the symmetry remains permanently broken as δ approaches 2. The Green's functions for this theory are $G_N = (-i)^N\Gamma(3/4)/\Gamma(3/4 - N/4)$. In the presence of an external source J, the vacuum functional $Z[J]$ for the theory has a convergent Taylor series in J:

$$Z[J] = Z[0]\sum_{N=0}^{\infty}\frac{J^N G_N}{N!} = \frac{\pi}{2}\sum_{N=0}^{\infty}\frac{J^N(-i)^N}{N!\,\Gamma\left(\frac{3-N}{4}\right)}. \tag{10}$$

Note that $Z[J]$ obeys the third-order differential equation

$$Z'''[J] + JZ[J]/4 = 0. \tag{11}$$

This is the $D = 0$ form of the functional Schwinger-Dyson equation for the theory. If J is rotated by $e^{i\pi/4}$, Eq. (11) becomes the differential equation satisfied by the vacuum functional $\tilde{Z}[J]$ for a conventional zero-dimensional ϕ^4 theory in the presence of an external source:

$$\tilde{Z}[J] = \int_{-\infty}^{\infty} dx\, e^{-x^4+Jx}. \tag{12}$$

This model does not exhibit symmetry breaking.

The Schwinger-Dyson equation (11) does not specify the vacuum functional uniquely [9,10]. Indeed, there are four distinct solutions to the rotated version of Eq. (11):

$$Z_j[J] = \int_{C_j} dx\, e^{-x^4+Jx} \quad (j = 1,\ 2,\ 3,\ 4). \tag{13}$$

Here, C_j are contours in the complex-x plane; C_1 joins $i\infty$ to $+\infty$, C_2 joins $+\infty$ to $-i\infty$, C_3 joins $-i\infty$ to $-\infty$, and C_4 joins $-\infty$ to $i\infty$. (The four solutions $Z_j[J]$ are not linearly independent; their sum is zero.) The vacuum functional $\tilde{Z}[J]$ in Eq. (12) is given by $Z_4[J] + Z_1[J]$. The vacuum functional for the symmetry-broken theory in Eq. (10) is $e^{i\pi/4}Z_1[e^{3i\pi/4}J]$. The symmetry-broken theory corresponding to $\delta = 2$ in Eq. (6) is distinguished from the conventional unbroken theory in Eq. (12) in that the sign of the x^4 term is reversed (reflecting the factor of $e^{3i\pi/4}$ in the source term above).

Next, let us examine the analytic behavior of Eq. (6) as a function of complex δ. The delta expansion of the one-point Green's function G_1, whose nonvanishing signals the presence of symmetry breaking, is

$$\begin{aligned} G_1 = -i\sqrt{\pi}\ (\ &\delta/2 + \delta^2(\gamma - 2 - 2\log 2)/8 - \delta^3[48 + \pi^2 \\ &-6(\gamma - 4 - 2\log 2)^2]/384 + \ldots\ )\ . \end{aligned} \tag{14}$$

The radius of convergence of this expansion is 2 because there is one singularity in the δ-plane at $\delta = -2$. [One might think (incorrectly) that as $\delta \to -1$ the integral for Z in Eq. (6) would become a representation of a Dirac delta function, which is singular. However, it is not generally true that when an integral representation for a function ceases to exist, the function exhibits a singularity [11].] G_N is analytic for $\mathrm{Re}\,\delta > -2$ because the integration path in Eq. (6) *is an implicit function of* δ. Indeed, as delta ranges through real values, the paths of integration of the two integrals in the first line of Eq. (7) *rotate in opposite directions* [12]. The path of integration of the first integral connects 0 and ∞ in the complex-x plane along the straight line

$$\text{path 1}: \quad \arg x = 3\pi/2 - \pi/(2+\delta). \tag{15}$$

The second integration path runs from 0 to ∞ along

$$\text{path 2}: \quad \arg x = \pi/(2+\delta) - \pi/2. \tag{16}$$

When $\delta = 0$ (free field theory), path 1 connects $-\infty$ to 0 and path 2 connects 0 to ∞ along the real-x axis. For this case G_N is *real* and there is *no symmetry breaking*. As δ increases, path 1 rotates anticlockwise and path 2 rotates clockwise. The two paths slope downward at 45° angles when $\delta = 2$ (ϕ^4 field theory). As $\delta \to \infty$, both paths connect the origin to $-i\infty$. Since the paths overlap and the integrations are in *opposite* senses, the two integrals in Eq. (7) cancel. [This cancellation becomes evident if we set $\delta = \infty$ in Eq. (8).] However, for this ϕ^∞ theory the expectation values G_N approach finite limits:

