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Analytic perturbation theory in QCD and Schwinger's connection between the β-function and the spectral density

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Abstract

We argue that a technique called analytic perturbation theory leads to a well-defined method for analytically continuing the running coupling constant from the spacelike to the timelike region, which allows us to give a self-consistent definition of the running coupling constant for timelike momentum. The corresponding β-function is proportional to the spectral density, which confirms a hypothesis due to Schwinger.

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I. INTRODUCTION

An outstanding problem in QCD is the extrapolation of limited perturbation-theory information so as to make contact with experiment. An important example is given by the running coupling constant, in which low-order calculations are summed by the renormalization group. It has been known since the early 1950's that this is really not self-consistent, because of the appearance of an unphysical spacelike singularity, the "ghost-pole." However, as it has been argued in Refs. [1,2], a possible way to resolve the ghost-pole problem for the QCD running coupling constant can be found by imposing Källén–Lehmann analyticity. This method, which was elaborated early in the development of QED [3,4] leads to the definition of the analytic running coupling constant in the complex q^2-plane with a cut along the negative part of the real axis. (In this paper we use a metric with signature $(-1,1,1,1)$, so that $q^2>0$ corresponds to a spacelike momentum transfer.) According to Refs. [1,2] the connection between the analytic coupling $\bar{a}(q^2)$ and the spectral density $\rho(\sigma)$ is given by the following spectral representation (the overbar signifies the analytically improved quantity)

$$\bar{a}(q^2) \equiv \frac{\bar{\alpha}_s(q^2)}{4\pi} = \frac{1}{\pi}\int_0^\infty d\sigma \frac{\rho(\sigma)}{\sigma + q^2 - \mathrm{i}\epsilon}. \tag{1}$$

(The questions about the validity of the spectral representation, which is rather obvious in QED, are resolved in [5] for the general case.) For instance, in the one-loop approximation to the spectral function,

$$\rho^{(1)}(\sigma) = \mathrm{Im}\,\bar{a}^{(1)}(-\sigma - \mathrm{i}\epsilon) = \mathrm{Im}\,\frac{a}{1 + a\beta_0 \ln(-\sigma/\mu^2 - \mathrm{i}\epsilon)} = \frac{a^2\beta_0\pi}{[1 + a\beta_0\ln(\sigma/\mu^2)]^2 + a^2\beta_0^2\pi^2}, \tag{2}$$

the corresponding analytic running coupling constant has the form

$$\bar{a}^{(1)}(q^2) = \frac{1}{\beta_0}\left[\frac{1}{\ln q^2/\Lambda^2} + \frac{\Lambda^2}{\Lambda^2 - q^2}\right], \tag{3}$$

where $\beta_0 = 11 - 2/3\,N_f$ is the first coefficient of the β-function with N_f active flavors, and Λ is the QCD scale. The analytically-improved coupling constant (3) has no ghost pole at $q^2 = \Lambda^2$, and its correct analytic properties are provided by the nonperturbative contribution, the second term in (3), which has appeared automatically through use of the spectral representation (1). The analytic running coupling constant obtained in such a way turns out to be remarkable stable in the infrared region with respect to higher loop corrections and has the universal infrared limit at $q^2 = 0$: $\bar{a}(0) = 1/\beta_0$, which does not depend on the value of Λ, being a universal constant.

II. SPACELIKE AND TIMELIKE RUNNING COUPLINGS

The method described above defines the running coupling constant in the Euclidean (spacelike) range of momentum, $q^2 > 0$, where $\overline{a}(q^2)$ is real. In this paper we wish to parametrize processes with timelike momentum transfer, for example, the process of e^+e^- annihilation into hadrons. To do so, we must make use of some nontrivial analytic continuation procedure from the spacelike to the timelike region. To this end one usually applies the dispersion relation for the Adler D-function, defined in terms of the correlation function for the quark vector current, $\Pi(q^2)$, as follows

$$D(q^2) = -q^2 \frac{d\Pi(-q^2)}{dq^2}. \tag{4}$$

This "vacuum polarization" satisfies an unsubtracted dispersion relation,

$$\Pi(-q^2) = \text{const.} + \int_0^\infty \frac{ds}{s+q^2} R(s), \tag{5}$$

with the e^+e^- annihilation ratio is given by

$$R(s) = \frac{1}{2\pi \mathrm{i}} \left[\Pi(s+\mathrm{i}\epsilon) - \Pi(s-\mathrm{i}\epsilon)\right]. \tag{6}$$

