$\mathcal{PT}$-Symmetric Versus Hermitian Formulations of Quantum Mechanics

Carl M. Bender,* Jun-Hua Chen,† and Kimball A. Milton‡

Department of Physics, Washington University, St. Louis, MO 63130, USA

(Dated: November 27, 2005)

Abstract

A non-Hermitian Hamiltonian that has an unbroken $\mathcal{PT}$ symmetry can be converted by means of a similarity transformation to a physically equivalent Hermitian Hamiltonian. This raises the following question: In which form of the quantum theory, the non-Hermitian or the Hermitian one, is it easier to perform calculations? This paper compares both forms of a non-Hermitian ix^3 quantum-mechanical Hamiltonian and demonstrates that it is much harder to perform calculations in the Hermitian theory because the perturbation series for the Hermitian Hamiltonian is constructed from divergent Feynman graphs. For the Hermitian version of the theory, dimensional continuation is used to regulate the divergent graphs that contribute to the ground-state energy and the one-point Green's function. The results that are obtained are identical to those found much more simply and without divergences in the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian. The $\mathcal{O}(g^4)$ contribution to the ground-state energy of the Hermitian version of the theory involves graphs with overlapping divergences, and these graphs are extremely difficult to regulate. In contrast, the graphs for the non-Hermitian version of the theory are finite to all orders and they are very easy to evaluate.

PACS numbers: 11.30.Er, 11.25.Db, 02.30.Mv, 11.10.Gh

*Electronic address: cmb@wustl.edu

†Electronic address: jchena@artsci.wustl.edu

‡Electronic address: milton@nhn.ou.edu; Permanent address: Oklahoma Center for High Energy Physics and Homer L. Dodge Department of Physics and Astronomy, University of Oklahoma, Norman, OK 73019, USA

I. INTRODUCTION

In 1998 it was shown using perturbative and numerical arguments that the non-Hermitian Hamiltonians

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \geq 0) \tag{1}$$

have real positive spectra [1, 2]. It was argued in these papers that the reality of the spectrum was due to the unbroken $\mathcal{PT}$ symmetry of the Hamiltonians. A rigorous proof of reality was given by Dorey *et al.* [3].

Later, in 2002 it was shown that the Hamiltonian in (1) describes unitary time evolution [4]. In Ref. [4] it was demonstrated that it is possible to construct a new operator called $\mathcal{C}$ that commutes with the Hamiltonian H. It was shown that the Hilbert space inner product with respect to the $\mathcal{CPT}$ adjoint has a positive norm and that the time evolution operator e^{iHt} is unitary. Evidently, Dirac Hermiticity of the Hamiltonian is not a necessary requirement of a quantum theory; unbroken $\mathcal{PT}$ symmetry is sufficient to guarantee that the spectrum of H is real and positive and that the time evolution is unitary. (In this paper we indicate that a Hamiltonian is Hermitian in the Dirac sense by writing $H = H^\dagger$, where the symbol $\dagger$ indicates Dirac Hermitian conjugation, that is, the combined operations of complex conjugation and matrix transposition: $H^\dagger \equiv H^{*\mathrm{T}}$.)

A recipe for constructing $\mathcal{C}$ was given in Ref. [5]. The procedure is to solve the three simultaneous algebraic equations satisfied by $\mathcal{C}$:

$$\mathcal{C}^2 = 1, \quad [\mathcal{C}, \mathcal{PT}] = 0, \quad [\mathcal{C}, H] = 0. \tag{2}$$

The recipe in Ref. [5] has been used to find the $\mathcal{C}$ operator for various quantum field theories [6–8]. This recipe produces the $\mathcal{C}$ operator as a product of the exponential of an antisymmetric Hermitian operator Q and the parity operator $\mathcal{P}$:

$$\mathcal{C} = e^Q \mathcal{P}. \tag{3}$$

As an example, we consider the $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + ix. \tag{4}$$

For this Hamiltonian, the exact Q operator is given by

$$Q = -2p. \tag{5}$$

A natural question to ask is whether there is a Hamiltonian that is Hermitian in the Dirac sense and is equivalent to a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian H. Mostafazadeh has shown that there is a Hermitian operator ρ that may to used to perform a similarity transformation on H,

$$h = \rho^{-1} H \rho, \tag{6}$$

to produce a new Hamiltonian h that is Hermitian in the Dirac sense [9]. The operator ρ is just the square-root of the (positive) $\mathcal{CP}$ operator:

$$\rho = e^{Q/2}. \tag{7}$$

The Hamiltonian h that results from the similarity transformation (6) has been studied perturbatively by Jones [10] and Mostafazadeh [11].

We summarize briefly the work in Refs. [10, 11]. One can verify that the Hamiltonian h produced by the similarity transformation (6) is Hermitian by taking the Hermitian conjugate of h:

$$h^\dagger = \left(e^{-Q/2} H e^{Q/2}\right)^\dagger = e^{Q/2} H^\dagger e^{-Q/2}. \tag{8}$$

Next, one uses the $\mathcal{PT}$ symmetry of H to replace $H^\dagger$ by $\mathcal{P}H\mathcal{P}$,

$$h^\dagger = e^{Q/2} \mathcal{P} H \mathcal{P} e^{-Q/2}, \tag{9}$$

and one uses the identity (3) to rewrite (9) as

$$h^\dagger = e^{-Q/2} \mathcal{C} H \mathcal{C} e^{Q/2}. \tag{10}$$

But $\mathcal{C}$ commutes with H, so

$$h^\dagger = e^{-Q/2} H e^{Q/2} = h, \tag{11}$$

which establishes the Hermiticity of h.

