Casimir Energies: Temperature Dependence, Dispersion, and Anomalies

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Abstract

There has been an ongoing controversy concerning the Casimir effect between metallic plates, which has now broadened to include semiconductors and dielectrics including small, but nonzero, conductivity. To help address such concerns, we here re-examine general issues involving the temperature and dispersion effects in real materials. We begin by reconsidering the free energy and internal energy for ideal metallic plates separated by a dielectric medium, and show that the results agree with the nondispersive electromagnetic energy. This raises the question of how properly to deal with dispersion, where the energy density should be described by the well-known expression involving $d[\omega\varepsilon(\omega)]/d\omega$. We show that the derivative of the permittivity with respect to frequency should *not* appear in the expression for the Casimir energy. As a final topic, we go on to consider an anomaly, associated with local surface divergences, encountered in the calculation of the Casimir energy for higher spacetime dimensions, $D > 4$, essentially generalizing recent calculations of Alnes *et al.* [J. Phys. A: Math. Theor. **40**, F315 (2007)] to the case of a medium-filled cavity between two hyperplanes.

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I. INTRODUCTION

For many years, the Casimir effect [1] was a theoretical curiosity, although it had evident applications to van der Waals forces [2] and models of hadrons [3]. The Casimir formula for the quantum vacuum force between conducting plates was generalized to dielectrics by Lifshitz [4], which was verified at the 1% level by Sabisky and Anderson [5].

But the renaissance in studies of the Casimir effect began in 1997 with the work of Lamoreaux [6]. He measured the Casimir force between a conducting plate and a spherical lens, which through the proximity force approximation [7–9], agreed with expectations at something like the 5% level. (The accuracy of this measurement remains under some dispute, because various corrections, such as the effects of surface roughness, patch potentials, and finite conductivity, were not adequately taken into account.) In subsequent years, a variety of experiments were carried out, some of much greater accuracy and at considerably shorter distances, down to less than 100 nm [10–19], which have incorporated various corrections [20]. The zero-temperature Casimir-Lifshitz theory seems to have been confirmed to 1% accuracy over a range from 100 nm to a few micrometers.

The major controversy in this subject lies in the temperature dependence of the Casimir force between real metal surfaces. Although an ambiguity in the process of extracting this dependence was recognized early, it was resolved with a prescription [21]. Because of inaccessibility of the effect to precise experiments, it was not reconsidered until the modern era, when Boström and Sernelius [22] recognized that this prescription could not be correct, and that necessarily the transverse electric reflection coefficient at zero frequency must vanish for metals. This led to a reduction by a factor of two in the prediction for the slope of the linear high-temperature behavior (which would only be visible in experiments carried out at several microns, where the Casimir force is very small), but it would predict a new linear temperature term at low temperatures, resulting in a 15% correction to the result found by Lamoreaux. Lamoreaux believes that his experiment could not be in error to this extent [23]. More heatedly, Mostepanenko and collaborators have insisted that this behavior is inconsistent with thermodynamics (the Nernst heat theorem), because it would predict, for an ideal metal, that the free energy has a linear temperature term at low temperature, and hence that the entropy would not vanish at zero temperature [24, 25]. Moreover, they assert that the precision Purdue experiments, which claim better than 1% accuracy at distances down to 60 nm, rule out the linear temperature term in the low temperature expansion [17].

We and others have responded that real metals do not exhibit this thermodynamic inconsistency, and that most probably the experiments are not so accurate as claimed [26–28]. The situation is summarized in recent reviews [29, 30]. In particular, the lack of a thermodynamic inconsistency has been conclusively demonstrated [31, 32], by showing that the free energy for a Casimir system made from real metal plates has a quadratic temperature dependence at low temperature.

Further evidence for the validity of the notion of excluding the TE zero mode for metals comes from the recent work of Buenzli and Martin [33], corroborating earlier work by these authors and others [34, 35], who show from a microscopic viewpoint that the high-temperature behavior of the Casimir force is half that of an ideal metal, a rather conclusive demonstration that the TE zero mode is not present.