Consequently, the dispersion relation for the Adler function is

$$D(q^2) = q^2 \int_0^\infty \frac{ds}{(s+q^2)^2} R(s), \tag{7}$$

The D-function is an analytic function in the complex q^2 plane with a cut along the negative real axis. Taking into account these analytic properties we can write down the inverse relation for $R(s)$,

$$R(s) = -\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\epsilon}^{s+\mathrm{i}\epsilon} \frac{dz}{z} D(-z), \tag{8}$$

where the contour goes from the point $z = s - \mathrm{i}\epsilon$ to the point $z = s + \mathrm{i}\epsilon$ and lies in the region of analyticity of the function $D(z)$.

Let us define effective coupling constants $\bar{a}^{\text{eff}}(q^2)$ in the spacelike region and $\bar{a}_s^{\text{eff}}(s)$ in the timelike region based on the following expressions for $D(q^2)$ and $R(s)$

$$D(q^2) \propto \left[1 + d_1 \bar{a}^{\text{eff}}(q^2)\right], \tag{9}$$

$$R(s) \propto \left[1 + r_1 \bar{a}_s^{\text{eff}}(s)\right], \tag{10}$$

where d_1 and r_1 are the first coefficients of perturbative expansions. (The superscript eff refers to the summation of all the remaining terms in the perturbative expansion of these quantities.) In fact, $d_1 = r_1$. The subscript s in (10) means "s-channel" (the timelike region). From (7) and (8), one finds the connections between these effective coupling constants in the spacelike and timelike regions:

$$\bar{a}^{\text{eff}}(q^2) = q^2 \int_0^\infty \frac{ds}{(s+q^2)^2} \bar{a}_s^{\text{eff}}(s) \tag{11}$$

and

$$\bar{a}_s^{\text{eff}}(s) = -\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\epsilon}^{s+\mathrm{i}\epsilon} \frac{dz}{z} \bar{a}^{\text{eff}}(-z). \tag{12}$$

These equations serve to define the effective coupling $\bar{a}_s^{\text{eff}}(s)$ which parametrizes the $R(s)$ ratio and plays the role of the running coupling in the timelike region. One usually applies the standard perturbative approximation for $a^{\text{eff}}(z)$ to derive the effective coupling in the s-channel from (12). This way leads to the so-called π^2-terms which play an important role in the phenomenological analysis of various processes [6]. However, the perturbative approximation of $a^{\text{eff}}(z)$ breaks the analytic properties mentioned above. For example, in the one-loop approximation the function $a^{\text{eff}}(z)$ has the form $1/[\beta_0 \ln(z/\Lambda^2)]$ with a ghost pole at $z = \Lambda^2$, which contradicts the assumption that $a^{\text{eff}}(z)$ is an analytic function in the cut z-plane. A consequence of this problem is the fact that if $a_s^{\rm eff}(s)$, obtained in such a way, is substituted into (11), the original one-loop formula in the spacelike region is not reproduced.

III. ANALYTIC PERTURBATION THEORY

This difficulty can be avoided in the framework of what we call analytic perturbation theory, in which the running coupling constant is forced to have the correct analytic properties.[1] We define $\bar{a}^{\rm eff}(q^2)$ in terms of the spectral density according to (1), that is, the effective coupling in the spacelike region is given by

$$\bar{a}^{\rm eff}(q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + q^2}\rho(\sigma), \tag{13}$$

As a result, from (12) and (13), the effective coupling in the timelike region is given by the following elegant expression:

$$\bar{a}_s^{\rm eff}(s) \;=\; \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\;\rho(\sigma)\,. \tag{14}$$

It is clear that both coupling constants $\bar{a}^{\rm eff}(q^2)$ and $\bar{a}_s^{\rm eff}(s)$ have the same universal limit at $q^2 = +0$ and $s = +0$ and a similar tails as $q^2 \to \infty$ and $s \to \infty$. However, in the intermediate region the effect of analytic continuation becomes important. As we will see in perturbation theory, below, the distinction between the different effective coupling constants is several percent, which may be important for extracting the QCD coupling constant from various experimental data.

