Casimir Energy for a Dielectric Cylinder

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Abstract

In this paper we calculate the Casimir energy for a dielectric-diamagnetic cylinder with the speed of light differing on the inside and outside. Although the result is in general divergent, special cases are meaningful. The well-known results for a uniform speed of light are reproduced. The self-stress on a purely dielectric cylinder is shown to vanish through second order in the deviation of the permittivity from its vacuum value, in agreement with the result calculated from the sum of van der Waals forces.

Key words: Casimir energy, van der Waals forces, electromagnetic fluctuations
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1 Introduction

Interest in quantum vacuum phenomena, subsumed under the general rubric of the Casimir effect, is increasing at a rapid pace. Status of work in the field is summarized in recent review articles and monographs [1–3]. The theoretical developments have been largely driven by experimental and technological developments, where it is becoming evident that Casimir forces may present fundamental limits and opportunities in nanomechanical devices [4] and nanoelectronics [5]. Thus it is imperative to understand fundamental aspects of the theory, such as the sign of the effect, which, at present, cannot be predicted without a detailed calculation. This paper represents an incremental increase in our list of solved examples of Casimir energies with nontrivial boundaries.

The Casimir energy for an uniform dielectric sphere was first calculated in 1979 by Milton [6] and later generalized to the case when both the electric

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permittivity and the magnetic permeability are present [7]. It was later observed [8] that, in the special dilute dielectric case where $\mu = 1$ and $|\varepsilon - 1| \ll 1$, the series expansion in $\varepsilon - 1$ has a leading term that perfectly matches the "renormalized" energy obtained by summing the van der Waals interactions [9].

The same analysis for the case of the circular cylinder has been attempted on several occasions; however, the difficulty of the geometry and the fact that the TE and TM modes do not decouple makes the problem considerably more complex. Only in the case when the speeds of light inside and outside the cylinder coincide is the result completely unambiguous [10–13]. This includes the classic case of a perfectly conducting cylindrical shell [14] where the energy per unit length is found to be

$$\mathcal{E} = -\frac{0.01356}{a^2}, \tag{1}$$

where a is the radius of the cylinder. The minus sign indicates that the Casimir self-stress is attractive, unlike the Boyer repulsion for a sphere [15].

We present here the calculation for an infinite circular dielectric-diamagnetic cylinder with electric permittivity ϵ and magnetic permeability μ inside the cylinder which is surrounded by vacuum with permittivity 1 and permeability 1 so $\varepsilon\mu \neq 1$. It is shown that the corresponding Casimir energy is divergent, as expected, but, for $\mu = 1$, the coefficient of $(\varepsilon - 1)^2$ in the expansion for the dilute approximation is finite and yields the surprising zero result found by summing the van der Waals energies between the molecules that make up the material [10,16].

The paper is laid out as follows. In Sec. 2 we calculate the dyadic Green's functions that will allow us to compute the one-loop vacuum expectation values of the quadratic field products. This enables us to calculate the vacuum expectation value of the stress tensor, the discontinuity of which across the surface gives the stress on the cylinder, computed in Sec. 3. The bulk Casimir stress, which would be present if either medium filled all space, is computed in Sec. 4 and must be subtracted from the stress found in Sec. 3. Finally, the case of a dilute dielectric cylinder is considered in Sec. 5, and by detailed analytic and numerical calculations in Sec. 6 it is shown that the Casimir stress vanishes both in order $\varepsilon - 1$ and $(\varepsilon - 1)^2$. The significance of all this is briefly discussed in the Conclusions.

2 Green's Function Derivation of the Casimir Energy

In order to write down the Green's dyadic equations, we introduce a polarization source $\mathbf{P}$ whose linear relation with the electric field defines the Green's dyadic as

$$\mathbf{E}(x) = \int (\,\mathrm{d}x')\boldsymbol{\Gamma}(x,x')\cdot\mathbf{P}(x'). \tag{2}$$

Since the response is translationally invariant in time, we introduce the Fourier transform at a given frequency ω,

$$\boldsymbol{\Gamma}(x,x') = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\exp\left[-\mathrm{i}\omega(t-t')\right]\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}',\omega). \tag{3}$$

We can then write the dyadic Maxwell's equations in a medium characterized by a dielectric constant ε and a permeability μ, both of which may be functions of frequency (see Ref. [6,17,18]):

$$\boldsymbol{\nabla}\times\boldsymbol{\Gamma}' - \mathrm{i}\omega\mu\boldsymbol{\Phi} = \frac{1}{\varepsilon}\boldsymbol{\nabla}\times\mathbf{1}, \tag{4a}$$

$$-\boldsymbol{\nabla}\times\boldsymbol{\Phi} - \mathrm{i}\omega\varepsilon\boldsymbol{\Gamma}' = \mathbf{0}, \tag{4b}$$

where

$$\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}',\omega) = \boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}',\omega) + \frac{\mathbf{1}}{\varepsilon(\omega)}, \tag{5}$$

and where the unit dyadic $\mathbf{1}$ includes a three-dimensional δ function,

$$\mathbf{1} = \mathbf{1}\delta(\mathbf{r}-\mathbf{r}'). \tag{6}$$

The two dyadics are solenoidal,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} = \mathbf{0}, \tag{7a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}' = \mathbf{0}. \tag{7b}$$

The corresponding second order equations are

$$(\nabla^2 + \omega^2\varepsilon\mu)\boldsymbol{\Gamma}' = -\frac{\mathbf{1}}{\varepsilon}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{1}), \tag{8a}$$

$$(\nabla^2 + \omega^2\varepsilon\mu)\boldsymbol{\Phi} = \mathrm{i}\omega\boldsymbol{\nabla}\times\mathbf{1}. \tag{8b}$$

Quantum mechanically, these Green's dyadics give the one-loop vacuum expectation values of the product of fields at a given frequency ω,

$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \frac{\hbar}{\mathrm{i}}\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}'), \tag{9a}$$

$$\langle \mathbf{H}(\mathbf{r})\mathbf{H}(\mathbf{r}')\rangle = -\frac{\hbar}{\mathrm{i}}\frac{1}{\omega^2\mu^2}\overrightarrow{\boldsymbol{\nabla}} \times \boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}') \times \overleftarrow{\boldsymbol{\nabla}'}. \tag{9b}$$

Thus, from the knowledge of the classical Green's dyadics, we can calculate the vacuum energy or stress.

We now introduce the appropriate partial wave decomposition for a cylinder, in terms of cylindrical coordinates (r,θ,z), a slight modification of that given for a conducting cylindrical shell [14] [1]:

$$\begin{aligned}\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}';\omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&(\boldsymbol{\nabla}\times\hat{\mathbf{z}})f_m(r;k,\omega)\chi_{mk}(\theta,z)\\ &+\frac{\mathrm{i}}{\omega\varepsilon}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\hat{\mathbf{z}})g_m(r;k,\omega)\chi_{mk}(\theta,z)\bigg\},\end{aligned} \tag{10a}$$

$$\begin{aligned}\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}';\omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&(\boldsymbol{\nabla}\times\hat{\mathbf{z}})\tilde{g}_m(r;k,\omega)\chi_{mk}(\theta,z)\\ &-\frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\hat{\mathbf{z}})\tilde{f}_m(r;k,\omega)\chi_{mk}(\theta,z)\bigg\},\end{aligned} \tag{10b}$$

where the cylindrical harmonics are

$$\chi(\theta,z) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{e}^{\mathrm{i}kz}, \tag{11}$$

and the dependence of f_m etc. on $\mathbf{r}'$ is implicit. Notice that these are vectors in the second tensor index. Because of the presence of these harmonics we have

$$\boldsymbol{\nabla}\times\hat{\mathbf{z}} \to \hat{\mathbf{r}}\frac{\mathrm{i}m}{r} - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial r} \equiv \boldsymbol{\mathcal{M}}, \tag{12a}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\hat{\mathbf{z}}) \to \hat{\mathbf{r}}\mathrm{i}k\frac{\partial}{\partial r} - \hat{\boldsymbol{\theta}}\frac{mk}{r} - \hat{\mathbf{z}}d_m \equiv \boldsymbol{\mathcal{N}}, \tag{12b}$$

in terms of the cylinder operator

$$d_m = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r^2}. \tag{13}$$

[1] It might be thought that we could immediately use the general waveguide decomposition of modes into those of TE and TM type, for example as given in Ref. [19]. However, this is here impossible because the TE and TM modes do not separate. See Ref. [20].

