Casimir Energy for a Purely Dielectric Cylinder by the Mode Summation Method

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Abstract

We use the mode summation method together with zeta-function regularization to compute the Casimir energy of a dilute dielectric cylinder. The method is very transparent, and sheds light on the reason the resulting energy vanishes.

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I. MOTIVATION

A few years ago a calculation of the sum of van der Waals interactions for a purely dielectric cylinder in the dilute-dielectric approximation led to a surprising null result [1]. (This was first found by an unpublished calculation by the first author of the present paper, and later independently confirmed by calculations by Milonni [2] and by Barton [3].) This unexpected finding produced a quest to calculate the corresponding Casimir energy with the aim of verifying the predicted equality between both quantities, which was recently established in Ref. [4]. However, a physical understanding of why this Casimir energy should vanish (as also does that for a dilute dielectric-diamagnetic cylinder [1]) remains elusive. In addition to the physical interest of this subject, we recall that the authors of Ref. [1] commented at some length upon the comparative advantages and shortcomings of Green's function formalisms and mode summation methods for the evaluation of Casimir energies. A formal relation between these two approaches was given in appendix A of that paper.

The first procedure is essentially a calculation of the expectation value of the stress-energy tensor expressed in terms of Green's functions and their transforms. This approach has proven to be remarkably fruitful and enlightening from the perspective of physical interpretation, as the quantities computed bear close relationships to easily identifiable observables expressed in terms of sources and fields. This was the method applied in Ref. [4] to the object of our present study, i.e., the Casimir energy for a dielectric cylinder with the light velocity different on the inside and the outside.

In contrast, the mode summation method rests on the concept of zero point energy and its basic expression as an infinite sum of eigenfrequencies. Even though this sum is a rather abstruse concept, from a prosaic viewpoint it has the appeal of simplicity. Since these eigenfrequencies stem from classical problems and are in many cases already known (e.g. through textbooks like Ref. [5]), the only remaining task is to perform their summation. Once this question has been mathematically posed, the answer sought after is rather easily within one's grasp.

In view of these considerations, the reflections in Ref. [1] itself, and the examples offered by many related works, it is sensible to say that the two procedures may complement each other. This is especially so, since these calculations still present a number of subtleties, and it is to be hoped that approaching the problems from disparate viewpoints may lead to improved physical insight. Specifically, the well-known divergence difficulties, which have recently received much attention [6], encourage us to examine the problem anew. With this idea in mind, we shall tackle some aspects of the problem treated in Ref. [4] by a particular variant of the mode summation method.

II. MODE SUM

According to Refs. [1, 5], the eigenfrequencies ω_{m,p,k_z} of the Maxwell equations for an infinite material cylinder of radius a, oriented along the z-axis, with permittivity and permeability (ε_1, μ_1), surrounded by a medium with permittivity and permeability (ε_2, μ_2), are the zeros of the following function:

$$f_m(k_z, \omega) = 0, \quad m = 0, \pm 1, \pm 2, \ldots, \quad k_z \in \mathbb{R}, \tag{2.1a}$$

$$f_m(k_z,\omega) \equiv \frac{1}{\Delta^2}\left[\Delta_m^{\rm TE}(x,y)\,\Delta_m^{\rm TM}(x,y) - m^2\frac{a^4\omega^2k_z^2}{x^2y^2}(\varepsilon_1\mu_1-\varepsilon_2\mu_2)^2 J_m^2(x)\,H_m^2(y)\right], \quad (2.1b)$$

with the following abbreviations

$$\begin{array}{rcl} & \Delta = -\dfrac{2i}{\pi}, \quad x=\lambda_1 a, \quad y=\lambda_2 a, & \\ \Delta_m^{\rm TE}(x,y) &=& \mu_1 y J_m'(x)\,H_m(y) - \mu_2 x J_m(x)\,H_m'(y), \\ \Delta_m^{\rm TM}(x,y) &=& \varepsilon_1 y J_m'(x)\,H_m(y) - \varepsilon_2 x J_m(x)\,H_m'(y), \\ & \lambda_i^2 = \varepsilon_i\mu_i\omega^2 - k_z, \quad i=1,2. & \end{array} \quad (2.2)$$