We illustrate this transformation by using the Hamiltonian (4). The similarity transformation (11) using (5) gives

$$h = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}, \tag{12}$$

which is clearly Hermitian.

To see that H and h have the same spectra, one can multiply the eigenvalue equation for H, $H\Phi_n = E_n\Phi_n$, on the left by $e^{-Q/2}$:

$$e^{-Q/2} H e^{Q/2} e^{-Q/2} \Phi_n = E_n e^{-Q/2} \Phi_n. \tag{13}$$

Thus, the eigenvalue problem for h reads $h\phi_n = E_n\phi_n$, where the eigenvectors ϕ_n are given by $\phi_n \equiv e^{-Q/2}\Phi_n$. More generally, the association between states $|A\rangle$ in the Hilbert space for the $\mathcal{PT}$-symmetric theory and states $|a\rangle$ in the Hilbert space for the Hermitian theory is given by

$$|a\rangle = e^{-Q/2}|A\rangle. \tag{14}$$

The Hermitian theory whose dynamics is specified by h has the standard Dirac inner product:

$$\langle a|b\rangle \equiv (|a\rangle)^\dagger \cdot |b\rangle. \tag{15}$$

However, the inner product for the non-Hermitian theory whose dynamics is governed by H is the $\mathcal{CPT}$ inner product explained in Ref. [4]:

$$\langle A|B\rangle_{\mathcal{CPT}} \equiv (\mathcal{CPT}|A\rangle)^{\mathrm{T}} \cdot |B\rangle. \tag{16}$$

If $|a\rangle$ and $|b\rangle$ are related to $|A\rangle$ and $|B\rangle$ by $|a\rangle = e^{-Q/2}|A\rangle$ and $|b\rangle = e^{-Q/2}|B\rangle$ according to (14), then the two inner products in (15) and (16) are identical. To show this one can argue as follows:

$$\begin{aligned}\langle a| = (|a\rangle)^\dagger = (|a\rangle)^{*\mathrm{T}} = (\mathcal{T}|a\rangle)^{\mathrm{T}} = \left(\mathcal{T}e^{-Q/2}|A\rangle\right)^{\mathrm{T}} = \left(e^{Q/2}\mathcal{T}|A\rangle\right)^{\mathrm{T}} \\ = \left(e^{-Q/2}e^{Q}\mathcal{PPT}|A\rangle\right)^{\mathrm{T}} = \left(e^{-Q/2}\mathcal{CPT}|A\rangle\right)^{\mathrm{T}} = (\mathcal{CPT}|A\rangle)^{\mathrm{T}} e^{Q/2}.\end{aligned} \tag{17}$$

Thus, $\langle a|b\rangle = \langle A|B\rangle_{\mathcal{CPT}}$.

In this paper we discuss the Hermitian Hamiltonian corresponding to the cubic non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + igx^3. \tag{18}$$

This is the quantum-mechanical analog of the field-theoretic Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + ig\varphi^3, \tag{19}$$

which is a non-Hermitian scalar quantum field theory that has appeared often in the literature. This quantum field theory describes the Lee-Yang edge singularity [12] and arises in Reggeon field theory [13]. The construction given in Ref. [5] of the $\mathcal{C}$ operator for this quantum field theory demonstrates that this model is a physical unitary quantum theory and not an unrealistic mathematical curiosity.

The question to be addressed in this paper is whether the Hermitian form of the Hamiltonian (18) is more useful or less useful than the non-Hermitian form. To answer this question, in Sec. II we calculate the ground-state energy to order g^2 using Feynman graphical methods for both the Hermitian and the non-Hermitian versions of the theory. We focus on graphical methods here because only graphical methods can be used as a perturbative approach in quantum field theory. We find that for the non-Hermitian version of the theory the Feynman rules are simple and the calculation is utterly straightforward. In contrast, for the Hermitian version of the theory the Feynman rules are significantly more complicated and lead to divergent integrals that must be regulated. In Sec. III we show how to calculate the one-point Green's function in both versions of the theory to order g^3. Again, we encounter divergent graphs in the Hermitian theory, and these graphs must be regulated to obtain the correct answer. In Sec. IV we show that the Feynman rules in the Hermitian theory become increasingly complicated as one goes to higher orders in perturbation theory. One is inevitably led to very difficult divergent integrals that involve overlapping divergences. In contrast, the calculation for the $\mathcal{PT}$-symmetric version of the theory is extremely simple and only contains finite graphs. We conclude in Sec. V that the Hermitian version of the $\mathcal{PT}$-symmetric theory is impractical.

II. CALCULATION OF THE GROUND-STATE ENERGY TO ORDER g^2

The Schrödinger eigenvalue problem corresponding to the quantum-mechanical Hamiltonian (18) is easy to solve perturbatively, and we can calculate the ground-state energy as a series in powers of g^2. The fourth-order result is

$$E_0 = \tfrac{1}{2} + \tfrac{11}{8}g^2 - \tfrac{465}{32}g^4 + \mathcal{O}(g^6). \tag{20}$$

However, our ultimate objective is to study $\mathcal{PT}$-symmetric quantum field theories, and therefore we need to construct Feynman-diagrammatic methods to solve for the Green's functions of the theory.