Now if we use the Maxwell equation (4b) we conclude [2]

$$\tilde{g}_m = g_m, \tag{14a}$$
$$(d_m - k^2)\tilde{f}_m = -\omega^2 \mu f_m. \tag{14b}$$

From the other Maxwell equation (4a), we deduce (we now make the second, previously suppressed, position arguments explicit; the prime on the differential operator signifies action on the second primed argument)

$$d_m \mathcal{D}_m \tilde{\mathbf{f}}_m(r; r', \theta', z') = \frac{\omega^2\mu}{\varepsilon} \boldsymbol{\mathcal{M}}'^{*} \frac{1}{r}\delta(r - r')\chi^*_{mk}(\theta', z'), \tag{15a}$$
$$d_m \mathcal{D}_m \mathbf{g}_m(r; r', \theta', z') = -\mathrm{i}\omega \boldsymbol{\mathcal{N}}'^{*} \frac{1}{r}\delta(r - r')\chi^*_{mk}(\theta', z'), \tag{15b}$$

where the Bessel operator appears,

$$\mathcal{D}_m = d_m + \lambda^2, \qquad \lambda^2 = \omega^2 \varepsilon\mu - k^2. \tag{16}$$

Now we separate variables in the second argument, [3]

$$\tilde{\mathbf{f}}_m(r, \mathbf{r}') = \left[\boldsymbol{\mathcal{M}}'^{*} F_m(r, r'; k, \omega) + \frac{1}{\omega}\boldsymbol{\mathcal{N}}'^{*} \tilde{F}_m(r, r'; k, \omega)\right] \chi^*_{mk}(\theta', z'), \tag{17a}$$
$$\mathbf{g}_m(r, \mathbf{r}') = \left[-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{N}}'^{*} G_m(r, r'; k, \omega) - \mathrm{i}\boldsymbol{\mathcal{M}}'^{*}\tilde{G}_m(r, r'; k, \omega)\right] \chi^*_{mk}(\theta', z'), \tag{17b}$$

where we have introduced the two scalar Green's functions F_m, G_m, which satisfy

$$d_m \mathcal{D}_m F_m(r, r') = \frac{\omega^2\mu}{\varepsilon}\frac{1}{r}\delta(r - r'), \tag{18a}$$
$$d_m \mathcal{D}_m G_m(r, r') = \omega^2 \frac{1}{r}\delta(r - r'), \tag{18b}$$

while $\tilde{F}_m$ and $\tilde{G}_m$ are annihilated by the operator $d_m\mathcal{D}_m$,

$$d_m \mathcal{D}_m \tilde{F}(r, r') = d_m \mathcal{D}_m \tilde{G}(r, r') = 0. \tag{19}$$

The Green's dyadics have now the form:

[2] The ambiguity in solving for these equations is absorbed in the definition of subsequent constants of integration.

[3] Note that here and in the following there are slight changes in notation, and numerous corrections, to the treatment sketched in Ref. [1].

$$
\begin{aligned}
\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}';\omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}\left(-\frac{d_m-k^2}{\omega^2\mu}\right)F_m(r,r')\\
&+\frac{1}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^{*}\left(-\frac{d_m-k^2}{\omega^2\mu}\right)\tilde{F}_m(r,r')+\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'\frac{1}{\omega^2\varepsilon}G_m(r,r')\\
&+\frac{1}{\omega\varepsilon}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^{*}\tilde{G}_m(r,r')\bigg\}\chi_{mk}(\theta,z)\chi'^{*}_{mk}(\theta',z'),
\end{aligned}
\tag{20a}
$$

$$
\begin{aligned}
\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}';\omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^{*}G_m(r,r')-\mathrm{i}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}\tilde{G}_m(r,r')\\
&-\frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^{*}F_m(r,r')-\frac{\mathrm{i}\varepsilon}{\omega^2\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^{*}\tilde{F}_m(r,r')\bigg\}\\
&\times\chi_{mk}(\theta,z)\chi'^{*}_{mk}(\theta',z').
\end{aligned}
\tag{20b}
$$

In the following, we will apply these equations to a dielectric-diamagnetic cylinder of radius a, where the interior of the cylinder is characterized by a permittivity ε and permeability μ, while the outside is vacuum, so $\varepsilon=\mu=1$ there. Let us consider the case that the source point is outside, $r'>a$. If the field point is also outside, $r,r'>a$, the scalar Green's functions $F'_m, G'_m, \tilde{F}', \tilde{G}'$ that make up the above Green's dyadics (we designate with primes the outside scalar Green's functions or constants) obey the differential equations (18a), (18b), and (19). To solve these fourth-order differential equations we introduce auxiliary Green's functions $\mathcal{G}_m^{F'(G')}(r,r')$ and $\mathcal{G}_m^{\tilde{F}'(\tilde{G}')}(r,r')$, satisfying ($m\neq 0$)

$$
d_m\mathcal{G}_m^{F'(G')}(r,r') = \frac{1}{r}\delta(r-r'), \tag{21a}
$$

$$
d_m\mathcal{G}_m^{\tilde{F}'(\tilde{G}')}(r,r') = 0, \tag{21b}
$$

which therefore have the general form

$$
\mathcal{G}_m^{F'(G')}(r,r') = a_m'^{F(G)}(r')\frac{1}{r^{|m|}}-\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|}, \tag{22a}
$$

$$
\mathcal{G}_m^{\tilde{F}'(\tilde{G}')}(r,r') = a_m'^{\tilde{F}(\tilde{G})}(r')\frac{1}{r^{|m|}}, \tag{22b}
$$

where $r_<(r_>)$ is the lesser (greater) of r, r' and we discarded a possible $r^{|m|}$ term because we seek a solution which vanishes at infinity. Thus $F'_m, G'_m, \tilde{F}'$ and $\tilde{G}'$ satisfy the second-order differential equations

$$
\mathcal{D}_m F'_m = \omega^2\mathcal{G}_m^{F'}, \qquad \mathcal{D}_m G'_m = \omega^2\mathcal{G}_m^{G'}, \tag{23a}
$$

$$
\mathcal{D}_m \tilde{F}'_m = \omega^2\mathcal{G}^{\tilde{F}'}, \qquad \mathcal{D}_m \tilde{G}'_m = \omega^2\mathcal{G}^{\tilde{G}'}, \tag{23b}
$$

where we recall that for the outside we have $\varepsilon = \mu = 1$. Now, from (23a) and the first identity in (16) we learn that ($\lambda'^2 = \omega^2 - k^2$)

$$F'_m - \frac{\omega^2}{\lambda'^2}\mathcal{G}^{F'}_m = A'^F_m(r')H_m(\lambda' r) - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' r_<)H_m(\lambda' r_>), \tag{24}$$

while G'_m obeys a similar expression with the replacement $F \to G$. Similarly, from (23b)

$$\tilde{F}'_m - \frac{\omega^2}{\lambda'}\mathcal{G}^{\tilde{F}}_m = A'^{\tilde{F}}_m(r')H_m(\lambda' r), \tag{25}$$

and for $\tilde{G}'_m$ replace $F \to G$. Here, to have the appropriate outgoing-wave boundary condition at infinity, we have used $H_m(\lambda' r) = H^{(1)}_m(\lambda' r)$.

The dependence of the constants on the second variable r' can be deduced by noticing that, naturally, the Green's dyadics have to satisfy Maxwell's equations in their second variable. Thus, by imposing the Helmholtz equations in the second variable together with the boundary conditions at $r' = \infty$, it is easy to see that

$$a'^F_m(r') = a'^F_m\frac{1}{r'^{|m|}} + b'^F_m H_m(\lambda' r'), \tag{26a}$$

$$A'^F_m(r') = A'^F_m\frac{1}{r'^{|m|}} + B'^F_m H_m(\lambda' r'), \tag{26b}$$

and with similar relations for $a'^G_m(r'), A'^G_m(r'), a'^{\tilde{G}}_m(r')$, and so on. Then, the outside Green's functions have the form

$$\begin{aligned} F'_m(r,r') = {} & \frac{\omega^2}{\lambda'^2}\left[\frac{a'^F_m}{r'^{|m|}} + b'^F_m H_m(\lambda' r')\right] r^{-|m|} - \frac{\omega^2}{\lambda'^2}\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} \\ & + \left[\frac{A'^F_m}{r'^{|m|}} + B'^F_m H_m(\lambda' r')\right] H_m(\lambda' r) - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' r_<)H_m(\lambda' r_>), \end{aligned} \tag{27}$$

while G'_m has the same form with the constants $a'^F_m, b'^F_m, A'^F_m, B'^F_m$ replaced by $a'^G_m, b'^G_m, A'^G_m, B'^G_m$, respectively. The homogeneous differential equations have solutions

$$\tilde{F}'_m(r,r') = \frac{\omega^2}{\lambda'^2}\left[\frac{a'^{\tilde{F}}_m}{r'^{|m|}} + b'^{\tilde{F}}_m H_m(\lambda' r')\right] r^{-|m|} + \left[\frac{A'^{\tilde{F}}_m}{r'^{|m|}} + B'^{\tilde{F}}_m H_m(\lambda' r')\right] H_m(\lambda' r), \tag{28}$$

while in $\tilde{G}'_m$ we replace $a'^{\tilde{F}} \to a'^{\tilde{G}}$, etc.