As usual, m is the azimuthal quantum number, k_z denotes the momentum along the axis of the cylinder, and p labels the zeroes of $f_m(k_z,\omega)$. Here, for $y>0$, $H_m(y) \equiv H_m^{(1)}(y)$. Note that $f_m = -\Delta^{-2}\Xi$, where Ξ is the same denominator introduced in Ref. [4] and the Δ^{-2} factor has been introduced for convenience. Recalling that the velocities of light in each media are given by $c_i = (\varepsilon_i\mu_i)^{-1/2}$, $i=1,2$, we see that Eqs. (2.1a), (2.1b) exhibit a peculiar feature of this situation: if $c_1 \neq c_2$, the second term in (2.1b) comes into play, and it is no longer possible to decompose the mode set into zeros of $\Delta_m^{\rm TE}$ and zeros of $\Delta_m^{\rm TM}$; that is, the transverse electric and magnetic modes become entangled. When medium 1 is purely dielectric and medium 2 is vacuum, $\varepsilon_1=\varepsilon$, $\mu_1=1$, $\varepsilon_2=\mu_2=1$, one may write $\omega = a^{-1}(y^2+\widehat{k}^2)^{1/2}$ and $x^2 = y^2+(\varepsilon-1)(y^2+\widehat{k}^2)$, where $\widehat{k}\equiv k_z a$. We will also set $c_2=c=1$.

Formally speaking, the Casimir energy per unit length is given by the mode sum

$$\mathcal{E}_C = \frac{1}{2}\hbar\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{m,p}\omega_{m,p,k_z}. \quad (2.3)$$

This quantity is related to the Casimir radial pressure P_C through

$$P_C = \frac{1}{\pi a^2}\mathcal{E}_C. \quad (2.4)$$

The sum in (2.3) is divergent and will be regularized by changing the power of ω, which power becomes the argument of the associated zeta function. By Ref. [7] we already know that, up through the order of $(\varepsilon-1)^2$, there should be no ambiguities, i.e., no logarithms of arbitrary scales, because the heat kernel coefficient which would multiply them is of $\mathcal{O}((\varepsilon-1)^3)$. Therefore, we are permitted to define

$$\mathcal{E}_C(s) = \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\sum_{m,p}\omega_{m,p,k_z}^{-s} = \frac{\hbar}{2}\,a^{s-1}\int_{-\infty}^{\infty}\frac{d\widehat{k}}{2\pi}\sum_{m,p}(y_{m,p}^2+\widehat{k}^2)^{-\frac{s}{2}}, \quad (2.5)$$

as a function of the complex variable s, without any additional mass scale. It is proposed to make sense of Eq. (2.3) by means of analytic continuation of this function to $s=-1$, namely,

$$\mathcal{E}_C = \lim_{s\to -1}\mathcal{E}_C(s). \quad (2.6)$$

For any given values of $\widehat{k}$, m, the remaining sum over p may be rewritten, with the help of the residue theorem, as a contour integral in complex y plane. Thus,

$$\mathcal{E}_C(s) = \frac{\hbar}{2}\,a^{s-1}\int_{-\infty}^{\infty}\frac{d\widehat{k}}{2\pi}\sum_{m=-\infty}^{\infty}\frac{s}{2\pi i}\int_C dy\,y\,(y^2+\widehat{k}^2)^{-\frac{s+2}{2}}\,\ln f_m, \quad (2.7)$$

where C is a circuit which encloses all the y values corresponding to the positive zeroes of f_m. This approach is often referred to as the argument principle [8]. Quite often in the application of this method, an asymptotic form $f_{m,\mathrm{as}}$ of f_m is found and then $\ln f_{m,\mathrm{as}}$ is subtracted from $\ln f_m$ in the integrand. Actually, the factors introduced in (2.1b) relative to the original f_m of Ref. [1] amount to dividing that function by the leading term of its asymptotic behaviour. At any rate, the full form of this asymptotic behaviour is presumably related to the limit of unbounded space, which is already available as the bulk contribution calculated in Ref. [4].