Evidently, it is appropriate to review theoretical underpinnings. In this paper, we first re-examine the temperature dependence of the Casimir energy for ideal metal plates, and establish the connection between the free energy, internal energy, and the electromagnetic

energy (Secs. II–IV). The question of dispersion is looked at anew (Sec. V). As a third point, we discuss in Sec. VI an anomaly encountered in the calculation of the Casimir energy in higher-dimensional spacetimes, $D > 4$, reflecting the breakdown of conformal symmetry associated with surface divergences in the local energy density recently found by Alnes *et al.* [36], here generalized to the case of a medium-filled cavity between two hyperplanes.

II. FREE ENERGY F

Consider the usual Casimir configuration, namely two infinitely large plates situated at $z = 0$ and $z = a$, with a homogeneous and isotropic medium in between. We take this intermediate medium to have permittivity ε and permeability μ. The medium to the left $(z < 0)$ is taken to have material constants ε_1, μ_1, and the medium to the right $(z > a)$ is taken to have material constants ε_2, μ_2. The free energy F per unit surface is

$$F = \frac{1}{2\pi\beta} \sum_{m=0}^{\infty} {}' \int_{n\zeta_m}^{\infty} \kappa_3 d\kappa_3 \left[\ln(1 - r_{\text{TE}}^2 e^{-2\kappa_3 a}) + \ln(1 - r_{\text{TM}}^2 e^{-2\kappa_3 a}) \right], \tag{2.1}$$

where $n = \sqrt{\varepsilon\mu}$ is the refractive index of the intervening medium, $\beta = 1/T$, $\zeta_m = 2\pi m/\beta$, and $\kappa_i^2 = k_\perp^2 + \varepsilon_i\mu_i\zeta^2$ for $i = 1, 2, 3$ (the index $i = 3$ refers to the intermediate region). The prime on the summation sign means that the $m = 0$ term is counted with half-weight. The squared reflection coefficients are

$$r_{\text{TE}}^2 = \left(\frac{\kappa_1\mu_3 - \kappa_3\mu_1}{\kappa_1\mu_3 + \kappa_3\mu_1} \right)^2, \tag{2.2a}$$

$$r_{\text{TM}}^2 = \left(\frac{\kappa_1\varepsilon_3 - \kappa_3\varepsilon_1}{\kappa_1\varepsilon_3 + \kappa_3\varepsilon_1} \right)^2. \tag{2.2b}$$

In the following we assume for simplicity the ideal metal (IM) model, for which $r_{\text{TE}}^2 = r_{\text{TM}}^2 = 1$ for all m, including $m = 0$. (The breakdown of this assumption for the r^{TE} coefficient at $m = 0$ is the crux of the temperature controversy for real metals.) Moreover, as mentioned, $\varepsilon_3 \equiv \varepsilon$, $\mu_3 \equiv \mu$. The free energy thereby becomes

$$F = \frac{1}{\pi\beta} \sum_{m=0}^{\infty} {}' \int_{n\zeta_m}^{\infty} \kappa_3 d\kappa_3 \ln\left(1 - e^{-2\kappa_3 a}\right), \tag{2.3}$$

for arbitrary T.

III. INTERNAL ENERGY U

The internal energy per unit area is constructed from the thermodynamical formula

$$U = \frac{\partial(\beta F)}{\partial\beta}. \tag{3.1}$$

From Eq. (2.3) it is apparent that β appears only in the lower limit of the integral in the expression for βF. Since $\partial\zeta_m/\partial\beta = -2\pi m/\beta^2$, we get

$$U = \sum_m \frac{\partial\zeta_m}{\partial\beta} \frac{\delta(\beta F)}{\delta\zeta_m} = \frac{4\pi n^2}{\beta^3} \sum_{m=0}^{\infty} {}' m^2 \ln\left(1 - e^{-\alpha m}\right), \tag{3.2}$$

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where

$$\alpha = \frac{4\pi na}{\beta} = 4\pi naT. \tag{3.3}$$

The $m = 0$ term does not contribute. One way of processing the expression (3.2) is to expand the logarithm,

$$m^2 \ln \left(1 - e^{-\alpha m}\right) = -\sum_{k=1}^{\infty} \frac{1}{k} m^2 e^{-\alpha km}, \tag{3.4}$$

and then sum over m,

$$\sum_{m=1}^{\infty} m^2 e^{-\alpha km} = \frac{1}{\alpha^2} \frac{\partial^2}{\partial k^2} \frac{e^{-\alpha k}}{1 - e^{-\alpha k}} = \frac{\coth(2\pi nkaT)}{4 \sinh^2(2\pi nkaT)}. \tag{3.5}$$