When the source point is outside and the field point is inside, all the Green's functions satisfy the homogeneous equations (19), and then, following the above scheme we have that

$$d_m \mathcal{G}_m^F = d_m \mathcal{G}_m^G = d_m \mathcal{G}_m^{\tilde{F}} = d_m \mathcal{G}_m^{\tilde{G}} = 0, \tag{29}$$

and

$$\mathcal{G}^F(r, r') = a_m^F(r') r^{|m|}, \tag{30}$$

since now r can be 0. Also $\mathcal{D}_m F_m = \omega^2 \mathcal{G}_m^F$ and

$$F_m - \frac{\omega^2}{\lambda^2} \mathcal{G}_m^F = A_m^F(r') J_m(\lambda r). \tag{31}$$

$G_m, \tilde{F}_m$ and $\tilde{G}_m$ have the same form, and the constants $a_m^F(r'), A_m^F(r')$, etc. follow the pattern in (26a) and (26b). Now, we may write for $r < a, r' > a$

$$F_m(r, r') = \frac{\omega^2}{\lambda^2} \left[\frac{a_m^F}{r'^{|m|}} + b_m^F H_m(\lambda' r') \right] r^{|m|} + \left[\frac{A_m^F}{r'^{|m|}} + B_m^F H_m(\lambda' r') \right] J_m(\lambda r), \tag{32}$$

and similarly for $G_m, \tilde{F}_m, \tilde{G}_m$, with the corresponding change of constants. In all of the above, the outside and inside forms of λ are given by

$$\lambda'^2 = \omega^2 - k^2, \qquad \lambda^2 = \omega^2 \mu \varepsilon - k^2. \tag{33}$$

The various constants are to be determined, as far as possible, by the boundary conditions at $r = a$. The boundary conditions at the surface of the dielectric cylinder are the continuity of tangential components of the electric field, of the normal component of the electric displacement, of the normal component of the magnetic induction, and of the tangential components of the magnetic field (we assume that there are no surface charges or currents):

$$\begin{array}{ll} \mathbf{E}_t \text{ is continuous,} & \varepsilon E_n \text{ is continuous,} \\ \mathbf{H}_t \text{ is continuous,} & \mu H_n \text{ is continuous.} \end{array} \tag{34}$$

These conditions are redundant, but we will impose all of them as a check of consistency. In terms of the Green's dyadics, the conditions read

$$\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\Gamma}' \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{35a}$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\Gamma}' \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{35b}$$

$$\hat{\boldsymbol{r}} \cdot \varepsilon \boldsymbol{\Gamma}' \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{35c}$$

$$\hat{\boldsymbol{r}} \cdot \mu \boldsymbol{\Phi} \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{35d}$$

$$\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\Phi} \Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{35e}$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\Phi} \Big|_{r=a-}^{r=a+} = \mathbf{0}. \tag{35f}$$

We can also impose the Helmholtz equations (8a) and (8b). From those we learn that the coefficients of terms with powers of r are related in the following way

$$\hat{a}'^F + \hat{a}'^G = 0, \tag{36a}$$

$$b'^G - (\mathrm{sgn} m)\frac{k}{\omega} b'^{\tilde{F}} = 0, \tag{36b}$$

$$b'^{\tilde{G}} - (\mathrm{sgn} m)\frac{k}{\omega} b'^F = 0, \tag{36c}$$

for the Green's dyadics outside the cylinder and equivalent expressions for the inside (no primes)

$$\frac{\varepsilon}{\mu}\hat{a}^F - \hat{a}^G = 0, \tag{37a}$$

$$b^G + (\mathrm{sgn} m)\frac{\varepsilon}{\mu}\frac{k}{\omega} b^{\tilde{F}} = 0, \tag{37b}$$

$$b^{\tilde{G}} + (\mathrm{sgn} m)\frac{\varepsilon}{\mu}\frac{k}{\omega} b^F = 0, \tag{37c}$$

where we have introduced the abbreviations for any constant K

$$\hat{K}^F = K^F - (\mathrm{sgn} m)\frac{k}{\omega} K^{\tilde{F}}, \qquad \hat{K}^G = K^G - (\mathrm{sgn} m)\frac{\omega}{k} K^{\tilde{G}}, \tag{38}$$

and the same for $\hat{K}'^F$ and $\hat{K}'^G$ (the outside). Then, from the boundary conditions we can solve for the remaining constants. Notice that, due to the tensorial character of the Green's dyadics, each of the above six boundary conditions

(35a), (35b), (35c), (35d), (35e), (35f) are in fact three equations corresponding to the three prime coordinates. From (35a) we get the following three equations:

$$
\begin{aligned}
-\varepsilon\lambda a J_m'(\lambda a)B_m^{\tilde{F}} - \frac{mk}{\omega\varepsilon}J_m(\lambda a)B_m^G &= -\lambda' a H_m'(\lambda' a)B_m'^{\tilde{F}} \\
&\quad - \frac{mk}{\omega}H_m(\lambda' a)B_m'^G + \frac{mk\omega}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' a), \qquad (39a)\\
-\varepsilon|m|\lambda a J_m'(\lambda a)A_m^F + \frac{mk\varepsilon}{\omega}\lambda a J_m'(\lambda a)A_m^{\tilde{F}} &+ \frac{m^2k^2}{\omega^2\varepsilon}J_m(\lambda a)A_m^G \\
-\frac{m|m|k}{\omega\varepsilon}J_m(\lambda a)A_m^{\tilde{G}} &= -|m|\lambda' a H_m'(\lambda' a)A_m'^F + \frac{mk}{\omega}\lambda' a H_m'(\lambda' a)A_m'^{\tilde{F}} \\
&\quad + \frac{m^2k^2}{\omega^2}H_m(\lambda' a)A_m'^G - \frac{m|m|k}{\omega}H_m(\lambda' a)A_m'^{\tilde{G}}, \qquad (39b)\\
\varepsilon\lambda a J_m'(\lambda a)B_m^F + \frac{mk}{\omega\varepsilon}J_m(\lambda a)B_m^{\tilde{G}} &= \lambda' a H_m'(\lambda' a)B_m'^F \\
&\quad - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\lambda' a J_m'(\lambda' a) + \frac{mk}{\omega}H_m(\lambda' a)B_m'^{\tilde{G}}. \qquad (39c)
\end{aligned}
$$

The three equations following from (35b) are:

$$
\begin{aligned}
B_m^G &= \varepsilon\left(\frac{\lambda'}{\lambda}\right)^2\left[B_m'^G\frac{H_m(\lambda' a)}{J_m(\lambda a)} - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{J_m(\lambda a)}\right], \qquad (40a)\\
|m|A_m^G - m\frac{\omega}{k}A_m^{\tilde{G}} &= \varepsilon\left(\frac{\lambda'}{\lambda}\right)^2\frac{H_m(\lambda' a)}{J_m(\lambda a)}\left[|m|A_m'^G - m\frac{\omega}{k}A_m'^{\tilde{G}}\right], \qquad (40b)\\
B_m^{\tilde{G}} &= \varepsilon\left(\frac{\lambda'}{\lambda}\right)^2\frac{H_m(\lambda' a)}{J_m(\lambda a)}B_m'^{\tilde{G}}, \qquad (40c)
\end{aligned}
$$

and those coming from (35c) are:

$$
\begin{aligned}
m\varepsilon^2 J_m(\lambda a)B_m^{\tilde F} + \frac{k}{\omega}\lambda a J_m'(\lambda a)B_m^G &= mH_m(\lambda' a)B_m'^{\tilde F} + \frac{k}{\omega}\lambda' a H_m'(\lambda' a)B_m'^G \\
&\quad - \frac{k\omega}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\lambda' a J_m'(\lambda' a), \qquad (41a)\\
\varepsilon^2 m^2 J_m(\lambda a)A_m^F - \frac{m|m|k}{\omega}\varepsilon^2 J_m(\lambda a)A_m^{\tilde F} &- \frac{k^2|m|}{\omega^2}\lambda a J_m'(\lambda a)A_m^G \\
+\frac{mk}{\omega}\lambda a J_m'(\lambda a)A_m^{\tilde G} &= m^2 H_m(\lambda' a)A_m'^F - \frac{m|m|k}{\omega}H_m(\lambda' a)A_m'^{\tilde F} \\
&\quad - \frac{|m|k^2}{\omega^2}\lambda' a H_m'(\lambda' a)A_m'^G + \frac{mk}{\omega}\lambda' a H_m'(\lambda' a)A_m'^{\tilde G}, \qquad (41b)\\
\varepsilon^2 m J_m(\lambda a)B_m^F + \frac{k}{\omega}\lambda a J_m'(\lambda a)B_m^{\tilde G} &= mH_m(\lambda' a)B_m'^F - m\frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' a) \\
&\quad + \frac{k}{\omega}\lambda' a H_m'(\lambda' a)B_m'^{\tilde G}. \qquad (41c)
\end{aligned}
$$