Next, the dilute-dielectric approximation will be made by expanding the logarithm function of Eq. (2.7) in powers of $(\varepsilon - 1)$, choosing y as the independent variable and taking x as a function of y and $\widehat{k}$:

$$\begin{aligned}\ln f_m = &\left[L^0_{m1}(y) + L^1_{m1}(y)(y^2+\widehat{k}^2)\right](\varepsilon-1)\\ &+\left[L^{00}_{m2}(y) + L^{10}_{m2}(y)(y^2+\widehat{k}^2) + L^{20}_{m2}(y)(y^2+\widehat{k}^2)^2 + L^{11}_{m2}(y)(y^2+\widehat{k}^2)\,\widehat{k}^2\right](\varepsilon-1)^2\\ &+\mathcal{O}((\varepsilon-1)^3),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
L^0_{m1}(y) &= \frac{1}{\Delta} y\, J'_m(y) H_m(y),\\
L^1_{m1}(y) &= \frac{1}{\Delta y}\Delta^{(1,0)}_m(y),\\
L^{00}_{m2}(y) &= -\frac{1}{2\Delta^2} y^2\, {J'_m}^2(y) H^2_m(y),\\
L^{10}_{m2}(y) &= -\frac{1}{2\Delta^2}\left[\Delta^{(1,0)}_m(y) J'_m(y) H_m(y) + \frac{\Delta}{y}\left(J'_m(y) + y\left(1-\frac{m^2}{y^2}\right)J_m(y)\right)H_m(y)\right],\\
L^{20}_{m2}(y) &= L^{20A}_{m2}(y) + L^{20B}_{m2}(y), \quad \begin{cases} L^{20A}_{m2}(y) = \dfrac{1}{4\Delta y^2}\left(\Delta^{(2,0)}_m(y) - \dfrac{1}{y}\Delta^{(1,0)}_m(y)\right),\\ L^{20B}_{m2}(y) = -\dfrac{1}{4\Delta^2 y^2}\left(\Delta^{(1,0)}_m(y)\right)^2,\end{cases}\\
L^{11}_{m2}(y) &= -\frac{m^2}{\Delta^2 y^4} J^2_m(y) H^2_m(y).
\end{aligned} \tag{2.9}$$

Here, the notation is

$$\Delta^{(j,0)}_m(y) = \left.\frac{\partial^j \Delta_m(x,y)}{\partial x^j}\right|_{\varepsilon_1=\varepsilon_2=1,\mu_1=\mu_2=1,x=y}, \quad \text{for } j=1,2, \tag{2.10}$$

where Δ_m stands for either Δ^{TE}_m or Δ^{TM}_m (in the free space limit there is no difference). Moreover, note that

$$\Delta_m(x,y)|_{\varepsilon_1=\varepsilon_2=1,\mu_1=\mu_2=1,x=y} = -y\, W[J_m(y), H_m(y)] = \Delta, \tag{2.11}$$

where W denotes the Wronskian. Explicitly,

$$\begin{aligned}
\Delta^{(1,0)}_m(y) &= -\frac{1}{y}\left[y^2 J'_m(y) H'_m(y) + (y^2-m^2) J_m(y) H_m(y)\right] - (J_m(y) H_m(y))',\\
\Delta^{(2,0)}_m(y) &= \left(\Delta^{(1,0)}_m(y)\right)' - \left(1-\frac{m^2+1}{y^2}\right)\Delta,
\end{aligned} \tag{2.12}$$

where $(\Delta_m^{(1,0)}(y))' \equiv \dfrac{d}{dy}\Delta_m^{(1,0)}(y) \neq \Delta_m^{(2,0)}(y)$, as shown. Once this task has been accomplished, we insert Eq. (2.8) into Eq. (2.7) and perform the $\widehat{k}$ integration.