Then,

$$U = -\pi n^2 T^3 \sum_{m=1}^{\infty} \frac{1}{m} \frac{\coth(2\pi nmaT)}{\sinh^2(2\pi nmaT)}. \tag{3.6}$$

When $n = 1$, this agrees with Eq. (18) of Ref. [37]. The expansion (3.6) is most convenient at high temperatures, $aT \gg 1$. By including only the $m = 1$ term, one gets

$$U = -4\pi n^2 T^3 e^{-4\pi naT}, \quad aT \gg 1. \tag{3.7}$$

It is apparent that $U \to 0$ when $T \to \infty$. This is as we should expect physically: The Casimir energy measures the change in energy induced by the boundaries, and these constraints decrease in importance when the classical thermal energy becomes high.

To get a convenient expression at low T one may perform a Poisson resummation, along the same lines as discussed in Ref. [21]. Define the quantity $b(m)$ by

$$b(m) = m^2 \ln \left(1 - e^{-\alpha|m|}\right), \tag{3.8}$$

for all m. In accordance with Eq. (3.2) we thus have

$$U = 2\pi n^2 T^3 \sum_{m=-\infty}^{\infty} b(m). \tag{3.9}$$

Further, define $c(q)$ as

$$c(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} b(x) e^{-iqx} dx = \frac{1}{\pi} \int_0^{\infty} x^2 \cos qx \ln \left(1 - e^{-\alpha x}\right) dx. \tag{3.10}$$

Then the Poisson formula says

$$\sum_{m=-\infty}^{\infty} b(m) = 2\pi \sum_{m=-\infty}^{\infty} c(2\pi m) = 2 \int_0^{\infty} x^2 \ln \left(1 - e^{-\alpha x}\right) dx$$

$$+ 4 \sum_{m=1}^{\infty} \int_0^{\infty} x^2 \cos(2\pi mx) \ln \left(1 - e^{-\alpha x}\right) dx. \tag{3.11}$$

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The first term is evaluated as

$$\int_0^\infty x^2 \ln\left(1 - e^{-\alpha x}\right) dx = -\frac{\pi^4}{45\alpha^3} = -\frac{\pi}{2880(naT)^3}. \tag{3.12}$$

In the second term in Eq. (3.11) we may perform a partial integration,

$$\int_0^\infty x^2 \cos(2\pi mx) \ln\left(1 - e^{-\alpha x}\right) dx = -\frac{\alpha}{2\pi m} \int_0^\infty \frac{x^2 \sin(2\pi mx)}{e^{\alpha x} - 1} dx$$
$$+ \frac{2\alpha}{(2\pi m)^3} \int_0^\infty \frac{\sin(2\pi mx)}{e^{\alpha x} - 1} dx - \frac{2\alpha}{(2\pi m)^2} \int_0^\infty \frac{x \cos(2\pi mx)}{e^{\alpha x} - 1} dx. \tag{3.13}$$

We may here use the following formulas (Ref. [38], sec. 3.951), which hold for $b > 0$,

$$\int_0^\infty \frac{x^{2m} \sin bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{2} \coth \pi b - \frac{1}{2b}\right], \tag{3.14a}$$

$$\int_0^\infty \frac{x^{2m+1} \cos bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{2} \coth \pi b - \frac{1}{2b}\right], \tag{3.14b}$$

and so we get

$$U = 2\pi n^2 T^3 \left[-\frac{\pi}{1440(naT)^3} + \frac{naT}{\pi^3} \sum_{m=1}^\infty \frac{1}{m^4} \left\{ -3 + \frac{\pi m}{2naT} \coth \frac{\pi m}{2naT} \right.\right.$$
$$\left.\left. + \frac{\left(\frac{\pi m}{2naT}\right)^2}{\sinh^2\left(\frac{\pi m}{2naT}\right)} \left[1 + \frac{\pi m}{2naT} \coth \frac{\pi m}{2naT}\right] \right\} \right]. \tag{3.15}$$