$$\lim_{\delta\to\infty} G_N = (-i)^N. \tag{17}$$

The case $\delta < 0$ is more interesting. As δ decreases below 0, path 1 rotates clockwise and path 2 rotates anticlockwise. (The paths rotate infinitely fast as $\delta \to -2$.) Whenever the two paths are horizontal, the resulting $\phi^{2/(2k+1)}$ theory is real and there is *no symmetry breaking*. This happens for the special set of values

$$\delta = -4k/(2k+1) \quad (k = 0,\ 1,\ 2,\ 3, \ldots). \tag{18}$$

For this case we have

$$\lim_{\delta\to -4k/(2k+1)} G_N = \begin{cases} 0 & (N \text{ odd}), \\ \frac{\Gamma\left[(N+1)\left(\frac{1}{2}+k\right)\right]}{\Gamma\left(\frac{1}{2}+k\right)} & (N \text{ even}). \end{cases} \tag{19}$$

If the two paths overlap (this occurs when the paths are vertical and go up or down the imaginary axis), there is cancellation. This happens for a $\phi^{1/k}$ theory when

$$\delta = (1-2k)/k \quad (k = 0,\ 1,\ 2,\ 3, \ldots). \tag{20}$$

For this case we have

$$\lim_{\delta\to(1-2k)/k} G_N = (-i)^N(-1)^{kN}[k(N+1)]!/k!, \tag{21}$$

which reduces to Eq. (17) for the special case $k = 0$.

Next, we examine theories in *one-dimensional* space-time (quantum mechanics). In general, the Hamiltonian

$$H = p^2/2 + (ix)^{2+\delta} \tag{22}$$

represents a broken-symmetry theory because H is not invariant under the parity operation $\mathcal{P}$, whose effect is to make the replacements $p \to -p$ and $x \to -x$. (While H is not symmetric under time reversal $\mathcal{T}$, which makes the replacements $p \to -p$, $x \to x$, and $i \to -i$, H *is* symmetric under $\mathcal{PT}$.) We emphasize that there are both broken- and unbroken-symmetric phases in zero- and one-dimensional Euclidean space-time, but there is no transition between these different phases. Transitions can only occur when the dimension is two or more.

As in the case of zero dimensions, there is a set of values of δ [see Eq. (18)] corresponding to quantum-mechanical theories for which there is no broken symmetry. These special theories are described by the Hamiltonians

$$H = p^2/2 - x^{2/(2k+1)} \quad (k = 0,\ 1,\ 2,\ 3, \ldots), \tag{23}$$

where we have chosen $i^{2/(2k+1)} = -1$. The Hamiltonians in Eq. (23) have a deep connection with *quasi-exactly solvable* theories in quantum mechanics [13]. To express our theories in a more familiar form we make the change of independent variable $x = t^{2k+1}$ in the Schrödinger equation $H\psi(x) = E\psi(x)$, followed by the multiplicative change of dependent variable $\psi = t^k y$ to eliminate one-derivative terms. We obtain the Schrödinger equation

$$y''(t) = \left[(k^2+k)t^{-2} - 2(2k+1)^2 t^{4k}(t^2+E)\right] y(t). \tag{24}$$

Equation (24) represents a particle of zero energy in a rotationally symmetric quasi-exactly solvable potential having a centrifugal barrier.

Finally, we use our new delta-expansion to calculate the one-point Green's function G_1 in a symmetry-broken scalar quantum field theory in D-dimensional Euclidean space. The vacuum-persistence amplitude for the Lagrangian density in Eq. (4) is

$$Z = \int \mathcal{D}\phi\, e^{\int d^Dx \left[-\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + g(i\phi)^{2+\delta}\right]}. \tag{25}$$

Since the field ϕ has dimensions $M^{(D-2)/2}$ and the coupling constant g has dimensions $M^{2+\delta-\delta D/2}$, where M is a mass, we re-express Eq. (25) using dimensionally explicit quantities by substituting $g = \frac{1}{2}M^{2+\delta-\delta D/2}$. Expanding the exponent to first order in δ gives

$$\begin{aligned} Z &= \int \mathcal{D}\phi\, e^{-S_0 - \frac{1}{2}\int d^Dx \left[\delta M^2\phi^2 \log\left(i\phi M^{1-\frac{D}{2}}\right) + \mathrm{O}(\delta^2)\right]} \\ &= \int \mathcal{D}\phi\, e^{-S_0} \Big[1 - \frac{1}{2}\delta M^2 \int d^Dt\, \phi^2(t) \\ &\qquad \times \log\left(i\phi(t) M^{1-\frac{D}{2}}\right) \Big] + \mathrm{O}(\delta^2), \end{aligned} \tag{26}$$

where $S_0 = \frac{1}{2}\int d^Dx[(\nabla\phi)^2 + \mu^2\phi^2]$ and $\mu^2 \equiv m^2 + M^2$.