Consideration of formulas (2.7)–(2.12) shows the need of dealing with integrals of the type

$$I \equiv \int_{-\infty}^{\infty} d\widehat{k} \int_C dy\, y\, F(y)\, (y^2 + \widehat{k}^2)^{-\alpha}\, \widehat{k}^{2\beta}, \tag{2.13}$$

where C is the contour specified above, and F satisfies $F(-iv) = F(iv)$ for $v \in \mathbb{R}$, as well as having adequate asymptotic properties (see below). Looking at the $(y^2 + \widehat{k}^2)$ powers in Eqs. (2.7), (2.8), we see that in the required cases $\alpha = s/2+1, s/2, s/2-1$, and $\beta = 0$ except for one integral with $\beta = 1$. Since α is just a translation of $s/2$, analytic continuation in s amounts to analytic continuation in α.

Straightforward integration for $\widehat{k}$ yields

$$I = \mathrm{B}\left(\beta + \frac{1}{2}, \alpha - \beta - \frac{1}{2}\right) \int_C dy\, y^{2-2\alpha+2\beta}\, F(y), \tag{2.14}$$

where B stands for the Euler beta function $\mathrm{B}(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Now we rotate the C contour so that it consists of a straight line parallel to and just to the right of the imaginary axis, closed by a semicircle of infinitely large radius on the right.[1] The branch line of the $y^{2-2\alpha+2\beta}$ function, which starts at the origin, is placed so that the circuit does not cross it, that is, it lies along the negative real axis. (In the limit where the vertical part of C overlaps the axis, the origin could be avoided by a small semicircle, and, eventually, the integration along this infinitesimal part would vanish as $s \to -1$). Further, it is assumed that, combining the asymptotic behaviour of F and the possibility of varying the α value as necessary, the contribution from the large semicircle vanishes when its radius tends to infinity. Thus, the y integral reduces to an integration along the vertical parts of C, where $y = \pm iv$, and one has $y^{2-2\alpha+2\beta} = e^{\pm i\pi(1-\alpha+\beta)} v^{2-2\alpha+2\beta}$ on the upper and lower segments. As a result,

$$I = -2i\,\mathrm{B}\left(\beta + \frac{1}{2}, \alpha - \beta - \frac{1}{2}\right) \sin\left(\pi\left(\alpha - \beta - \frac{1}{2}\right)\right) \int_0^{\infty} dv\, v^{2-2\alpha+2\beta}\, F(iv). \tag{2.15}$$

Applying the reflection formula $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ to the gamma functions in the Euler beta function, we re-express I as

$$I = -2i\,\mathrm{B}\left(\beta + \frac{1}{2}, 1 - \alpha\right) \sin(\pi\alpha) \int_0^{\infty} dv\, v^{2-2\alpha+2\beta}\, F(iv). \tag{2.16}$$

Note that for the α values corresponding to $s = -1$ ($\alpha = 1/2, -1/2, -3/2$), and for $\beta = 0, 1$, the pole of the B function and the zero of the sine function in Eq. (2.15) combine to give a finite product, while in Eq. (2.16) each of these two factors is already finite.

[1] Another way to describe this rotation is in terms of a purely mathematical transformation, based on the required analyticity of the underlying Green's function. See Refs. [9, 10].

Applying formula (2.16) to Eqs. (2.7), (2.8) yields the following result:

$$\mathcal{E}_C(s) = \mathcal{E}_{C1}(\varepsilon-1) + \mathcal{E}_{C2}(s)(\varepsilon-1)^2 + \mathcal{O}((\varepsilon-1)^3), \tag{2.17}$$

where

$$\mathcal{E}_{C1}(s) = \mathcal{E}^0_{C1}(s) + \mathcal{E}^1_{C1}(s),$$

$$\begin{cases} \mathcal{E}^0_{C1}(s) &= -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2},-\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{-s} L^0_{m1}(iv), \\ \mathcal{E}^1_{C1}(s) &= -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2},1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{2-s} L^1_{m1}(iv), \end{cases} \tag{2.18}$$