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[1]The correct analytic properties of the D-function can also be maintained in the framework of the so-called variational perturbation theory [7] which is based on a new small expansion parameter [8].

IV. ONE-LOOP RESULTS

Let us consider this problem at the one-loop level. The perturbative contribution of the leading logarithms to the effective coupling in the spacelike region can be written as follows:

$$\bar{a}^{(1)}(q^2) = a \sum_{n=0}^{\infty} \left(-a\beta_0 \ln \frac{q^2}{\mu^2} \right)^n . \tag{15}$$

In any finite order this function has the correct analytic properties. The ghost pole appears due to the naive sum of the infinite geometrical series in (15). However, we should consider the series in (15) as an asymptotic series and try to find its sum in such a way to maintain the required analytic properties, taking into account the fact that the sum of an asymptotic series is not unique. To this end, let us consider the correlation function for which, from (4) and (9), the contribution of the leading logarithms has the following form:

$$\Pi(q^2) \propto -\ln \frac{q^2}{\mu^2} + \mathrm{const.} + \frac{d_1}{\beta_0} \sum_{n=0}^{\infty} \frac{1}{n+1} \left(-a\beta_0 \ln \frac{q^2}{\mu^2} \right)^{n+1} . \tag{16}$$

Carrying out the sum and taking the imaginary part, we immediately find from (6)

$$R(s) \propto 1 + \frac{r_1}{\pi\beta_0} \arccos \frac{1 + a\beta_0 L}{\sqrt{(1 + a\beta_0 L)^2 + (a\pi\beta_0)^2}}, \tag{17}$$

where $L = \ln s/\mu^2$. By introducing the QCD parameter $\Lambda^2 = \mu^2 \exp(-1/a\beta_0)$, we obtain for the running coupling constant in the s-channel

$$\bar{a}_s^{(1)}(s) = \frac{1}{\pi\beta_0} \arccos \frac{\ln(s/\Lambda^2)}{\sqrt{\ln^2(s/\Lambda^2) + \pi^2}}. \tag{18}$$

The same result for the running coupling constant in the timelike region can be obtained by substituting the one-loop spectral density (2) into (14). Moreover, the substitution of (18) into (11) reproduces the one-loop analytic running coupling (3) which parametrizes the D-function and has the asymptotic expansion (15). Thus, the summation of the leading logarithms for the physical quantity $R(s)$ leads to a D-function with the correct analytic properties.

Let us compare the two one-loop couplings $\bar{a}_s^{(1)}(s)$, given by (18), and $\bar{a}^{(1)}(q^2)$, given by (3). As noted above, they have a universal value at 0,

$$\bar{a}_s^{(1)}(0) = \bar{a}^{(1)}(0) = \frac{1}{\beta_0}, \tag{19}$$

and in fact are exactly the same at $s = \Lambda^2$, $q^2 = \Lambda^2$. Asymptotically, for large spacelike and timelike momenta, respectively,

$$\bar{a}^{(1)}(q^2) \sim \frac{1}{\beta_0}\frac{1}{\ln q^2/\Lambda^2}, \quad q^2 \gg \Lambda^2, \tag{20}$$

$$\bar{a}_s^{(1)}(s) \sim \frac{1}{\beta_0}\frac{1}{\ln s/\Lambda^2}\left(1 - \frac{\pi^2}{3}\frac{1}{\ln^2 s/\Lambda^2}\right), \quad s \gg \Lambda^2, \tag{21}$$

exhibiting the fact that the t- and s-channel couplings differ in three-loop order. In general, these two couplings, in their respective regimes, agree numerically quite closely, as shown in Fig. 1, with the relative difference being no more than 9%, as shown in Fig. 2. Similar features hold in two-loop order, the discrepancy between the couplings in the intermediate region dropping to about 5%.