This expression, in principle valid for all temperatures, is most convenient to use when $aT \to 0$. Then, the expression between curly braces becomes approximately

$$\{ \} \to -3 + \frac{\pi m}{2naT}, \tag{3.16}$$

where the corrections are exponentially small. Now using that $\sum_1^\infty m^{-4} = \pi^4/90$, $\sum_1^\infty m^{-3} = \zeta(3)$, we get

$$U = -\frac{\pi^2}{720na^3} \left[1 - 720 \left(\frac{naT}{\pi}\right)^3 \zeta(3) + 48(naT)^4\right], \quad aT \ll 1. \tag{3.17}$$

Again, this agrees with the low-temperature expression obtained earlier, for instance in Ref. [26], when $n = 1$. It is to be noted that U, as well as the corresponding low-temperature expression for F,

$$F = -\frac{\pi^2}{720na^3} \left[1 + 360 \left(\frac{naT}{\pi}\right)^3 \zeta(3) - (2naT)^4\right], \tag{3.18}$$

up to exponentially small corrections, contain a term that is independent of a, which means that this term does not contribute to the force between the plates.

IV.   ELECTROMAGNETIC ENERGY W—NONDISPERSIVE CASE

Now consider first the electromagnetic energy density w in the intermediate region $0 < z < a$. In this section we take the medium to be nondispersive. The classical expression for the energy density w is

$$w = \frac{1}{2}\varepsilon(E_z^2 + E_\perp^2) + \frac{1}{2}\mu(H_z^2 + H_\perp^2). \tag{4.1}$$

Obviously it is the electromagnetic energy per unit area $W = wa$ that is to be compared with the thermodynamical energy U calculated above. (In this case it will turn out that w is independent of position between the two media 1 and 2.) Quantum mechanically, the product $E_z^2(\mathbf{r})$ is to be replaced by the expectation value $\langle E_z(\mathbf{r})E_z(\mathbf{r}')\rangle$ in the limit when $\mathbf{r}' \to \mathbf{r}$. Similarly for the other components. We assume first that $T = 0$. According to the fluctuation-dissipation theorem in Fourier space we have

$$i\langle E_i(\mathbf{r})E_k(\mathbf{r}')\rangle_\omega = \text{Im}\,\Gamma_{ik}(\mathbf{r},\mathbf{r}';\omega), \tag{4.2a}$$

$$i\langle H_i(\mathbf{r})H_k(\mathbf{r}')\rangle_\omega = \frac{1}{\mu^2\omega^2}\text{curl}_{ij}\,\text{curl}'_{kl}\,\text{Im}\,\Gamma_{jl}(\mathbf{r},\mathbf{r}';\omega), \tag{4.2b}$$

where $\text{curl}_{ik} \equiv \epsilon_{ijk}\partial_j$, ϵ_{ijk} being the Levi-Cività symbol. Further, $\mathbf{\Gamma}$ is the Green's function as defined by Schwinger et al. [21], in terms of a polarization source $\mathbf{P}$,

$$\mathbf{E}(x) = \int d^4x'\,\mathbf{\Gamma}(x,x')\cdot\mathbf{P}(x'), \tag{4.3}$$

with

$$\mathbf{\Gamma}(x,x') = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\,\mathbf{\Gamma}(\mathbf{r},\mathbf{r}';\omega), \tag{4.4}$$

and $\tau = t - t'$. Introducing a transverse Fourier transform,

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}';\omega) = \int\frac{d^2k_\perp}{(2\pi)^2}\,e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')}\,\mathbf{g}^E(z,z';\mathbf{k}_\perp,\omega), \tag{4.5}$$

we can write

$$g_{xx}^E = -\frac{\kappa}{\varepsilon}\frac{1}{d}\cosh\kappa(z-z'), \tag{4.6a}$$

$$g_{yy}^E = \frac{\mu\omega^2}{\kappa}\frac{1}{d}\cosh\kappa(z-z'), \tag{4.6b}$$

$$g_{zz}^E = \frac{k_\perp^2}{\kappa\varepsilon}\frac{1}{d}\cosh\kappa(z-z'), \tag{4.6c}$$

where

$$d = e^{2\kappa a} - 1, \quad \kappa^2 = k_\perp^2 - n^2\omega^2. \tag{4.7}$$

(Details are given in Ref. [39].) (Note that the notation is slightly different than that given in Ref. [40].)