To evaluate the one-point Green's function G_1 we insert a factor of $\phi(0)$ into the integrand of Eq. (26) and divide by Z. By symmetry, the resulting expression vanishes at $\delta = 0$. To first order in δ we have

$$G_1 = -\delta \frac{\int \mathcal{D}\phi\, e^{-S_0}\phi(0) \int d^Dt\, \phi^2(t) \log\left(i\phi(t) M^{1-\frac{D}{2}}\right)}{2M^{-2}\int \mathcal{D}\phi\, e^{-S_0}}.$$

We now show that G_1 is pure imaginary: We average the above expression with that obtained by setting ϕ to $-\phi$ at every lattice point. *Provided that the integrals exist*, we obtain the explicitly imaginary result

$$G_1 = -i\pi\delta M^2 \frac{\int \mathcal{D}\phi\, e^{-S_0}\phi(0)\int d^Dt\, \phi(t)|\phi(t)|}{4\int \mathcal{D}\phi\, e^{-S_0}}. \tag{27}$$

It is not easy to evaluate this functional integral because it contains $|\phi(t)|$, so we employ a limiting process: $\lim_{\alpha\to 1/2}\phi^{2\alpha} = |\phi|$. Before taking the limit we temporarily treat α as an integer. Thus, this limiting process involves continuing off the integers. [We cannot prove the validity of this technique, but we justify this procedure in part by comparing the final result with the first term in Eq. (14) for the case $D = 0$.] To evaluate this limit we introduce an external source $J(x)$, and express G_1 in terms of functional derivatives with respect to $J(x)$:

$$
\begin{aligned}
G_1 = -i\frac{\pi}{4}\delta M^2 \int d^D t\, \frac{\delta}{\delta J(0)} \\
\times \lim_{\alpha\to 1/2} \left(\frac{\delta}{\delta J(t)}\right)^{1+2\alpha} \left.\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}\right|_{J=0},
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
\mathcal{Z}[J] &\equiv \int \mathcal{D}\phi\, e^{\int d^D x\,\left[-\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\mu^2\phi^2 + J\phi\right]} \\
&= \mathcal{Z}[0] e^{\frac{1}{2}\int\int d^D x\, d^D y\, J(x)J(y)\Delta(x,y)},
\end{aligned}
\tag{29}
$$

and $\Delta(x,y)$, the free propagator in D-dimensional Euclidean space, is the Fourier transform of $1/(p^2+\mu^2)$:

$$
\begin{aligned}
\Delta(x,y) &= \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(x-y)} \frac{1}{p^2+\mu^2} \\
&= \frac{1}{2\pi}\left(\frac{2\pi|x-y|}{\mu}\right)^{1-\frac{D}{2}} K_{-1+\frac{D}{2}}(\mu|x-y|).
\end{aligned}
\tag{30}
$$

For integer α we use Eqs. (29) and (30) to evaluate Eq. (28) and obtain the final result for G_1:

$$
\begin{aligned}
G_1 &= -i\frac{\pi}{4}\delta M^2 \lim_{\alpha\to 1/2} \int d^D t\, \frac{\delta}{\delta J(0)} \left(\frac{\delta}{\delta J(t)}\right)^{1+2\alpha} \\
&\quad \times \left[\frac{1}{2}\int\int d^D x\, d^D y\, J(x)J(y)\Delta(x,y)\right]^{\alpha+1} /(\alpha+1)! \\
&= -i\frac{\pi}{4}\delta M^2 \lim_{\alpha\to 1/2} \frac{(2\alpha+2)!\Delta^\alpha(0,0)\int d^D t\, \Delta(0,t)}{2^{\alpha+1}(\alpha+1)!} \\
&= -i\delta M^2 \sqrt{\pi\Delta(0,0)/2} \int d^D t\, \Delta(0,t) \\
&= -i\delta M^2 \mu^{-3}(\mu^2/4\pi)^{D/4}\sqrt{\pi\Gamma(1-D/2)/2},
\end{aligned}
\tag{31}
$$

where we have used $\int d^D t\, \Delta(0,t) = \mu^{-2}$ and

$$\Delta(0,0) = \mu^{D-2} 2^{-D} \pi^{-D/2} \Gamma(1 - D/2) \quad (D < 2). \tag{32}$$

At $D = 0$, $m = 0$, and $M = \mu = \sqrt{2}$, Eq. (31) reduces to the first term in Eq. (14). Note that as D increases past 2, G_1 in Eq. (31) becomes *real* because the gamma function changes sign. For $D < 2$, $\Delta(0,0)$ is finite and positive, but for $D > 2$ it is a divergent integral regulated by continuing through complex dimension. Thus, for $D > 2$, the sign is *a priori* unpredictable.

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