From the set of equations involving the magnetic part, Φ, we find that (35d) gives us

$$
\begin{aligned}
\mu m J_m(\lambda a)B_m^G + \frac{\varepsilon k}{\omega}\lambda a J_m'(\lambda a)B_m^{\tilde F} &= mH_m(\lambda' a)B_m'^G \\
&\quad - m\frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' a) + \frac{k}{\omega}\lambda' a H_m'(\lambda' a)B_m'^{\tilde F}, \qquad (42a)\\
\frac{\mu m|m|k}{\omega}J_m(\lambda a)A_m^G - \mu m^2 J_m(\lambda a)A_m^{\tilde G} &- \frac{\varepsilon k m}{\omega}\lambda a J_m'(\lambda a)A_m^F \\
+\frac{\varepsilon|m|k^2}{\omega^2}\lambda a J_m'(\lambda a)A_m^{\tilde F} &= \frac{m|m|k}{\omega}H_m(\lambda' a)A_m'^G - m^2 H_m(\lambda' a)A_m'^{\tilde G} \\
&\quad - \frac{mk}{\omega}\lambda' a H_m'(\lambda' a)A_m'^F + \frac{|m|k^2}{\omega^2}\lambda' a H_m'(\lambda' a)A_m'^{\tilde F}, \qquad (42b)\\
-\mu m J_m(\lambda a)B_m^{\tilde G} - \frac{\epsilon k}{\omega}\lambda a J_m'(\lambda a)B_m^F &= -mH_m(\lambda' a)B_m'^{\tilde G} \\
&\quad + \frac{k\omega}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\lambda' a J_m'(\lambda' a) - \frac{k}{\omega}\lambda' a H_m'(\lambda' a)B_m'^F. \qquad (42c)
\end{aligned}
$$

By imposing (35e) we get the conditions

$$\lambda a J'_m(\lambda a) B^G_m + \frac{\varepsilon m k}{\omega\mu} J_m(\lambda a) B^{\tilde{F}}_m = \lambda' a H'_m(\lambda' a) B'^G_m + \frac{mk}{\omega} H_m(\lambda' a) B'^{\tilde{F}}_m - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}} \lambda' a J'_m(\lambda' a), \tag{43a}$$

$$-\frac{|m|k}{\omega}\lambda a J'_m(\lambda a) A^G_m + m\lambda a J'_m(\lambda a) A^{\tilde{G}}_m + \frac{m^2 k\varepsilon}{\omega\mu} J_m(\lambda a) A^F_m - \frac{m|m|\varepsilon k^2}{\omega^2\mu} J_m(\lambda a) A^{\tilde{F}}_m = -\frac{|m|k}{\omega}\lambda' a H'_m(\lambda' a) A'^G_m + m\lambda' a H'_m(\lambda' a) A'^{\tilde{G}}_m + \frac{m^2 k}{\omega} H_m(\lambda' a) A'^F_m - \frac{m|m|k^2}{\omega^2} H_m(\lambda' a) A'^{\tilde{F}}_m, \tag{43b}$$

$$\lambda a J'_m(\lambda a) B^{\tilde{G}}_m + \frac{mk\varepsilon}{\omega\mu} J_m(\lambda a) B^F_m = \lambda' a H'_m(\lambda' a) B'^{\tilde{G}}_m - \frac{\omega m k}{\lambda'^2}\frac{\pi}{2\mathrm{i}} J_m(\lambda' a) + \frac{mk}{\omega} H_m(\lambda' a) B'^F_m. \tag{43c}$$

And finally (35f) gives us

$$B^{\tilde{F}}_m = \frac{\mu}{\varepsilon}\left(\frac{\lambda'}{\lambda}\right)^2 \frac{H_m(\lambda' a)}{J_m(\lambda a)} B'^{\tilde{F}}_m, \tag{44a}$$

$$-A^F_m + \frac{k}{\omega}\frac{|m|}{m} A^{\tilde{F}}_m = \frac{\mu}{\varepsilon}\left(\frac{\lambda'}{\lambda}\right)^2 \frac{H_m(\lambda' a)}{J_m(\lambda a)} \left[-A'^F_m + \frac{k}{\omega}\frac{|m|}{m} A'^{\tilde{F}}_m\right], \tag{44b}$$

$$B^F_m = \frac{\mu}{\varepsilon}\left(\frac{\lambda'}{\lambda}\right)^2 \left[B'^F_m \frac{H_m(\lambda' a)}{J_m(\lambda a)} - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{J_m(\lambda a)}\right]. \tag{44c}$$

By combining these equations we find the remaining constants, but the equations are not all independent. First, from (40b), (44b), (39b) and (41b) we learn that the coefficients of terms involving Bessel functions and $r'^{-|m|}$ cancel among themselves in a way such that the ones from the outside do not mix with those from the inside:

$$\hat{A}^F_m = \hat{A}^G_m = 0, \tag{45a}$$

$$\hat{A}'^F_m = \hat{A}'^G_m = 0. \tag{45b}$$

The same can be found if we use (43b) and (42b) instead of (39b) and (41b).

Next we determine the coefficients of functions involving just Bessel functions. From (43c) and (41c) we find using (44c) and (40c) that

$$B_m^{\tilde{G}} = -\frac{\varepsilon^2}{\mu}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' D}J_m(\lambda a)H_m(\lambda' a)B_m^F, \tag{46a}$$

$$B_m'^{\tilde{G}} = -\left(\frac{\lambda}{\lambda'}\right)^2 \frac{\varepsilon}{\mu}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' D}J_m^2(\lambda a)B_m^F, \tag{46b}$$

$$B_m'^{F} = \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{H_m(\lambda' a)} + \left(\frac{\lambda}{\lambda'}\right)^2 \frac{\varepsilon}{\mu}\frac{J_m(\lambda a)}{H_m(\lambda' a)}B_m^F, \tag{46c}$$

all in terms of

$$B_m^F = -\frac{\mu}{\varepsilon}\frac{\omega^2}{\lambda\lambda'},\frac{D}{\Xi} \tag{47}$$

found by subtracting $\frac{k}{\omega}$ times equation (41c) from (43c) and using (42c)[4]. The denominators occurring here are

$$\Xi = (1-\varepsilon\mu)^2\frac{m^2k^2\omega^2}{\lambda^2\lambda'^2}J_m^2(\lambda a)H_m^2(\lambda' a) - D\tilde{D}, \tag{48a}$$

$$D = \varepsilon\lambda' a J_m'(\lambda a)H_m(\lambda' a) - \lambda a H_m'(\lambda' a)J_m(\lambda a), \tag{48b}$$

$$\tilde{D} = \mu\lambda' a J_m'(\lambda a)H_m(\lambda' a) - \lambda a H_m'(\lambda' a)J_m(\lambda a). \tag{48c}$$

The second set of constants is found using (42a), (39a), (44a) and (40a):

$$B_m^{\tilde{F}} = -\frac{\mu}{\varepsilon^2}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' \tilde{D}}J_m(\lambda a)H_m(\lambda' a)B_m^G, \tag{49a}$$

$$B_m'^{\tilde{F}} = -\left(\frac{\lambda}{\lambda'}\right)^2 \frac{1}{\varepsilon}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' \tilde{D}}J_m^2(\lambda a)B_m^G, \tag{49b}$$

$$B_m'^{G} = \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{H_m(\lambda' a)} + \left(\frac{\lambda}{\lambda'}\right)^2 \frac{1}{\varepsilon}\frac{J_m(\lambda a)}{H_m(\lambda' a)}B_m^G, \tag{49c}$$

in terms of

$$B_m^G = -\varepsilon\frac{\omega^2}{\lambda\lambda'}\frac{\tilde{D}}{\Xi} \tag{50}$$

coming from (49b) and (41a)[5].