For any quantum field theory the perturbation expansion of the ground-state energy is the negative sum of the connected Feynman graphs having no external lines. To evaluate Feynman graphs we must first determine the Feynman rules, which are obtained from the Lagrangian. Thus, we begin by constructing the Lagrangian corresponding to the Hamiltonian H in (18):

$$L = \tfrac{1}{2}(p\dot{x} + \dot{x}p) - H. \tag{21}$$

Because the interaction term is local (it depends only on x and not on p), the formula for $\dot{x}$ is simple:

$$\dot{x} = p. \tag{22}$$

Thus, we have

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 - igx^3. \tag{23}$$

From (23) we read off the Euclidean Feynman rules: The three-point vertex amplitude is $-6ig$. In coordinate space a line connecting vertices at x and y is represented by $\frac{1}{2}e^{-|x-y|}$ and in momentum space the line amplitude is $\frac{1}{p^2+1}$. These Feynman rules are illustrated in Fig. 1.

In order g^2 there are two connected graphs that contribute to the ground-state energy, and these are shown in Fig. 2. The symmetry number for graph (a1) is $\frac{1}{8}$ and the symmetry number for graph (a2) is $\frac{1}{12}$. Both graphs have vertex factors of $-36g^2$. The evaluation of the Feynman integrals for (a1) and (a2) gives $V/4$ and $V/12$, respectively, where $V = \int dx$ is the volume of coordinate space. Thus, the sum of the graph amplitudes is $-\frac{11}{8}g^2V$. The contribution to the ground-state energy is the negative of this amplitude divided by V: $E_2 = \frac{11}{8}g^2$, which easily reproduces the g^2 term in (20).

We showed in Sec. I that the energy levels of the Hermitian Hamiltonian h that is obtained by means of the similarity transformation (6) are identical to those of H. Our objective here is to recalculate the g^2 term in the expansion of the ground-state energy in (20) using

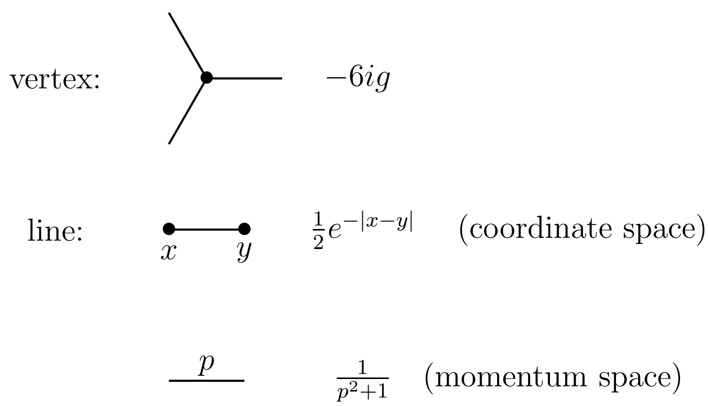


FIG. 1: Feynman rules for the Lagrangian (23). For this simple local trilinear interaction the Feynman graphs are built from three-point vertices connected with lines. The line amplitudes in both coordinate space and momentum space are shown.

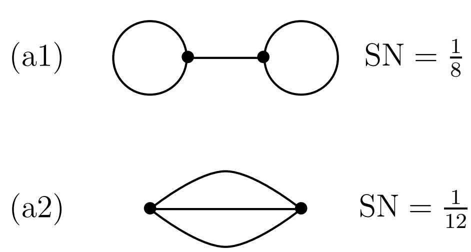


FIG. 2: The two connected vacuum graphs, labeled (a1) and (a2), contributing to the ground-state energy of H in (18) to order g^2. The symmetry numbers for each graph are indicated.

the Feynman rules obtained from the transformed Hamiltonian h. The first step in this calculation is to construct the operator Q, which is given in Ref. [5] as

$$Q = \left(-\tfrac{4}{3}p^3 - 2S_{1,2}\right) g + \left(\tfrac{128}{15}p^5 + \tfrac{40}{3}S_{3,2} + 8S_{1,4} - 12p\right) g^3 + \mathcal{O}(g^5), \tag{24}$$

where the symbol $S_{m,n}$ represents a totally symmetric combination of m factors of p and n factors of x.

One can use (6) – (7) to construct h. The result given in Refs. [10, 11] is

$$\begin{aligned} h \;=\; & \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + \left(\tfrac{3}{2}x^4 + 3S_{2,2} - \tfrac{1}{2}\right) g^2 \\ & + \left(-\tfrac{7}{2}x^6 - \tfrac{51}{2}S_{2,4} - 36S_{4,2} + 2p^6 + \tfrac{15}{2}x^2 + 27p^2\right) g^4 + \mathcal{O}(g^6). \end{aligned} \tag{25}$$