and

$$\mathcal{E}_{C2}(s) = \mathcal{E}^{00}_{C2}(s) + \mathcal{E}^{10}_{C2}(s) + \mathcal{E}^{20A}_{C2}(s) + \mathcal{E}^{20B}_{C2}(s) + \mathcal{E}^{11}_{C2}(s),$$

$$\begin{cases} \mathcal{E}^{00}_{C2}(s) &= -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2},-\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{-s} L^{00}_{m2}(iv), \\ \mathcal{E}^{10}_{C2}(s) &= -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2},1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{2-s} L^{10}_{m2}(iv), \\ \mathcal{E}^{20A,B}_{C2}(s) &= -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{1}{2},2-\dfrac{s}{2}\right)\sin\left(-\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{4-s} L^{20A,B}_{m2}(iv), \\ \mathcal{E}^{11}_{C2}(s) &= -\dfrac{\hbar}{2}\dfrac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\dfrac{3}{2},1-\dfrac{s}{2}\right)\sin\left(\pi\dfrac{s}{2}\right)\displaystyle\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{4-s} L^{11}_{m2}(iv). \end{cases} \tag{2.19}$$

A. First order in $(\varepsilon-1)$

Taking $\mathcal{E}^0_{C1}(s)$ from (2.18), and $L^0_{m1}(iv)$ from (2.9) one is led to

$$\mathcal{E}^0_{C1}(s) = -\frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\frac{1}{2},-\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\sum_{m=-\infty}^{\infty}\int_0^\infty dv\, v^{1-s} I'_m(v)K_m(v). \tag{2.20}$$

The beta and sine functions are finite at $s=-1$. As for the integral, it will be rewritten by introducing the factor $1 = -vW[I_m(v),K_m(v)] = -v[I_m(v)K'_m(v) - I'_m(v)K_m(v)]$ for each m:

$$\begin{aligned} &\int_0^\infty dv\, v^{1-s}\sum_{m=-\infty}^{\infty} I'_m(v)K_m(v) = \\ -&\int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty} I_m(v)I'_m(v)K_m(v)K'_m(v) + \int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty} {I'_m}^2(v)K^2_m(v). \end{aligned} \tag{2.21}$$

We carry out the summation over m by using the addition theorem for the modified Bessel functions:

$$\begin{aligned} \sum_{m=-\infty}^{\infty} I_m(kr)K_m(k\rho)\,e^{im\phi} &= K_0(kR(r,\rho,\phi)) \\ R(r,\rho,\phi) &= \sqrt{r^2+\rho^2-2r\rho\cos\phi}, \quad \rho > r. \end{aligned} \tag{2.22}$$

Expressions for products of four Bessel functions are found by multiplying differentiated versions of the Bessel function addition theorem (2.22), integrating over ϕ, and setting $kr = k\rho \equiv v$. A change of variable $u = \sin\frac{\phi}{2}$ is made. Next, instead of proceeding with the evaluation of the resulting u integral, the expression is multiplied by a v power involving s, and integrated over v, with the help of

$$\int_0^\infty dx\, x^{-\lambda} K_\mu(x) K_\nu(x) = 2^{-2-\lambda} \frac{\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)}{\Gamma(1-\lambda)} \tag{2.23}$$

(from result (6.576.4) in Ref. [11]). Then the u integration is carried out afterwards. Thus, the emerging expressions depend on s only through gamma functions (see also Refs. [4, 12]). In fact, the results in formulæ (81) of Ref. [4] may be viewed as intermediate steps, the final result being for example

$$\begin{aligned} \int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} {I'_m}^2(v) K_m^2(v) &= \int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} {K'_m}^2(v) I_m^2(v) \\ = \int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} I_m(v) I'_m(v) K_m(v) K'_m(v) &= \frac{1}{8\pi^{1/2}} \frac{\Gamma\left(\frac{5-s}{2}\right)\Gamma^2\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(3-s)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}. \end{aligned} \tag{2.24}$$

Even if the left hand side of each such integral is not initially defined for $s = -1$, the right hand side provides the desired extension to this value through the existing analytic continuations of the gamma functions themselves. Moreover, every result displays a single pole at $s = -1$ in the gamma function in the last factor of each denominator, while the rest of the expression is finite at this point. Therefore, each expression has a zero of order one at $s = -1$.