Defining the Fourier components $\langle..\rangle_{\omega\mathbf{k}}$ of the energy density according to

$$w = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}\,[\varepsilon\langle E^2\rangle_{\omega\mathbf{k}} + \mu\langle H^2\rangle_{\omega\mathbf{k}}], \tag{4.8}$$

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we first obtain for the electric part, letting $z' \to z$,

$$\frac{1}{2}\varepsilon \langle E^2 \rangle_{\omega \mathbf{k}} = \frac{1}{i}\frac{\varepsilon}{2}(g_{xx}^E + g_{yy}^E + g_{zz}^E) = \frac{n^2\omega^2}{i\kappa}\frac{1}{d}. \tag{4.9}$$

Then defining the magnetic counterpart g_{ik}^H to the electric part g_{ik}^E according to

$$g_{ik}^H = \frac{1}{\omega^2}\mathrm{curl}_{il}\,\mathrm{curl}'_{km}\,g_{lm}^E, \tag{4.10}$$

we obtain

$$g_{xx}^H = -\frac{\mu\kappa}{d}\cosh\kappa(z-z'), \tag{4.11a}$$

$$g_{yy}^H = \frac{n^2\mu\omega^2}{\kappa}\frac{1}{d}\cosh\kappa(z-z'), \tag{4.11b}$$

$$g_{zz}^H = \frac{\mu k_\perp^2}{\kappa}\frac{1}{d}\cosh\kappa(z-z'). \tag{4.11c}$$

Thus in the limit when $z' \to z$,

$$\frac{1}{2}\mu \langle H^2 \rangle_{\omega \mathbf{k}} = \frac{1}{i}\frac{1}{2\mu}(g_{xx}^H + g_{yy}^H + g_{zz}^H) = \frac{n^2\omega^2}{i\kappa}\frac{1}{d}. \tag{4.12}$$

The electric and magnetic contributions to the energy are equal, as we would expect. Now adding the expressions (4.9) and (4.12) we obtain for the electromagnetic energy $W = wa$ per unit surface area, at zero temperature,

$$W = -\frac{n^2 a}{\pi^2}\int_0^\infty d\zeta\,\zeta^2 \int_0^\infty \frac{k_\perp dk_\perp}{\kappa d}. \tag{4.13}$$

Here a standard frequency rotation $\omega \to i\zeta$ has been performed. This expression can be further processed by introducing new coordinates $X = k_\perp = \kappa\cos\theta$, $Y = n\zeta = \kappa\sin\theta$, with $\kappa = \sqrt{k_\perp^2 + n^2\zeta^2}$. The integral over θ from 0 to $\pi/2$ is elementary, and we get

$$W = -\frac{1}{48\pi^2 na^3}\int_0^\infty \frac{z^3 dz}{e^z - 1} = -\frac{\pi^2}{720na^3}, \tag{4.14}$$

in accordance with Eqs. (3.17) and (3.18).

At arbitrary temperature T we replace $\zeta \to \zeta_m = 2\pi mT$, and make the conventional substitution

$$\int_0^\infty \frac{d\zeta}{2\pi} \to T\sum_{m=0}^\infty{}', \tag{4.15}$$

whereby we get

$$W = -8\pi n^2 aT^3 \sum_{m=1}^\infty m^2 \int_0^\infty \frac{k_\perp dk_\perp}{\kappa d}, \tag{4.16}$$

with $\kappa = \sqrt{k_\perp^2 + (2\pi nmT)^2}$. Alternatively, we may write

$$W = -4\pi n^2 T^3 \sum_{m=1}^\infty m^2 \int_{\alpha m}^\infty \frac{dz}{e^z - 1}, \tag{4.17}$$

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where $\alpha = 4\pi naT$ as before.

At high temperature, $aT \gg 1$, it is easy to check that W agrees with U calculated previously. We approximate the integral in Eq. (4.17) by $\int_{\alpha m}^{\infty} e^{-z}dz = e^{-\alpha m}$, and so get

$$W \to -4\pi n^2 T^3 e^{-4\pi naT} \tag{4.18}$$

when $m = 1$, in agreement with Eq. (3.7).