It might be thought that $m=0$ is a special case, and indeed

$$\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} \to \frac{1}{2}\ln\frac{r_<}{r_>}, \tag{51}$$

but just as the latter is correctly interpreted as the limit as $|m|\to 0$, so the coefficients in the Green's functions turn out to be just the $m=0$ limits for those given above, so the $m=0$ case is properly incorporated.

[4] (42c) is the same equation as (39c), which can easily be seen by using (44c).

[5] By using (44a) it can be seen that this equation is the same as (43a).

It is now easy to check that, as a result of the conditions (36a), (36b), (36c), (37a), (37b), (37c), (45a), and (45b), the terms in the Green's functions that involve powers of r or r' do not contribute to the electric or magnetic fields. As we might have anticipated, only the pure Bessel function terms contribute. (This observation was not noted in Ref. [14]).

3 Stress on the Cylinder

We are now in a position to calculate the pressure on the surface of the cylinder from the radial-radial component of the stress tensor

$$P = \langle T_{rr} \rangle (a-) - \langle T_{rr} \rangle (a+) \tag{52}$$

where

$$T_{rr} = \frac{1}{2} \left[\varepsilon (E_\theta^2 + E_z^2 - E_r^2) + \mu (H_\theta^2 + H_z^2 - H_r^2) \right]. \tag{53}$$

As a result of the boundary conditions (34), the pressure on the cylindrical walls are given by the expectation value of the squares of field components just outside the cylinder, therefore

$$\begin{aligned} T_{rr}\Big|_{r=a-} - T^{rr}\Big|_{r=a+} = \frac{\varepsilon - 1}{2} \left(E_\theta^2 + E_z^2 + \frac{E_r^2}{\varepsilon} \right) \Bigg|_{r=a+} \\ + \frac{\mu - 1}{2} \left(H_\theta^2 + H_z^2 + \frac{H_r^2}{\mu} \right) \Bigg|_{r=a+}. \end{aligned} \tag{54}$$

These expectation values are given by (9a), (9b), where the latter may also be written as

$$\langle \mathbf{H}(\mathbf{r})\, \mathbf{H}(\mathbf{r}') \rangle = -\frac{\hbar}{\omega \mu} \boldsymbol{\Phi}(\mathbf{r}, \mathbf{r}') \times \overleftarrow{\boldsymbol{\nabla}}'. \tag{55}$$

It is quite straightforward to write the vacuum expectation value of the fields occurring here in terms of the Green's functions,

$$\begin{aligned}\langle E_r(r)E_r(r')\rangle = \frac{\hbar}{\mathrm{i}}\Gamma_{rr'} = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{ &-\frac{m^2}{rr'}\frac{d_m-k^2}{\omega^2}F'_m(r,r')\\ &-\frac{mk}{\omega r}\frac{\partial}{\partial r'}\frac{d_m-k^2}{\omega^2}\tilde{F}'_m(r,r')+\frac{k^2}{\omega^2}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}G'_m(r,r)\\ &+\frac{km}{\omega r'}\frac{\partial}{\partial r}\tilde{G}'_m(r,r')\bigg\},\end{aligned} \tag{56a}$$

$$\begin{aligned}\langle E_\theta(r)E_\theta(r')\rangle = \frac{\hbar}{\mathrm{i}}\Gamma_{\theta\theta'} = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{ &-\frac{\partial}{\partial r}\frac{\partial}{\partial r'}\frac{d_m-k^2}{\omega^2}F'_m(r,r')\\ &-\frac{mk}{\omega r'}\frac{\partial}{\partial r}\frac{d_m-k^2}{\omega^2}\tilde{F}'_m(r,r')+\frac{m^2k^2}{\omega^2 rr'}G'_m(r,r')\\ &+\frac{mk}{\omega r}\frac{\partial}{\partial r'}\tilde{G}'_m(r,r')\bigg\},\end{aligned} \tag{56b}$$

$$\langle E_z(r)E_z(r')\rangle = \frac{\hbar}{\mathrm{i}}\Gamma_{zz'} = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{\omega^2}d_m d'_m G'_m(r,r'). \tag{56c}$$

According to (55) the magnetic field expectation values can be written as follows,

$$\begin{aligned}\langle H_r(r)H_r(r')\rangle = -\frac{\hbar}{2\pi}\frac{\mathrm{i}}{\omega\mu}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{ &-\frac{m^2}{\omega rr'}(d'_m-k^2)G'_m(r,r')\\ &+\frac{mk}{r}\frac{\partial}{\partial r'}\tilde{G}'_m(r,r')+\frac{k^2}{\omega}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}F'_m(r,r)\\ &-\frac{km}{\omega^2 r'}\frac{\partial}{\partial r}(d'_m-k^2)\tilde{F}'_m(r,r')\bigg\},\end{aligned} \tag{57a}$$

$$\begin{aligned}\langle H_\theta(r)H_\theta(r')\rangle = -\frac{\hbar}{2\pi}\frac{\mathrm{i}}{\omega\mu}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{ &-\frac{1}{\omega}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}(d'_m-k^2)G'_m(r,r')\\ &+\frac{mk}{r'}\frac{\partial}{\partial r}\tilde{G}'_m(r,r')+\frac{m^2k^2}{\omega rr'}F'_m(r,r')\\ &-\frac{mk}{\omega^2 r}\frac{\partial}{\partial r'}(d_m-k^2)\tilde{F}'_m(r,r')\bigg\},\end{aligned} \tag{57b}$$

$$\langle H_z(r)H_z(r')\rangle = -\frac{\hbar}{2\pi}\frac{\mathrm{i}}{\omega\mu}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{\omega}d_m d'_m F'_m(r,r'). \tag{57c}$$

When these vacuum expectation values are substituted into the stress expression (54), and the property of d_m exploited,

$$d_m r^{\pm|m|} = 0, \qquad d_m J_m(\lambda r) = -\lambda^2 J_m(\lambda r), \tag{58}$$

(of course, the later formula holds for H_m as well and the same for d'_m acting on the primed coordinate), we obtain the pressure on the cylinder as

$$
\begin{aligned}
P = \hbar\frac{\varepsilon-1}{4\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\lambda^2}{\Xi}\Bigg\{ & H_m'^2(x')J_m(x)J_m'(x)\lambda\lambda' x'(\omega^2\mu+k^2) \\
& + H_m'(x')J_m^2(x)H_m(x')\Bigg[\frac{m^2k^2\omega^2}{x'\varepsilon}\Bigg((2\varepsilon+2)(1-\varepsilon\mu) \\
& + \frac{\omega^2\varepsilon+k^2}{\lambda^2}(1-\varepsilon\mu)^2\Bigg) + x\lambda\lambda'\left(\frac{m^2}{x'^2}\left(k^2+\frac{\omega^2}{\varepsilon}\right)+\lambda'^2\right)\Bigg] \\
& - H_m'(x')J_m'^2(x)H_m(x')\mu\lambda'^2 x'(\omega^2\varepsilon+k^2) \\
& - J_m(x)J_m'(x)H_m^2(x')\lambda\lambda' x'\left[\frac{m^2}{x'^2}(k^2\mu+\omega^2)+\lambda'^2\mu\right]\Bigg\} \\
+ \hbar\frac{\mu-1}{4\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\lambda^2}{\Xi}\Big\{ & (\varepsilon\longleftrightarrow\mu)\Big\},
\end{aligned}
\tag{59}
$$

where $x=\lambda a$, $x'=\lambda' a$ and the last bracket indicates that the expression there is similar to the one for the electric part by switching ε and μ, showing manifest symmetry between the electric and magnetic parts.