Because Eqs. (2.24) show that the two integrals in (2.21) have the same value (even before setting $s = -1$), we conclude, in a neighborhood of $s = -1$,

$$\mathcal{E}^0_{C1}(s) = 0. \tag{2.25}$$

Equation (2.18) shows that the $\mathcal{E}^1_{C1}(s)$ contribution involves the integration of the $L^1_{m1}(iv)$ function, determined by Eqs. (2.9), (2.12). In consequence,

$$\begin{aligned} \mathcal{E}^1_{C1}(s) &= -\frac{\hbar}{2} \frac{s a^{s-1}}{2\pi^2} \,\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right) \sin\left(\pi \frac{s}{2}\right) \\ \times \sum_{m=-\infty}^{\infty} \int_0^\infty dv\, v^{2-s} &\left[I'_m(v) K'_m(v) - \left(1 + \frac{m^2}{v^2}\right) I_m(v) K_m(v) + \frac{1}{v} (I_m(v) K_m(v))' \right]. \end{aligned} \tag{2.26}$$

Bearing in mind the useful property

$$\frac{d}{dv}[v^2 I'_m(v) K'_m(v) - (v^2 + m^2) I_m(v) K_m(v)] = -2v I_m(v) K_m(v), \tag{2.27}$$

we apply partial integration omitting a 'boundary term' which vanishes for a given s range ($2 < \Re s < 3$). Doing so, we find that Eq. (2.26) becomes

$$\begin{aligned} \mathcal{E}^1_{C1}(s) &= -\frac{\hbar}{2} \frac{s a^{s-1}}{2\pi^2} \,\mathrm{B}\left(\frac{1}{2}, 1-\frac{s}{2}\right) \sin\left(\pi \frac{s}{2}\right) \\ \times &\left[\int_0^\infty dv\, v^{1-s} \sum_{m=-\infty}^{\infty} (I_m(v) K_m(v))' + \frac{2}{1-s} \int_0^\infty dv\, v^{2-s} \sum_{m=-\infty}^{\infty} I_m(v) K_m(v) \right]. \end{aligned} \tag{2.28}$$

Because of this s restriction, the integrals in (2.28) cannot be directly taken at $s = -1$. However, if this difficulty is disregarded, we may formally set $s = -1$ and find

$$\mathcal{E}_{C1}^{1}(s=-1) \to -\frac{\hbar}{8\pi a^2}\int_0^\infty dv\, v^2 \sum_{m=-\infty}^{\infty} (I_m(v)K_m(v))' - \frac{\hbar}{8\pi a^2}\int_0^\infty dv\, v^3 \sum_{m=-\infty}^{\infty} I_m(v)K_m(v). \tag{2.29}$$

One could argue that the first part can be dismissed as a mere contact term (since it may be shown from Eq. (2.22) that it is local in v). The second part of (2.29) gives a contribution which exactly cancels the bulk contribution found in Ref. [4], where the same type of formal expressions was employed. [See formulas (72), (78) there. In that paper what was evaluated was the Casimir radial pressure P_C, which is related to $\mathcal{E}_C$ through Eq. (2.4).]