In order to obtain the Feynman rules we must now construct the corresponding Hermitian Lagrangian ℓ. To do so, we must replace the operator p with the operator $\dot{x}$ by using the formula

$$p = \dot{x} - 6g^2 s_{1,2}, \tag{26}$$

where $s_{m,n}$ represents a totally symmetric combination of m factors of $\dot{x}$ and n factors of x. The result for the Hermitian Lagrangian ℓ is

$$\begin{aligned} \ell \;=\; & \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 - \left(\tfrac{3}{2}x^4 + 3s_{2,2} - \tfrac{1}{2}\right) g^2 \\ & + \left(\tfrac{7}{2}x^6 + \tfrac{87}{2}s_{2,4} + 36s_{4,2} - 2\dot{x}^6 - \tfrac{27}{2}x^2 - 27\dot{x}^2\right) g^4 + \mathcal{O}(g^6). \end{aligned} \tag{27}$$

From this Lagrangian we can read off the Euclidean-space Feynman rules. Unlike the $\mathcal{PT}$ version of the theory, increasingly many new vertices appear in every order of perturbation theory. The three vertices to order g^2 are shown in Fig. 3 and the six vertices to order g^4 are shown in Fig. 4. Note that some of the lines emerging from the vertices have tick marks. A tick mark indicates a derivative in coordinate space and a factor of ip in momentum space. The tick marks are a result of the *derivative* coupling terms in the Lagrangian ℓ. As we will see, the derivative coupling gives rise to divergent Feynman graphs.

We now use the Feynman rules in Fig. 3 to construct the vacuum graphs contributing to the ground-state energy in order g^2. These graphs are shown in Fig. 5. The simplest of these three graphs is (b3) because there is no Feynman integral to perform. This graph arises from the constant term in ℓ in (27). The value of this graph is simply $\frac{1}{2}g^2V$.

Graph (b1) has symmetry number $\frac{1}{8}$ and vertex factor $-36g^2$ and the Feynman integral in momentum space is

$$\left(\int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\frac{1}{p^2+1}\right)^2 = \frac{1}{4}. \tag{28}$$

The integrals associated with this graph are convergent. The value of graph (b1) is therefore $-\frac{9}{8}g^2V$, where the factor of V comes from the translation invariance of the graph.

The interesting graph is (b2). The symmetry number is $\frac{1}{4}$, the vertex factor is $12g^2$, and the Feynman integral in momentum space is

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\frac{p^2}{p^2+1} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\,\frac{1}{q^2+1}. \tag{29}$$

The q integral is convergent and gives the value $\frac{1}{2}$. However, the p integral is divergent. We therefore regulate it using dimensional continuation and represent its value as the limit as the number of dimensions approaches 1:

$$\lim_{D\to 1} 2\int_0^\infty \frac{r^{D-1}dr}{2\pi}\,\frac{r^2}{r^2+1} = \lim_{D\to 1}\frac{\Gamma\left(1+\frac{1}{2}D\right)\Gamma\left(-\frac{1}{2}D\right)}{2\pi} = -\frac{1}{2}. \tag{30}$$

Hence, the value of graph (b2) is $-\frac{3}{4}g^2V$, where again the volume factor V comes from translation invariance. Adding the three graphs (b1), (b2), and (b3), dividing by V, and changing the sign gives the result $\frac{11}{8}g^2$, which reproduces the result in (20). This is a more difficult calculation than that using the Feynman rules in Fig. 1 because we encounter a divergent graph. It is most surprising to find a divergent graph in one-dimensional quantum field theory (quantum mechanics). The infinite graph here is not associated with a renormalization of a physical parameter in the Lagrangian. Rather, it is an artifact of the derivative coupling terms that inevitably arise from the similarity transformation (6).

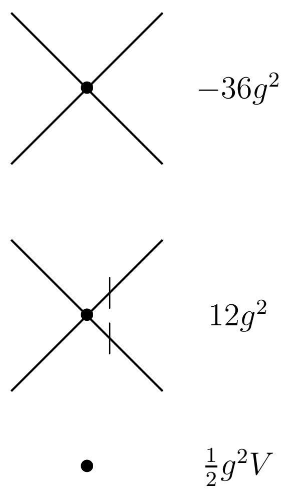


FIG. 3: The three Euclidean-space vertices to order g^2 for the Hermitian Lagrangian ℓ in (27). Note that the second vertex has tick marks on two of the legs. These tick marks indicate coordinate-space derivatives that arise because of derivative coupling. Derivative coupling results in divergent Feynman graphs.

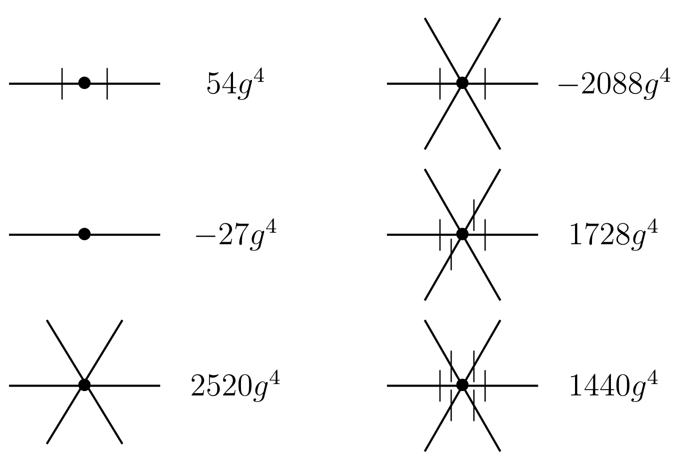


FIG. 4: The six Euclidean-space vertices to order g^4 for the Hermitian Lagrangian ℓ in (27). Note that four of the vertices have tick marks on the legs. These tick marks indicate derivative coupling.