In order to simplify this expression, we make an Euclidean rotation [21],

$$
\omega \to \mathrm{i}\zeta \qquad \lambda \to \mathrm{i}\kappa, \tag{60}
$$

so that the Bessel functions are replaced by the modified Bessel functions,

$$
J_m(x)H_m(x') \to \frac{2}{\pi\mathrm{i}}I_m(y)K_m(y'), \tag{61}
$$

where $y=\kappa a$ and $y'=\kappa' a$. Then (59) becomes

$$
\begin{aligned}
P = \frac{\varepsilon-1}{16\pi^3a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\zeta a\ \mathrm{d}ka\frac{\hbar}{\Xi}\Bigg\{ & K_m'^2(y')I_m(y)I_m'(y)y(k^2a^2-\zeta^2a^2\mu) \\
& - K_m'(y')I_m^2(y)K_m(y')\Bigg[\frac{m^2k^2a^2\zeta^2a^2}{y'^3\varepsilon}\Bigg(-2(\varepsilon+1)(1-\varepsilon\mu) \\
& + \frac{k^2a^2-\zeta^2a^2\varepsilon}{y^2}(1-\varepsilon\mu)^2\Bigg) - \frac{y^2}{y'}\left(\frac{m^2}{y'^2}\left(k^2a^2-\frac{\zeta^2a^2}{\varepsilon}\right)+y'^2\right)\Bigg] \\
& - K_m'(y')I_m'^2(y)K_m(y')\mu y'(k^2a^2-\zeta^2a^2\varepsilon) \\
& - I_m(y)I_m'(y)K_m^2(y')y\left[\frac{m^2}{y'^2}(k^2a^2\mu-\zeta^2a^2)+y'^2\mu\right]\Bigg\} + (\varepsilon\leftrightarrow\mu),
\end{aligned}
\tag{62}
$$

where

$$\tilde{\Xi} = \frac{m^2k^2a^2\zeta^2a^2}{y^2y'^2} I_m^2(y)K_m^2(y')(1-\varepsilon\mu)^2 + \Delta\tilde{\Delta}, \tag{63a}$$

$$\Delta = \varepsilon y' I_m'(y)K_m(y') - yK_m'(y')I_m(y) \tag{63b}$$

$$\tilde{\Delta} = \mu y' I_m'(y)K_m(y') - yK_m'(y')I_m(y) \tag{63c}$$

This result reduces to the well-known expression for the Casimir pressure when the speed of light is the same inside and outside the cylinder, that is, when $\varepsilon\mu = 1$. Then, it is easy to see that the denominator reduces to

$$\tilde{\Xi} = \Delta\tilde{\Delta} = \frac{(\varepsilon+1)^2}{4\varepsilon}\left[1 - \xi^2y^2[(I_mK_m)']^2\right], \tag{64}$$

where $\xi = (\varepsilon-1)/(\varepsilon+1)$. In the numerator introduce polar coordinates,

$$y^2 = k^2a^2 + \zeta^2a^2, \qquad ka = y\sin\theta, \qquad \zeta a = y\cos\theta, \tag{65}$$

and carry out the trivial integral over θ. The result is

$$P = -\frac{1}{8\pi^2a^4}\int_0^\infty \mathrm{d}y\, y^2 \sum_{m=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}y}\ln\left(1 - \xi^2[y(K_mI_m)']^2\right), \tag{66}$$

which is exactly the finite result derived in Ref. [10], and analyzed in a number of papers [11–13].

4 Bulk Casimir Stress

The expression derived above (62) is incomplete. It contains an unobservable "bulk" energy contribution, which the formalism would give if either medium, that of the interior with dielectric constant ε and permeability μ, or that of the exterior with dielectric constant and permeability unity, fills all the space. The corresponding stresses are computed from the free Green's functions which satisfy (18a) and (18b), therefore

$$F_m^{(0)}(r,r') = \frac{\mu}{\varepsilon}G_m^{(0)}(r,r') = -\frac{\omega^2\mu}{\varepsilon\lambda^2}\left[\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} + \frac{\pi}{2\mathrm{i}}J_m(\lambda r_<)H_m(\lambda r_>)\right], \tag{67}$$

where $0 < r, r' < \infty$. Notice that in this case, both $\tilde{F}_m^{(0)}$ and $\tilde{G}_m^{(0)}$ are zero and the Green's dyadics are given by

$$\begin{aligned}\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}';\omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}\left(-\frac{d_m-k^2}{\omega^2\mu}\right)F_m^{(0)}(r,r')\\ &+\frac{1}{\omega^2\varepsilon}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'G_m^{(0)}(r,r')\bigg\}\chi_{mk}(\theta,z)\chi_{mk}'^{*}(\theta',z'),\end{aligned}\tag{68a}$$

$$\begin{aligned}\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}';\omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^{*}G_m^{(0)}(r,r')\\ &-\frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^{*}F_m^{(0)}(r,r')\bigg\}\chi_{mk}(\theta,z)\chi_{mk}'^{*}(\theta',z').\end{aligned}\tag{68b}$$

It should be noticed that such Green's dyadics do not satisfy the appropriate boundary conditions, and therefore we cannot use (54), but rather one must compute the interior and exterior stresses individually by using (53). Because the two scalar Green's functions differ only by a factor of μ/ε in this case, for the electric part the inside stress tensor is

$$\begin{aligned}T_{rr}^{(0)}(a-) = \frac{\hbar}{2\pi\mathrm{i}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{\omega^2\varepsilon}\bigg[&\frac{\partial}{\partial r}\frac{\partial}{\partial r'}(-d_mG_m^{(0)})\\ &+\left(-d_m'-\frac{m^2}{rr'}\right)(-d_mG_m^{(0)})\bigg]\bigg|_{r=r'=a-},\end{aligned}\tag{69}$$

while the outside bulk stress is given by the same expression with $\lambda\to\lambda'=\omega^2-k^2$ and $\varepsilon=\mu=1$. When we substitute the appropriate interior and exterior Green's functions given in (67), and perform the Euclidean rotation, $\omega\to i\zeta$, we find a rather simple formula for the bulk contribution to the pressure

$$\begin{aligned}P^b =& T_{rr}^{(0)}(a-)-T_{rr}^{(0)}(a+)\\ =&\frac{\hbar}{16\pi^3a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\zeta a\ \mathrm{d}ka\bigg\{y^2I_m'(y)K_m'(y)-(y^2+m^2)I_m(y)K_m(y)\\ &-y'^2I_m'(y')K_m'(y')+(y'^2+m^2)I_m(y')K_m(y')\bigg\}.\end{aligned}\tag{70}$$

This term must be subtracted from the pressure given in (62). Note that this term is the direct analog of that found in the case of a dielectric sphere in Ref. [6]. Note also that $P^b=0$ in the special case $\varepsilon\mu=1$.

In the following, we are going to be interested in dilute dielectric media, where $\mu=1$ and $|\varepsilon-1|\ll 1$. We easily find that when the integrand in (70) is expanded in powers of $(\varepsilon-1)$ the leading terms yield

$$
\begin{aligned}
P^b = & -\frac{\hbar}{8\pi^3 a^4} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}ka \int_{-\infty}^{\infty} \mathrm{d}\zeta a \biggl\{ (\varepsilon-1)\zeta^2 a^2 I_m(y) K_m(y) \\
& + \frac{1}{4}(\varepsilon-1)^2 \frac{(\zeta a)^4}{y} [I_m(y)K_m(y)]' + \mathrm{O}\Bigl((\varepsilon-1)^3\Bigr) \biggr] \\
= & -\frac{\hbar}{8\pi^2 a^4} \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}y\, y^3 \biggl[(\varepsilon-1) I_m(y) K_m(y) \\
& + \frac{3(\varepsilon-1)^2}{16} y [I_m(y)K_m(y)]' + \mathrm{O}\Bigl((\varepsilon-1)^3\Bigr) \biggr],
\end{aligned} \tag{71}
$$

where in the last form we have introduced polar coordinates as in (65) and performed the angular integral.

5 Dilute Dielectric Cylinder

We now turn to the case of a dilute dielectric medium filling the cylinder, that is, set $\mu = 1$ and consider $\varepsilon - 1$ as small. We can then expand the integrand in (62) in powers of $(\varepsilon - 1)$ and, because the expression is already proportional to that factor, we need only expand the integrand to first order. Let us write it as

$$
P = \frac{(\varepsilon-1)\hbar}{16\pi^3 a^4} \int_{-\infty}^{\infty} \mathrm{d}\zeta a \int_{-\infty}^{\infty} \mathrm{d}ka \sum_{m=-\infty}^{\infty} \frac{N}{\Delta\tilde{\Delta}}, \tag{72}
$$

where we have noted that the $(\varepsilon-1)^2$ in $\tilde{\Xi}$ (63a) can be dropped. Expanding the numerator and denominator according to

$$
N = N^{(0)} + (\varepsilon-1)N^{(1)} + \dots, \qquad \Delta\tilde{\Delta} = 1 + (\varepsilon-1)\Delta^{(1)} + \dots, \tag{73}
$$

we can write

$$
P = \frac{(\varepsilon-1)\hbar}{16\pi^3 a^4} \int_{-\infty}^{\infty} \mathrm{d}\zeta a \int_{-\infty}^{\infty} \mathrm{d}ka \sum_{m=-\infty}^{\infty} \biggl\{ N^{(0)} + (\varepsilon-1)\Bigl(N^{(1)} - N^{(0)}\Delta^{(1)}\Bigr) + \dots \biggr\}, \tag{74}
$$