Perhaps a better argument is to again use the technique of multiplying each term in the m summation by $1 = -vW[I_m(v), K_m(v)]$, and turn the initial expression into a linear combination of integrals with summations of products of four Bessel functions [like Eq. (2.24)]. That linear combination would yield an identically null result — one that is zero for any s value — by virtue of the symmetries seen in Eqs. (2.24) under interchange of different Bessel function types (see also comment after Eqs. (80) in Ref. [4]). Hence, near $s = -1$,

$$\mathcal{E}_{C1}^{1}(s) = 0. \tag{2.30}$$

From another viewpoint, according to the reasonings in Ref. [13] (and references therein), which dealt with a similar problem for a dielectric ball, all linear terms in $(\varepsilon_2 - \varepsilon_1)$ have to be subtracted because they represent the self-energy of the electromagnetic field due to polarizable particles. Therefore, one simply must remove the linear part, no matter what precise form it has. This, of course, is the physical basis for removing the bulk energy contribution.

B. Second order in $(\varepsilon - 1)$

We start with the part called $\mathcal{E}_{C2}^{20A}(s)$, because its evaluation is most similar to that of the $\mathcal{E}_{C1}^{0}(s)$, $\mathcal{E}_{C1}^{1}(s)$ contributions. Using the fourth line in Eq. (2.19), the fifth in Eq. (2.9), expressions (2.12) with $y = iv$, introducing, again, a unit factor in terms of the Wronskian, and following the same argument that led to (2.30), one arrives at

$$\mathcal{E}_{C2}^{20A}(s) = 0, \tag{2.31}$$

near $s = -1$. Next, selecting the lines in Eqs. (2.19), which define $\mathcal{E}_{C2}^{00}(s)$, $\mathcal{E}_{C2}^{10}(s)$, $\mathcal{E}_{C2}^{20B}(s)$, $\mathcal{E}_{C2}^{11}(s)$, the parts of Eq. (2.9) which determine $L_{m2}^{00}(y)$, $L_{m2}^{10}(y)$, $L_{m2}^{20B}(y)$, $L_{m2}^{11}(y)$, the form of $\Delta_m^{(1,0)}(y)$ given by (2.12) (its square for the case of $L_{m2}^{20B}(y)$), and taking imaginary arguments $y = iv$, one finds

$$\mathcal{E}_{C2}^{00}(s) \;=\; \frac{\hbar}{2}\,\frac{sa^{s-1}}{4\pi^2}\,\mathrm{B}\left(\frac{1}{2}, -\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right)\int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty} {I'_m}^2(v)K_m^2(v), \tag{2.32a}$$

$$\mathcal{E}^{10}_{C2}(s) = \frac{\hbar}{2}\frac{sa^{s-1}}{4\pi^2}\,\mathrm{B}\left(\frac{1}{2},1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right) \times \int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty}\Big[2I_m(v)I'_m(v)K_m(v)K'_m(v)+v\,{I'_m}^2(v)K_m(v)K'_m(v) - \left(v+\frac{m^2}{v}\right)I_m^2(v)K_m(v)K'_m(v)\Big], \tag{2.32b}$$

$$\mathcal{E}^{20B}_{C2}(s) = \frac{\hbar}{2}\frac{sa^{s-1}}{8\pi^2}\,\mathrm{B}\left(\frac{1}{2},2-\frac{s}{2}\right)\sin\left(-\pi\frac{s}{2}\right) \times \int_0^\infty dv\, v^{2-s}\sum_{m=-\infty}^{\infty}\Big[{I'_m}^2(v)K_m^2(v)+I_m^2(v){K'_m}^2(v) + 2(1-v^2-m^2)I_m(v)I'_m(v)K_m(v)K'_m(v) + v^2{I'_m}^2(v){K'_m}^2(v)+\left(v^2+2m^2+\frac{m^4}{v^2}\right)I_m^2(v)K_m^2(v) + 2v\,{I'_m}^2(v)K_m(v)K'_m(v)+2v\,I_m(v)I'_m(v){K'_m}^2(v) - 2\left(v+\frac{m^2}{v}\right)I_m(v)I'_m(v)K_m^2(v)-2\left(v+\frac{m^2}{v}\right)I_m^2(v)K_m(v)K'_m(v)\Big], \tag{2.32c}$$

$$\mathcal{E}^{11}_{C2}(s) = \frac{\hbar}{2}\frac{sa^{s-1}}{2\pi^2}\,\mathrm{B}\left(\frac{3}{2},1-\frac{s}{2}\right)\sin\left(\pi\frac{s}{2}\right)\int_0^\infty dv\, v^{-s}\sum_{m=-\infty}^{\infty}m^2I_m^2(v)K_m^2(v). \tag{2.32d}$$