Here is a simple model that illustrates the use of dimensional continuation as a means of regulating Feynman graphs: Consider the quadratic Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}g\dot{x}^2. \tag{31}$$

The corresponding Hamiltonian is

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + \frac{g}{2-2g}p^2. \tag{32}$$

The ground-state energy E_0 for H in (32) is

$$E_0 = \tfrac{1}{2}(1-g)^{-1/2}. \tag{33}$$

The Euclidean Feynman rules for L in (31) are elementary. The amplitude for a line is given in Fig. 1 and there is a two-tick two-point vertex with amplitude g. (This vertex has the form of the first vertex shown in Fig. 4.) The graphs contributing to the ground-state energy are all polygons. The nth-order graph has n vertices; its symmetry number is $\frac{1}{2n}$ and its vertex amplitude is g^n. The total graphical contribution to the ground-state energy is

$$E_0 - \tfrac{1}{2} = -\sum_{n=1}^{\infty} \frac{g^n}{2n} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p^{2n}}{(p^2+1)^n}. \tag{34}$$

Each of the integrals in (34) is divergent, but we regulate the integrals as in (30):

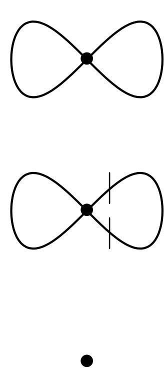
$$\lim_{D\to 1} 2\int_0^\infty \frac{r^{D-1}dr}{2\pi}\left(\frac{r^2}{r^2+1}\right)^n = \lim_{D\to 1} \frac{\Gamma\left(n+\frac{1}{2}D\right)\Gamma\left(-\frac{1}{2}D\right)}{2\pi(n-1)!} = -\frac{\Gamma\left(n+\frac{1}{2}\right)}{\pi^{1/2}(n-1)!}. \tag{35}$$

Therefore, (34) becomes

$$E_0 = \tfrac{1}{2}\sum_{n=0}^{\infty} g^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\pi^{1/2}n!} = \tfrac{1}{2}(1-g)^{-1/2}, \tag{36}$$

which verifies the result in (33).

This dimensional-continuation procedure is effective because it extracts the correct finite contribution from each of the divergent graphs. However, this procedure is much more difficult to apply when there are graphs having overlapping divergences, as we shall see in Sec. IV.



(b1) SN $= \frac{1}{8}$

(b2) SN $= \frac{1}{4}$

(b3) SN $= 1$

FIG. 5: The three graphs contributing to the ground-state energy of the Hermitian Lagrangian ℓ in (27) in order g^2. Note that while graphs (b1) and (b3) are finite, the Feynman integral for graph (b2) diverges and must be regulated to obtain a finite result.

III. CALCULATION OF THE ONE-POINT GREEN'S FUNCTION

The connection in (14) between states in the Hermitian and the non-Hermitian $\mathcal{PT}$-symmetric theories implies the following relation between an operator O in the non-Hermitian $\mathcal{PT}$-symmetric theory and the corresponding operator $\tilde{O}$ in the Hermitian theory:

$$\tilde{O} = e^{-Q/2} O e^{Q/2}. \tag{37}$$

Using this connection, we now calculate the one-point Green's function G_1 in both versions of the theory to order g^3. Again, we find that the calculation in the non-Hermitian theory is extremely simple, but that in the Hermitian theory the calculation again involves divergent graphs that must be regulated.

The graphs contributing to $G_1 = \langle 0|x|0\rangle_{\mathcal{CPT}}$ through order g^3 in the non-Hermitian theory defined by H in (18) or, equivalently, L in (23) are shown in Fig. 6. Each of these graphs is finite and is easily evaluated. The result is that

$$G_1 = -\tfrac{3}{2} i g + \tfrac{33}{2} i g^3 + \mathcal{O}(g^5). \tag{38}$$

Next, we calculate the identical one-point Green's function in the Hermitian theory. To do so, we need to transform the field x to the corresponding field $\tilde{x}$ in the Hermitian theory using (37):

$$\begin{aligned}
\tilde{x} &= e^{-Q/2} x e^{Q/2} \\
&= x - i\left(x^2 + 2p^2\right) g + \left(2S_{2,1} - x^3\right) g^2 + i\left(20p^4 + 24S_{2,2} + 5x^4 - 6\right) g^3 + \mathcal{O}(g^4) \\
&= x - i\left(x^2 + 2\dot{x}^2\right) g + \left(2s_{2,1} - x^3\right) g^2 + i\left(20\dot{x}^4 + 48s_{2,2} + 5x^4 - 6\right) g^3 + \mathcal{O}(g^4),
\end{aligned} \tag{39}$$

where we have replaced p in favor of $\dot{x}$ using (26). (This result may be found in Refs. [10, 11] to order g^2.)