where

$$
\begin{aligned}
N^{(0)} =& -(k^2a^2-\zeta^2a^2)K_m'(y)I_m'(y)\\
&-\left[\frac{m^2}{y^2}(k^2a^2-\zeta^2a^2)+y^2\right]K_m(y)I_m(y), && (75a)\\
N^{(1)} =& \frac{\zeta^2a^2}{2}\left(1+\frac{m^2}{y^2}\right)(k^2a^2-\zeta^2a^2)K_m'^2(y)I_m^2(y)\\
&+\frac{\zeta^2a^2}{2}(k^2a^2-\zeta^2a^2)K_m'^2(y)I_m'^2(y)\\
&-\frac{\zeta^2a^2}{2}\left[\frac{m^2}{y^2}(k^2a^2-\zeta^2a^2)+y^2\right]K_m^2(y)I_m'^2(y)\\
&-\frac{\zeta^2a^2}{2}\left(1+\frac{m^2}{y^2}\right)\left[\frac{m^2}{y^2}(k^2a^2-\zeta^2a^2)+y^2\right]K_m^2(y)I_m^2(y)\\
&+\zeta^2a^2\left[y\left(1+\frac{m^2}{y^2}\right)+\frac{m^2}{y^3}(k^2a^2-\zeta^2a^2)-\frac{4}{y^3}m^2k^2a^2\right]\\
&\quad\times K_m'(y)K_m(y)I_m^2(y)\\
&+\left[y^2\zeta^2a^2-\zeta^2a^2(k^2a^2-\zeta^2a^2)\right]K_m(y)K_m'(y)I_m(y)I_m'(y)\\
&+\left[y\zeta^2a^2+\frac{\zeta^2a^2}{y}(k^2a^2-\zeta^2a^2)\right]K_m(y)K_m'(y)I_m'^2(y), && (75b)\\
\Delta^{(1)} =& -\frac{1}{y}\zeta^2a^2[I_m(y)K_m(y)]'+yI_m'(y)K_m(y)-\zeta^2a^2I_m'(y)K_m'(y)\\
&+\zeta^2a^2\left(1+\frac{m^2}{y^2}\right)I_m(y)K_m(y). && (75c)
\end{aligned}
$$

When we introduce polar coordinates as in (65) and perform the trivial angular integrals, the straightforward reduction of (74) is

$$
\begin{aligned}
P =& -\frac{\hbar}{8\pi^2a^4}(\varepsilon-1)\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}y\biggl\{y^3K_m(y)I_m(y)\\
&-(\varepsilon-1)\frac{y^4}{2}\biggl[\frac{1}{2}K_m'^2(y)I_m'(y)I_m(y)\\
&+K_m'(y)I_m'^2(y)K_m(y)+K_m'^2(y)I_m'^2(y)\frac{y}{4}\\
&-K_m'^2(y)I_m^2(y)\frac{y}{4}\left(1+\frac{m^2}{y^2}\right)+K_m^2(y)I_m^2(y)\frac{y}{2}\left(1+\frac{m^2}{y^2}\right)\left(1-\frac{m^2}{2y^2}\right)\\
&+K_m^2(y)I_m'(y)I_m(y)\left(1+\frac{m^2}{2y^2}\right)-K_m^2(y)I_m'^2(y)\frac{y}{2}\left(1-\frac{m^2}{2y^2}\right)\biggr]\biggr\}.
\end{aligned}
\tag{76}
$$

The leading term in the pressure,

$$P^{(1)} = -\frac{\hbar}{8\pi^2 a^4}(\varepsilon - 1)\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}y\, y^3 K_m(y) I_m(y), \tag{77}$$

can also be obtained from (62) by setting $\varepsilon = \mu = 1$ everywhere in the integrand, and the denominator $\tilde{\Xi}$ is then unity. This is also exactly what is obtained to leading order $\mathrm{O}[(\varepsilon-1)^1]$ from the bulk stress (71). Thus the total stress vanishes in leading order:

$$P^{(1)} - P^b = \mathrm{O}[(\varepsilon - 1)^2], \tag{78}$$

which is consistent with the interpretation of the Casimir energy as arising from the pairwise interaction of dilutely distributed molecules.

6 Evaluation of the $(\varepsilon - 1)^2$ term

We now turn to the considerably more complex evaluation of the $(\varepsilon-1)^2$ term in (76).

6.1 Summation method

As a first approach to evaluating this second-order term, we first carry out the sum on m by use of the addition theorem

$$K_0(kP) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')}\, K_m(k\rho) I_m(k\rho'), \quad \rho > \rho', \tag{79}$$

where $P = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi')}$. Then by squaring this addition theorem and applying suitable differential operators, in the singular limit $\rho' \to \rho$ we obtain the following formal results:

$$\sum_{m=-\infty}^{\infty} K_m^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} K_0^2(z), \tag{80a}$$

$$\sum_{m=-\infty}^{\infty} m^2 K_m^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} [K_0'(z)]^2 (k\rho)^2 \cos^2\frac{\phi}{2}, \tag{80b}$$

$$\sum_{m=-\infty}^{\infty} m^4 K_m^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \left[K_0'(z)\frac{z}{4} - K_0''(z)(k\rho)^2\cos^2\frac{\phi}{2}\right]^2, \tag{80c}$$

$$\sum_{m=-\infty}^{\infty} K_m^2(k\rho)I_m(k\rho)I_m'(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} K_0(z)K_0'(z)\sin\frac{\phi}{2}, \tag{80d}$$

$$\sum_{m=-\infty}^{\infty} m^2 K_m^2(k\rho)I_m(k\rho)I_m'(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \frac{z}{2} K_0(z)K_0'(z) k\rho \cos^2\frac{\phi}{2}, \tag{80e}$$

$$\sum_{m=-\infty}^{\infty} K_m'^2(k\rho)I_m^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} [K_0'(z)]^2 \sin^2\frac{\phi}{2}, \tag{80f}$$

$$\sum_{m=-\infty}^{\infty} m^2 K_m^2(k\rho)I_m'^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \frac{z^2}{4} K_0^2(z)\cos^2\frac{\phi}{2}, \tag{80g}$$

$$\begin{aligned}\sum_{m=-\infty}^{\infty} I_m(k\rho)I_m'(k\rho)K_m'^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} K_0'(z)\sin\frac{\phi}{2}\\ \times\left[K_0(z)\sin^2\frac{\phi}{2} - \frac{K_0'(z)}{z}\right],\end{aligned} \tag{80h}$$

$$\sum_{m=-\infty}^{\infty} I_m'^2(k\rho)K_m'^2(k\rho) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \left[K_0(z)\sin^2\frac{\phi}{2} - \frac{K_0'(z)}{z}\right]^2. \tag{80i}$$

Here $z = 2k\rho\sin\frac{\phi}{2}$, and we recognize that in this singular limit (which omits delta functions, i.e., contact terms) terms with I_m and K_m interchanged in the sum have the same values.

When we put this all together, we obtain the following expression for the pressure at second order:

$$\begin{aligned}P^{(2)} &= \frac{(\varepsilon-1)^2}{4096\pi^2 a^4}\int_0^\infty \mathrm{d}z\, z^5 \int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\biggl\{\frac{1}{s^6}\left[K_0'^2(z) + K_0^2(z)\left(1-\frac{4}{z^2}\right)\right]\\ &\quad + \frac{2}{s^4}\left[\left(1-\frac{8}{z^2}\right)K_0^2(z) - 2\left(1+\frac{3}{z^2}\right)K_0'^2(z)\right] - \frac{16}{s^2 z^2}K_0^2(z)\biggr\}\\ &= \frac{(\varepsilon-1)^2}{15360\pi^2 a^4}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\left[\frac{5}{s^6} - \frac{66}{s^4} - \frac{20}{s^2}\right],\end{aligned} \tag{81}$$

where $s = \sin\phi/2$. Of course, the ϕ integrals in (81) are divergent. However, we will regulate them by continuing from the region where the integrals converge:

$$\int_0^{2\pi} \mathrm{d}\phi \left(\sin\frac{\phi}{2}\right)^s = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}, \tag{82}$$

which is valid for $\operatorname{Re} s > -1$. We will take the right side of (82) to define the angular integral for negative s. Then we see that those integrals vanish when $s = -2n$ where n is a positive integer. Thus, this analytic continuation procedure says that the result (81) is zero. As for the bulk term, the addition theorem (79) implies that the y integral in the second term in (71) reduces to

$$\sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}y\, y^4 \left(I_m(y)K_m(y)\right)' = \int_0^\infty \mathrm{d}y\, y^4 \frac{\mathrm{d}}{\mathrm{d}y} K_0(0) = 0. \tag{83}$$

This argument is exactly that given in Ref. [2] to show that the Casimir energy of a dilute dielectric-diamagnetic cylinder with $\varepsilon\mu = 1$ vanishes. However, it is not very convincing, because it seems to show no relevance of cancellations between various terms in the expressions for the pressure. That relevance will be established in the method which follows.