The v integrals are again evaluated in the manner illustrated in Eq. (2.24). All of the resulting formulas exhibit zeros of order one at $s=-1,-3,-5,\ldots$. Since all the beta and sine functions in Eq. (2.32) are finite at $s=-1$, each of these quantities will just yield $\mathcal{O}(s+1)$. In fact, after calculating the coefficients,

$$\mathcal{E}^{00}_{C2}(s) = \hbar\frac{1}{192\,\pi\,a^2}(s+1)+\mathcal{O}((s+1)^2), \tag{2.33a}$$

$$\mathcal{E}^{10}_{C2}(s) = -\hbar\frac{1}{288\,\pi\,a^2}(s+1)+\mathcal{O}((s+1)^2), \tag{2.33b}$$

$$\mathcal{E}^{20B}_{C2}(s) = \hbar\frac{7}{3840\,\pi\,a^2}(s+1)+\mathcal{O}((s+1)^2), \tag{2.33c}$$

$$\mathcal{E}^{11}_{C2}(s) = \hbar\frac{1}{2304\,\pi\,a^2}(s+1)+\mathcal{O}((s+1)^2). \tag{2.33d}$$

From Eqs. (2.31) and (2.33), we see that $\lim_{s\to-1}\mathcal{E}_{C2}(s)=0$, i.e., the $(\varepsilon-1)^2$ contribution to the Casimir energy per unit length in the dilute-dielectric approximation is zero.

III. CONCLUDING REMARKS

Applying an analytic regularization which changes the eigenmode power, we have calculated — to the order of $(\varepsilon-1)^2$ — what in Ref. [3] is called a pure Casimir term, i.e., the convergent component of the energy depending only on $\hbar, c$, the electrostatic polarizability of the material and the dimensions of the body. Using the words of Ref. [14], a forest of gamma functions has grown out of an analytic continuation. Although the applied technique might be regarded as somewhat physically opaque, its relations to more transparent regularizations have already been studied (see e.g. Refs. [15]). In Ref. [4] the only relevant contribution of the bulk part took place at the order of $(\varepsilon - 1)^1$. Since in the present regularization the corresponding term vanishes by itself, to omit or include a separate bulk part leaves the outcome unchanged. This remark is true as far as the analytic regularization method proposed in Sec. 6.1 of Ref. [4] is concerned, but not so for the (regulated) numerical method presented in Secs. 6.2 and 6.3 there, where a detailed numerical cancellation occurs between the bulk and cylinder parts. As a result, the pure Casimir term is seen to vanish to order $(\varepsilon - 1)^2$. Comparing this calculation to the one offered in Ref. [4], a decrease in difficulty can be appreciated in the derivation of the expression to be evaluated. However, we should stress that a substantial part of this merit is not in the nature of the method itself, but in the use of a previously known equation for the classical modes of this problem. Another key ingredient has been the exploitation of a Bessel function addition theorem [4, 12, 13].

Ref. [7] exhibits the presence of a divergence at the order of $(\varepsilon-1)^3$. Among all the possible divergences which show up by point-splitting or ultraviolet cutoffs, the ones which survive in analytic regularization schemes introduce an unavoidable ambiguity parametrized by an arbitrary length or mass scale. This issue is viewed with more or less concern, depending on the authors (see comments in Ref. [14]). The standpoint of Ref. [7] was to admit that the problem is simply ill-defined because its posing constitutes an idealization (the permittivity ε as a function of the radial coordinate is treated as a step function).

In any case, the reason why the Casimir energy term of order $(\varepsilon - 1)^2$ is zero for the cylinder, while it is finite for other geometries, remains rather mysterious. It is clear that we still have some way to go to understand quantum vacuum energies.

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