V. SCHWINGER'S IDENTIFICATION

More than two decades ago, Schwinger proposed [9] that the Gell-Mann–Low function, or the β-function, in QED could be represented by a spectral function for the photon propagator, which has direct physical meaning. The precise connection, of course, depends on the definition of the running coupling constant [9,10]. Remarkably, we find that this idea is realized in our proposal for the timelike coupling constant in QCD. For the β-function which corresponds to the coupling defined in the Euclidean region this statement is true through the two-loop approximation, but breaks down if one takes into account three-loop contributions. (The analogous thing happens in QED for the conventional charge definitions [10].) However, Schwinger's identification is certainly correct if we construct the β-function for the coupling (14) defined in the timelike region: indeed,

$$\beta_s = s\frac{d\bar{a}_s^{\rm eff}}{ds} = -\frac{\rho(s)}{\pi}. \tag{22}$$

As noted above, in perturbation theory, the difference between the couplings in the spacelike and timelike regions is given by three-loop diagrams and, therefore, $\beta = q^2 d\bar{a}^{\rm eff}/dq^2 =$

$-\rho(s)/\pi + O(3\text{-loop})$.

VI. CONCLUSIONS

We have considered the procedure of constructing the QCD running coupling constant by using analytic perturbation theory. The fundamental quantity here is the spectral density $\rho(\sigma)$, in terms of which the running coupling in the Euclidean region is expressed through the spectral representation (13), while the running coupling in the timelike region is expressed by (14). Both these couplings have the same universal infrared limit

$$\bar{a}^{\mathrm{eff}}(0) = \bar{a}_s^{\mathrm{eff}}(0) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma}\rho(\sigma) = \frac{1}{\beta_0}, \tag{23}$$

which turns out to be remarkably stable with respect to higher loop corrections [2]. Further, both coupling constants have the same leading asymptotic behavior. Thus, in comparison with standard perturbation theory, the strong requirement of analyticity modifies the theory in the infrared and intermediate domains significantly, which is particularly relevant for a physical description of the timelike regime. Finally, we have shown that Schwinger's proposed connection between the renormalization group β-function and the spectral density is valid for the coupling defined in the timelike region.