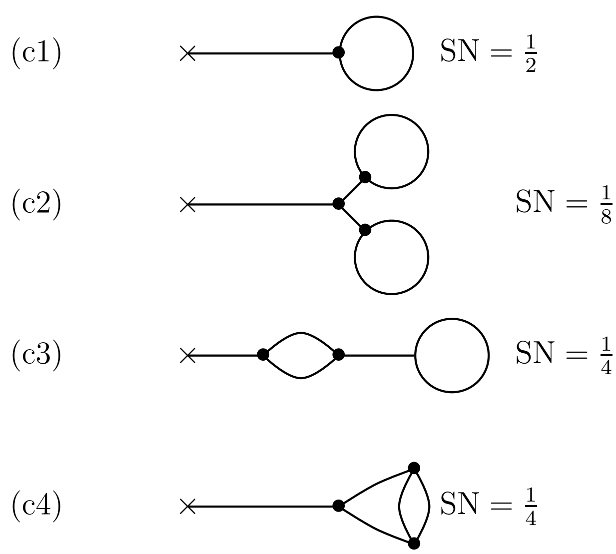


FIG. 6: The Feynman graphs contributing to the one-point Green's function G_1 of the non-Hermitian Hamiltonian H through order g^3. Graph (c1) is of order g and graphs (c2) – (c4) are of order g^3.

Using (39) we can construct the graphs contributing to G_1 to order g (see Fig. 7). Graph (d1) is finite and has the value $-\frac{1}{2}ig$. However, graph (d2) is infinite and must be regulated using dimensional continuation. The result is $-ig$. Combining these two graphs, we obtain the term of order g in (38).

The calculation of G_1 to order g^3 in the Hermitian theory is much more complicated. First, we must construct the six connected graphs arising from the expectation value of $\tilde{x}$ in (39) (see Fig. 8). Three of these graphs, (e1), (e3), and (e4), are finite. The remaining graphs are divergent and must be regulated using dimensional continuation. There are four more disconnected graphs arising from the expectation values of the terms $20\dot{x}^4$, $48s_{2,2}$, $5x^4$, and -6. Two of these graphs must also be regulated. Finally, combining the contributions of these ten graphs, we successfully reproduce the $\mathcal{O}(g^3)$ result $\frac{33}{2}ig^3$ in (38). We emphasize that the calculation for the Hermitian theory is orders of magnitude more difficult than the corresponding calculation for the non-Hermitian $\mathcal{PT}$-symmetric theory.

IV. HIGHER-ORDER CALCULATION OF THE GROUND-STATE ENERGY

In this section we extend the calculation of the ground-state energy that is described in Sec. II to next order in powers of g^2. We will see that this calculation is completely straightforward in the non-Hermitian theory, while it is nearly impossible in the Hermitian theory. We show that the difficulty is not just due to the arithmetic difficulty of sorting through large numbers of graphs, but rather is one of principle. The problem is that we

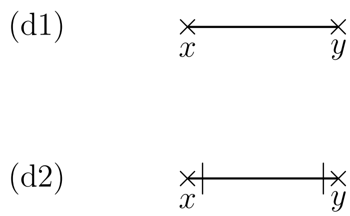


FIG. 7: The two graphs contributing to the one-point Green's function G_1 in the Hermitian theory to order g. Note that graph (d1) is finite, while graph (d2) diverges and must be regulated to give a finite result.

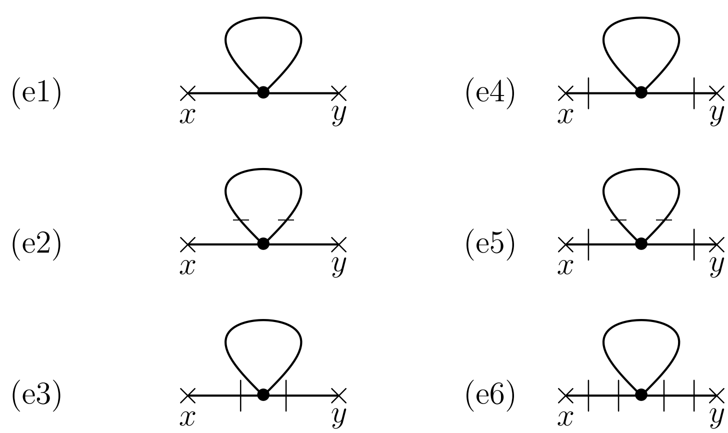


FIG. 8: The six connected graphs contributing to the one-point Green's function G_1 for the Hermitian theory to order g^3. Graphs (e1), (e3), and (e4) are finite, but the remaining graphs are divergent and must be regulated.

encounter two graphs with overlapping divergences, and calculating the numerical values of the corresponding regulated graphs remains an unsolved problem, even in one-dimensional field theory (quantum mechanics)!

There are five graphs (f1) – (f5) contributing in order g^4 to the ground-state energy of the non-Hermitian Hamiltonian H in (18). These are shown in Fig. 9. The symmetry numbers for these graphs are indicated in the figure. The vertex factors for all these graphs are $1296g^4$. The Feynman integrals for these graphs are $\frac{1}{16}V$ for (f1), $\frac{11}{864}V$ for (f2), $\frac{1}{8}V$ for (f3), $\frac{1}{36}V$ for (f4), and $\frac{1}{96}V$ for (f5). Thus, the sum of the graphs is $\frac{465}{32}V$. The negative of this amplitude divided by V is $E_4 = -\frac{465}{32}g^4$. This reproduces the order g^4 term in the perturbation expansion for the ground-state energy in (20).