6.2 Numerical analysis

We now turn to a detailed numerical treatment of the second-order terms in (76) and (71). It is based on use of the uniform asymptotic or Debye expansions for the Bessel functions, $m \gg 1$:

$$I_m(y) \sim \frac{1}{\sqrt{2\pi m}} t^{1/2}\, \mathrm{e}^{m\eta} \left(1 + \sum_{k=1}^\infty \frac{u_k(t)}{m^k}\right), \tag{84a}$$

$$K_m(y) \sim \sqrt{\frac{\pi}{2m}} t^{1/2}\, \mathrm{e}^{-m\eta} \left(1 + \sum_{k=1}^\infty (-1)^k \frac{u_k(t)}{m^k}\right), \tag{84b}$$

$$I'_m(y) \sim \frac{1}{\sqrt{2\pi m}} \frac{1}{z} t^{-1/2}\, \mathrm{e}^{m\eta} \left(1 + \sum_{k=1}^\infty \frac{v_k(t)}{m^k}\right), \tag{84c}$$

$$K'_m(y) \sim -\sqrt{\frac{\pi}{2m}} \frac{1}{z} t^{-1/2}\, \mathrm{e}^{-m\eta} \left(1 + \sum_{k=1}^\infty (-1)^k \frac{v_k(t)}{m^k}\right), \tag{84d}$$

where $y = mz$ and $t = 1/\sqrt{1+z^2}$. (The value of η is irrelevant here.) The polynomials in t appearing here are generated by

$$u_0(t) = 1, \quad v_0(t) = 1, \tag{85a}$$
$$u_k(t) = \frac{1}{2}t^2(1-t^2)u'_{k-1}(t) + \frac{1}{8}\int_0^t \mathrm{d}x\,(1-5x^2)u_{k-1}(x), \tag{85b}$$
$$v_k(t) = u_k(t) + t(t^2-1)\left(\frac{1}{2}u_{k-1}(t) + tu'_{k-1}(t)\right). \tag{85c}$$

Now suppose we write the second-order expression for the pressure as

$$P = \frac{(\varepsilon-1)^2}{16\pi^2 a^4}\sum_{m=0}^{\infty}{}'\int_0^\infty \mathrm{d}y\, y^4 g_m(y), \tag{86}$$

where the explicit form for $g_m(y)$ can be immediately read off from (76) and (71), and the prime on the summation sign means that the $m=0$ term is counted with half weight. Let us subtract and add the first five terms in the uniform asympotic expansion for g_m, $m \gg 1$:

$$g_m(y) \sim \frac{1}{2m^2}\sum_{k=1}^{5}\frac{1}{m^k}f_k(z), \tag{87}$$

where $z = y/m$ and

$$f_1(z) = \frac{4+z^2}{4z(1+z^2)^3}, \tag{88a}$$
$$f_2(z) = \frac{-8+8z^2+z^4}{8z(1+z^2)^{7/2}}, \tag{88b}$$
$$f_3(z) = \frac{16-84z^2+84z^4-16z^6-5z^8}{16z(1+z^2)^6}, \tag{88c}$$
$$f_4(z) = \frac{-64+1024z^2-1864z^4+504z^6-9z^8}{64z(1+z^2)^{13/2}}, \tag{88d}$$
$$f_5(z) = \frac{64-2416z^2+11808z^4-15696z^6+6856z^8-555z^{10}-15z^{12}}{64z(1+z^2)^9}. \tag{88e}$$

We note that when these function are inserted into (86) in place of g_m, the first three f_k give divergent integrals, logarithmically so for f_1 and f_3, and linearly divergent for f_2. We also note the crucial fact that

$$\int_0^\infty \mathrm{d}z\, z^4 f_4(z) = 0, \tag{89}$$

which means that $\zeta(1)$, which would indicate an unremovable divergence, does not occur in the summation over m. This is the content of the proof that the Casimir energy for a dilute dielectric cylinder is finite in this order, given by Bordag and Pirozhenko [22]. We also note that when the divergent part is removed from the f_2 integration we again get zero,

$$\int_0^\infty \mathrm{d}z \left(z^4 f_2(z) - \frac{1}{8} \right) = 0. \tag{90}$$

This means that this term may be simply omitted as a contact term.

However, the two logarithmically divergent terms, corresponding to f_1 and f_3, give finite contributions, because they are multiplied by formally zero values of the Riemann zeta function. The first one may be regulated by a small change in the power:

$$\lim_{s\to 0} \sum_{m=1}^\infty m^{2-s} \int_0^\infty \mathrm{d}z\, z^{4-s} f_1(z) = \lim_{s\to 0} \frac{1}{4}\zeta(-2+s)\frac{1}{s} = -\frac{\zeta(3)}{16\pi^2}. \tag{91a}$$

The f_3 term gives similarly

$$\lim_{s\to 0} {\sum_{m=0}^\infty}' m^{-s} \int_0^\infty \mathrm{d}z\, z^{4-s} f_3(z) = \zeta'(0) \left(-\frac{5}{16} \right) = \frac{5}{32} \ln 2\pi. \tag{91b}$$

The f_5 term is completely finite:

$$\sum_{m=1}^\infty \frac{1}{m^2} \int_0^\infty \mathrm{d}z\, z^4 f_5(z) = \frac{19\pi^2}{7680}. \tag{92}$$

Following the above prescription, we arrive at the following entirely finite expression for the pressure on the cylinder:

$$\begin{aligned} P = \frac{(\varepsilon-1)^2}{32\pi^2 a^4} \Bigg\{ & -\frac{\zeta(3)}{16\pi^2} + \frac{5}{32}\ln 2\pi + \frac{19\pi^2}{7680} \\ & + 2\sum_{m=1}^\infty \int_0^\infty \mathrm{d}y\, y^4 \left[g_m(y) - \frac{1}{2m^2} \sum_{k=1}^5 \frac{1}{m^k} f_k(y/m) \right] \\ & + \int_0^\infty \mathrm{d}y\, y^4 \left[g_0(y) - \frac{1}{16}\frac{1}{y^4} - \frac{1}{2} f_3(y) \right] \Bigg\}. \end{aligned} \tag{93}$$

All that remains is to do the integrals numerically. We do so for m from 0 through 4, after which we use the next nonzero term in the uniform asymptotic expansion,

$$\sum_{m=5}^\infty \int_0^\infty \mathrm{d}z\, z^4 \left[\frac{1}{m^3} f_6(z) + \frac{1}{m^4} f_7(z) \right] = -\frac{209}{64512} \sum_{m=5}^\infty \frac{1}{m^4}, \tag{94}$$

because, again, the integral over f_6 vanishes.

When all the above is included, to 6 decimal places, we obtain

$$P = \frac{(\varepsilon-1)^2}{32\pi^2 a^4}(-0.007612 + 0.287168 + 0.024417 - 0.002371 - 0.000012 \\ - 0.301590) = 0.000000, \tag{95}$$

where the successive terms come from (91a), (91b), (92), the numerical integral over the first 4 g_ms ($m > 0$), the remainder (94), and the numerical integral over g_0, respectively. This constitutes a convincing demonstration of the vanishing of the Casimir pressure in this case. It is similar to the numerical demonstration [10] of the seemingly coincidental vanishing of the Casimir energy for a dilute dielectric-diamagnetic cylinder, obtained by expanding (66) to order ξ^2.

7 Conclusions

Since the beginning of the subject, the identity of the Casimir force with van der Waals forces between individual molecules has been evident [23,24]. It is essentially just a change of perspective from action at a distance to local field fluctuations. So it was no surprise that the retarded dispersion force between molecules, the Casimir-Polder force, could be derived from the Lifshitz force between parallel dielectric surfaces [25,18]. However, the identity is not really that trivial, because both the van der Waals and the Casimir energies contain divergent contributions. This is particularly crucial when one is considering the self-energy of a single body rather than the energy of interaction of distinct bodies. Thus it was nontrivial when it was proved that the Casimir energy of a dilute dielectric sphere [8] coincided with that obtained by summing the van der Waals energies of the constituent molecules [9].

When it was shortly thereafter discovered that the sum of van der Waals forces vanished for a dielectric cylinder [16,10] it was universally believed that the corresponding Casimir energy, in the dilute approximation, must also vanish. Proving this turned out to be extraordinarily difficult. This paper is the result of a five-year-long effort. It should dispell any lingering doubts about the meaning of the Casimir force. The importance of this finding is impossible to evaluate at this point; a zero value suggests some underlying symmetry, which is certainly far from apparent. It probably has technological implications, for example in the physics of nanotubes, which will be explored in a subsequent publication.

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