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FIGURES

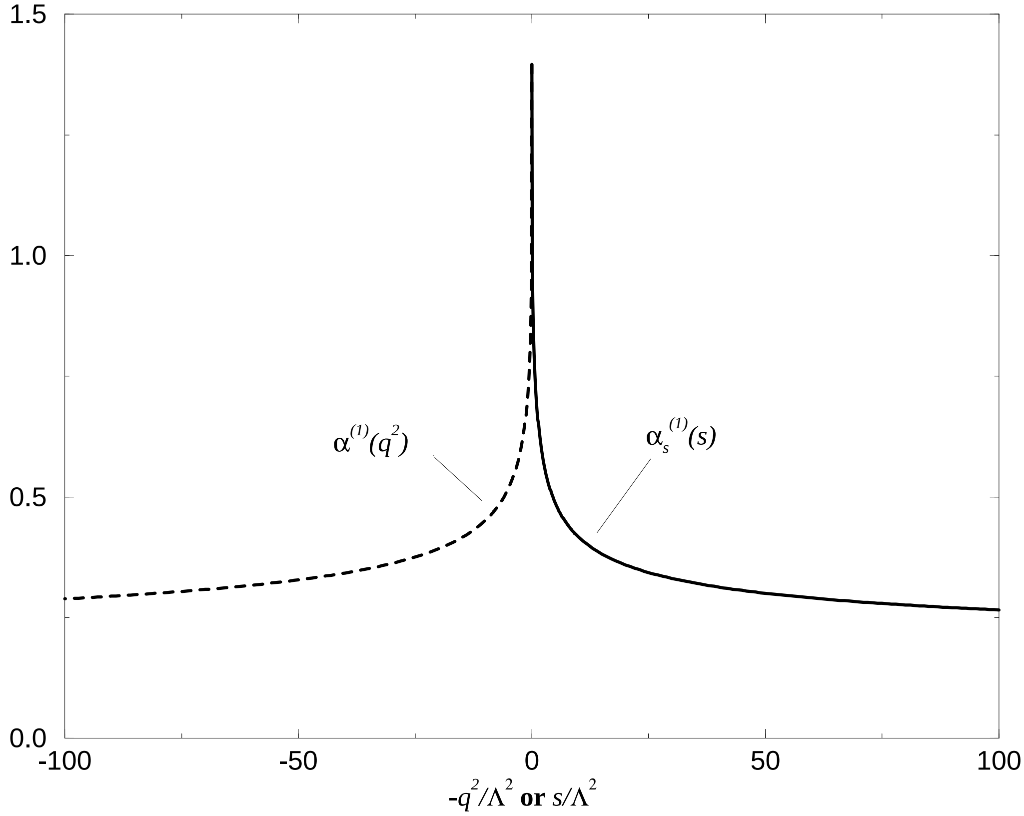

FIG. 1. Plot of the spacelike and timelike definitions of the one-loop running coupling constants, $\bar{\alpha}^{(1)}(q^2)$ and $\bar{\alpha}_s^{(1)}(s)$. The abscissa is respectively $-q^2/\Lambda^2$ and s/Λ^2 for the two functions. Here we have displayed $\alpha = 4\pi a$ and have used $N_f = 3$ in β_0.

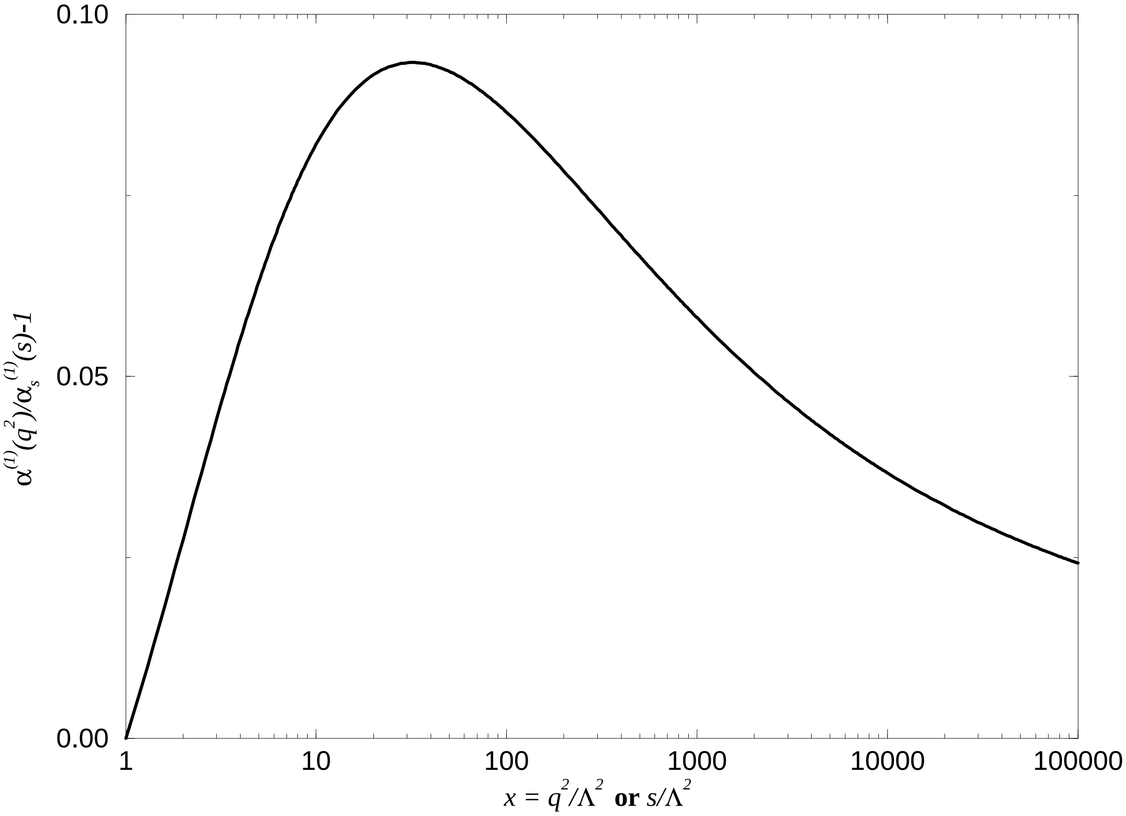

FIG. 2. Plot of the relative difference of the two coupling constants shown in Fig. 1, $\bar{\alpha}^{(1)}(x)/\bar{\alpha}_s^{(1)}(x) - 1$. Here the argument is $x = q^2/\Lambda^2$ for $\bar{\alpha}$ and $x = s/\Lambda^2$ for $\bar{\alpha}_s$.