We shall not delve further into a detailed study of the equality between W and U in the case of arbitrary T. The equality should be clear on physical grounds, since we are dealing with a closed thermodynamical system.

V. REMARKS ON THE DISPERSIVE CASE

Assume now that the medium in the region $0 < z < a$ is frequency dispersive, $\varepsilon = \varepsilon(\omega), \mu = \mu(\omega)$. The total energy density w_{disp} is known to be [41, 42]

$$w_{\mathrm{disp}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2 k_{\perp}}{(2\pi)^2} \left[\frac{d(\varepsilon\omega)}{d\omega} \langle E^2 \rangle_{\omega \mathbf{k}} + \frac{d(\mu\omega)}{d\omega} \langle H^2 \rangle_{\omega \mathbf{k}} \right]. \tag{5.1}$$

We can write this as a sum of two parts w_I and w_{II}, where w_I is the same expression as in Eq. (4.8) with $\varepsilon \to \varepsilon(\omega)$, $\mu \to \mu(\omega)$, and where

$$w_{II} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\omega \int \frac{d^2 k_{\perp}}{(2\pi)^2} \left[\frac{d\varepsilon}{d\omega} \langle E^2 \rangle_{\omega \mathbf{k}} + \frac{d\mu}{d\omega} \langle H^2 \rangle_{\omega \mathbf{k}} \right]. \tag{5.2}$$

Correspondingly, for the surface densities, $W_{\mathrm{disp}} = W_I + W_{II}$.

The first property to be noted in connection with Eq. (5.1) is that it is derived under the assumption of *negligible dissipation*. Some dissipation is always present—this being a consequence of Kramers-Kronig's relations—but it is a legitimate approximation to neglect it except in the neighborhood of eigenfrequencies in the cavity. This assumption means that the relaxation frequency in the dispersion relation can be set equal to zero, and we may adopt the usual dispersion relation for a dielectric,

$$\varepsilon(\omega) = 1 + \frac{\varepsilon_0 - 1}{1 - \omega^2/\omega_0^2}, \quad \mu = 1. \tag{5.3}$$

In the case of a general dissipative medium, neither the energy nor the stress tensor are derivable in terms of permittivity/permeability alone, and therefore cannot be given in a general form using macroscopic methods. (This point is discussed in detail by Ginzburg [43].)

Second, it is clear that the expression (5.1) is not intimately related to the Casimir effect as such. It is more natural to consider the problem as belonging to classical electrodynamics, namely a system of two conducting plates between which there are stationary electromagnetic oscillations. The expression (5.1) is actually obtained from the energy balance equation

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}} = 0. \tag{5.4}$$

(See, for example, Eq. (7.5) in Ref. [42].) In order to accumulate electromagnetic energy, one has to consider oscillations that are not purely monochromatic, but distributed within

a band of frequencies around each eigenfrequency. In this way external agencies, outside of the plates, are called for. It is natural here to regard the system to be a capacitor, linked to an external appropriately adjusted self-inductance L such that stationary oscillations become possible (external resistances are forbidden since we omit dissipation). That means, the plates with the intervening medium is thermodynamically a *non-closed* system. From this we can draw the important conclusion that the full dispersive energy W_{disp} is not to be identified with the thermodynamical energy $W = U$ calculated earlier. The laws of thermodynamics are applicable to *closed* systems only.

The mentioned model of a classical electromagnetic non-dissipative circuit is studied in Ref. [41]. It is instructive to consider the salient features of the argument also here:

Let the charges Q be supplied and withdrawn from the plates with frequency ω. The self-inductance of the circuit is L, as mentioned, and the electromotive force we call $\mathcal{E}$. The potential ϕ across the plates is determined from the equation

$$\phi = \mathcal{E} - L\dot{J}, \tag{5.5}$$

where $J = \dot{Q}$. The frequency of the circuit is

$$\omega = 1/\sqrt{LC(\omega)}, \tag{5.6}$$

where the capacitance $C(\omega)$ of the capacitor is determined by $\phi = Q/C(\omega)$. By considering almost monochromatic quantities [the same kind of argument that led to Eq. (5.1)], we get, when taking the average over a period,

$$\overline{\mathcal{E}J} = \frac{d}{dt}\left\{\frac{1}{2}L\overline{J^2} + \frac{1}{2}\frac{d(\omega C)}{d\omega}\overline{\phi^2}\right\}. \tag{5.7}$$