There are seventeen graphs of order g^4 contributing to the ground-state energy of the Hermitian Hamiltonian (25). These graphs, along with their symmetry numbers, are shown in Fig. 10. Seven of these graphs, (g1), (g3), (g7), (g8), (g10), (g11), and (g16) are finite and easy to calculate. The Feynman integrals for the remaining graphs are all infinite and must be regulated. Dimensional continuation can be readily implemented as in (30) except for the graphs (g15) and (g17). These two graphs are extremely difficult to regulate because they have overlapping divergences. It is most dismaying to find Feynman graphs having overlapping divergences in one-dimensional quantum field theory! Since the g^4 contribution to the ground-state energy is given in (20), we can deduce that the sum of the regulated values of these two graphs (multiplied by their respective symmetry numbers and vertex factors) must be $\frac{21}{16}g^4V$. However, we are unable to find a simple way to obtain this result.

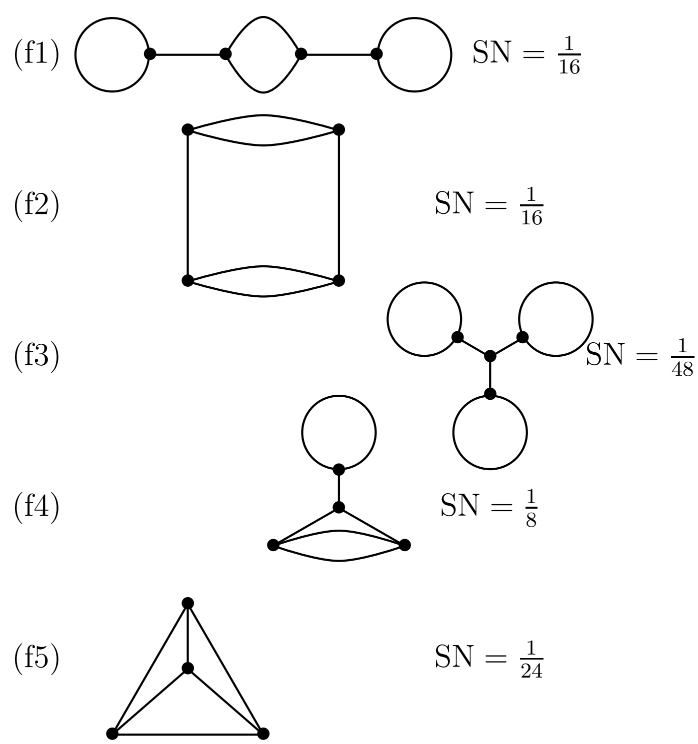


FIG. 9: The five vacuum graphs contributing to the ground-state energy of the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian H in (18) to order g^4. These graphs are all finite and very easy to evaluate.

V. CONCLUDING REMARKS

This study was motivated by the concern that the mechanics of solving problems in quantum field theory might not work in non-Hermitian theories. The usual techniques rely on the use of the Schwinger action principle, the construction of functional integrals, and