The expression between brackets is the circuit energy. From $J = -i\omega Q$ and Eq. (5.6) we get $\frac{1}{2}L\overline{J^2} = \frac{1}{2}C\overline{\phi^2}$ and so the circuit energy may be written

$$\overline{W}_{\mathrm{circ}} = \frac{1}{2\omega}\frac{d(\omega^2 C)}{d\omega}\overline{\phi^2}. \tag{5.8}$$

This expression, because of the derivative with respect to ω, is seen to be related to Eq. (5.1).

Now consider a small adiabatic displacement of the capacitor plates. As $\overline{W}_{\mathrm{circ}}/\omega$ is an adiabatic invariant,

$$\delta\overline{W}_{\mathrm{circ}} = \overline{W}_{\mathrm{circ}}\delta\omega/\omega. \tag{5.9}$$

By means of Eq. (5.6),

$$\frac{\delta\omega}{\omega} = -\frac{1}{2}\frac{\delta C}{C}. \tag{5.10}$$

The change in C consists of two parts,

$$\delta C = (\delta C)_{\mathrm{st}} + \frac{dC}{d\omega}\delta\omega, \tag{5.11}$$

where the first term is the static part and the second term depends on the frequency change. From Eqs. (5.10) and (5.11),

$$\delta C_{\mathrm{st}} = -\frac{1}{\omega^2}\frac{d(\omega^2 C)}{d\omega}\delta\omega. \tag{5.12}$$

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When Eq. (5.8) is substituted in Eq. (5.9) and (5.12) is used, $dC/d\omega$ disappears, and we get

$$\delta\overline{W}_{\rm circ} = -\frac{1}{2}\overline{\phi^2}(\delta C)_{\rm st} = -\frac{1}{2}\frac{\overline{Q^2}}{C^2}(\delta C)_{\rm st}. \tag{5.13}$$

This is the same expression as one obtains by taking the variation of the average of the energy $Q^2/2C$ of a thermally insulated capacitor. It means that when dispersion is present, the electromagnetic stress tensor contains no derivatives with respect to the frequency. The argument is general, and is not critically dependent on our choice of a capacitor model.

When applied to our case, we can thus conclude as follows:

1. The dispersive energy $W_{\rm disp}$ whose density is given in Eq. (5.1) refers thermodynamically to a non-closed system, and is therefore not to be identified with the internal energy U calculated in Sec. II starting from the free energy F, or the electromagnetic energy W calculated in Sec. III in the nondispersive case. As was demonstrated, when ε and μ are constants, $W = U$.

2. As for the electromagnetic stress tensor, the derivatives with respect to ω are not to be included. That is, the electromagnetic force can be calculated from Eq. (4.8) with $\varepsilon \to \varepsilon(\omega)$, $\mu \to \mu(\omega)$.

It may finally be noted that by inserting the simple form (5.3) for $\varepsilon(\omega)$ for a dielectric, we obtain for the dispersive correction $W_{II} = aw_{II}$ a divergent expression,

$$W_{II} = \frac{2a(\varepsilon_0 - 1)}{\omega_0^2} \int_0^\infty \frac{d\omega}{2\pi}\frac{\omega^2}{(1 - \omega^2/\omega_0^2)^2} \int \frac{d^2k_\perp}{(2\pi)^2}\langle E^2\rangle_{\omega\mathbf{k}}; \tag{5.14}$$

cf. Eq. (5.2).

Another way to to see that the dispersive medium should be treated without the frequency derivative of the permittivity is to recognize that the Casimir energy may be derived by a variation expression

$$\frac{\delta E}{A} = \frac{i}{2}\int \frac{d\omega}{2\pi}\frac{d^2k_\perp}{(2\pi)^2}dz\,\delta\varepsilon(z)g_{kk}^E(z, z, \mathbf{k}_\perp, \omega), \tag{5.15}$$

which is Eq. (2.26) of Ref. [21]. This starting point is equivalent to the variational argument recounted in this section.