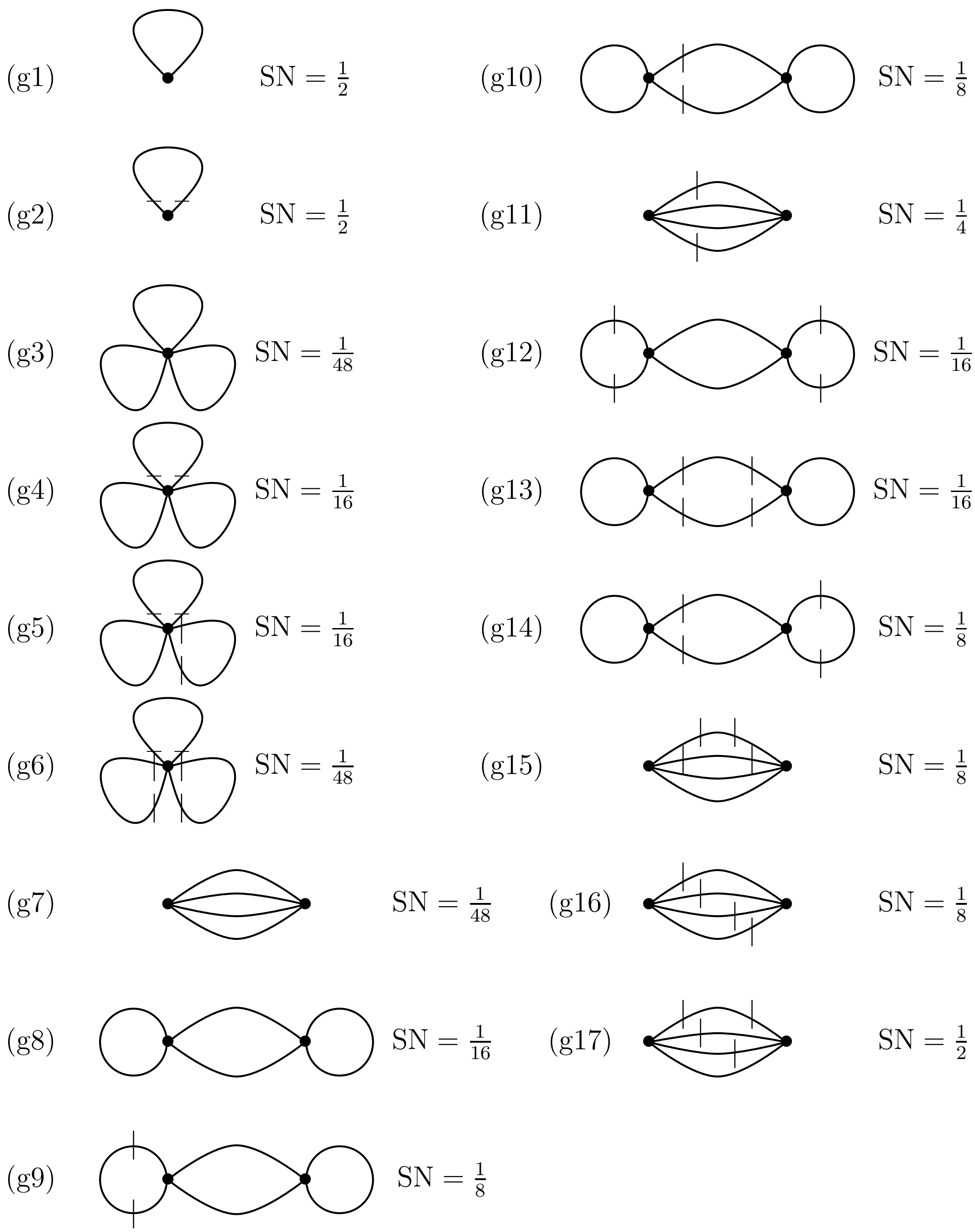


FIG. 10: The seventeen graphs contributing to the order g^4 term in the perturbation expansion for the ground-state energy of the Hermitian Hamiltonian h in (25). Note that ten of these graphs have divergent Feynman integrals. Of these ten, eight are relatively easy to regulate using dimensional continuation. However, graphs (g15) and (g17) have overlapping divergences, and are therefore extremely hard to evaluate.

the identification of and application of Feynman rules. These procedures are conventionally formulated in a Hermitian setting. The surprise is that all of these standard techniques work perfectly in a non-Hermitian context, but that they are much too difficult to apply if the non-Hermitian theory is first transformed to the equivalent Hermitian one.

We conclude by citing the comment of Jones in the second paper in Ref. [10] regarding the critique of $\mathcal{PT}$-symmetric theories in Ref. [14]. Jones writes, "Clearly, this [Eq. (25)] is not a Hamiltonian that one would have contemplated in its own regard were it not derived from [Eq. (18)]. It is for this reason that we disagree with the contention of Mostafazadeh [14] that, 'A consistent probabilistic $\mathcal{PT}$-symmetric quantum theory is doomed to reduce to ordinary quantum mechanics.'" Mostafazadeh appears to be correct in arguing that a $\mathcal{PT}$-symmetric theory can be transformed to a Hermitian theory by means of a similarity transformation. However, we have demonstrated that the difficulties with the Hermitian theory are severe and virtually insurmountable because this theory possesses a Feynman perturbation expansion that becomes increasingly divergent as one goes to higher order. The divergences are not removable by renormalization, but rather are due to increasingly singular derivative interactions. In contrast, the non-Hermitian $\mathcal{PT}$-symmetric theory is completely free from all such difficulties.

Acknowledgments

KAM is grateful for the hospitality of the Physics Department at Washington University. We thank the U.S. Department of Energy for partial financial support of this work.

---

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[2] C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. **40**, 2201 (1999).

[3] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. **34**, L391 (2001); *ibid.* **34**, 5679 (2001).

[4] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).

[5] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **93**, 251601 (2004) and Phys. Rev. D **70**, 025001 (2004).

[6] C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang, Phys. Rev. D **71**, 025014 (2005).

[7] C. M. Bender, I. Cavero-Pelaez, K. A. Milton, and K. V. Shajesh, Phys. Lett. B **613**, 97 (2005).

[8] C. M. Bender, H. F. Jones, and R. J. Rivers, Phys. Lett. B **625**, 333 (2005).

[9] A. Mostafazadeh, J. Math. Phys. **33**, 205 (2002) and J. Phys. A: Math. Gen. **36**, 7081 (2003).

[10] H. F. Jones, Czech. J. Phys. **54**, 1107 (2004); J. Phys. A: Math. Gen. **38**, 1741 (2005).

[11] A. Mostafazadeh, J. Phys. A: Math. Gen. **38**, 6557 (2005) and Erratum **38**, 8185 (2005).

[12] M. E. Fisher, Phys. Rev. Lett. **40**, 1610 (1978); J. L. Cardy, *ibid.* **54**, 1345 (1985); J. L. Cardy and G. Mussardo, Phys. Lett. B **225**, 275 (1989); A. B. Zamolodchikov, Nucl. Phys. B **348**, 619 (1991).

[13] H. D. I. Abarbanel, J. D. Bronzan, R. L. Sugar, and A. R. White, Phys. Rep. **21**, 119 (1975); R. Brower, M. Furman, and M. Moshe, Phys. Lett. B **76**, 213 (1978); B. Harms, S. Jones, and C.-I Tan, Nucl. Phys. **171**, 392 (1980) and Phys. Lett. B **91B**, 291 (1980).

[14] A. Mostafazadeh, arXiv:quant-ph/0310164.