VI. AN ANOMALY IN THE CASIMIR ENERGY FOR HIGHER DIMENSIONS

The last topic that we shall consider is an anomaly encountered in the case of higher spacetime dimensions, $D > 4$, reflecting the breaking of conformal symmetry. Let us assume two parallel hyperplanes with separation a, when the region $0 < z < a$ is filled with an isotropic medium of refractive index $n = \sqrt{\varepsilon\mu}$. The medium to the left ($z < 0$) is taken to be an ideal metal, as is the medium to the right ($z > a$). That is, we assume the same configuration as earlier, in Secs. II-IV, except that we allow now the number of spatial dimensions $d = D - 1$ to be greater than 3. Casimir forces in spacetimes with dimensions $D > 4$ were first extensively studied by Ambjørn and Wolfram [44], whereas the anomaly of

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the type considered below was studied recently by Alnes *et al.* [36]; cf. also Refs. [45, 46]. These works assumed a vacuum in the intermediate region. The new development below is that we allow for a dielectric in this region. The anomaly is present also in the case of zero temperature, so that we shall assume $T = 0$ in the following.

The appropriate electromagnetic energy-momentum tensor is the Minkowski expression, called $S_{\mu\nu}^M$,

$$S_{\mu\nu}^M = F_{\mu\alpha}H_{\nu\alpha} - \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}H_{\alpha\beta}; \tag{6.1}$$

cf., for instance, Refs. [47–49]. (We make use of the Euclidean time coordinate, $x_4 = it$.) Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $\mu, \nu = 1, 2, ...D$ is the field tensor, whereas $F_{4k} = iE_k$ with $k = 1, 2, ...d$ are the components of the d-dimensional electric field vector $\mathbf{E}$. The magnetic induction ($\mathbf{B}$ in the three-dimensional case) does not constitute a vector in the higher-dimensional case, but is given by the $d(d-1)/2$ components of the antisymmetric spatial tensor F_{ik}. Analogously, the second D-dimensional tensor $H_{\mu\nu}$ occurring in Eq. (6.1) is given by the vector components $H_{4k} = iD_k$, $\mathbf{D}$ being the d-dimensional induction vector, and by the $d(d-1)/2$ components of the spatial magnetic field tensor H_{ik} ($\mathbf{H}$ in the three-dimensional case). In analogy with three-dimensional theory, we assume constitutive relations in the form $H_{4k} = \varepsilon F_{4k}$ and $F_{ik} = \mu H_{ik}$ also when $D > 4$.

Turning now to the calculation of physical quantities in the higher-dimensional theory, we shall start with the surface pressure P on the hyperplane $z = 0$. We then first observe that the usual expression for F (cf. Eq. (2.3)) can easily be generalized to the case of d spatial dimensions. Taking into account that there are $(D-2)$ physical degrees in the field in the cavity, we have, for $T = 0$,

$$F = (D-2)\int_0^\infty \frac{d\zeta}{2\pi} \int \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}} \ln(1 - e^{-2\kappa a}), \tag{6.2}$$

where

$$\kappa^2 = k_\perp^2 + n^2\zeta^2, \quad k_\perp^2 \equiv k_x^2 + k_y^2 + ... + k_{D-2}^2. \tag{6.3}$$

The volume element in momentum space is $d^{d-1}k_\perp = \Omega_{d-2} k_\perp^{d-2}dk_\perp$, where the solid angle is determined by

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{6.4}$$

The pressure $P = -\partial F/\partial a$ can now be written

$$P = -\frac{2(D-2)}{(2\pi)^d}\Omega_{d-2}\int_0^\infty d\zeta \int \frac{\kappa k_\perp^{d-2}dk_\perp}{e^{2\kappa a} - 1}. \tag{6.5}$$

Here the double integral over ζ and $k_\perp$ can be further processed by introducing polar coordinates; cf. Ref. [21]. As in Sec. IV, we introduce $X = \kappa\cos\theta = k_\perp$, $Y = \kappa\sin\theta = n\zeta$, satisfying $X^2 + Y^2 = \kappa^2$. The area element in the XY plane is $\kappa d\kappa d\theta = n dk_\perp d\zeta$. The double integral therewith becomes

$$\int_0^\infty d\zeta \int \frac{\kappa k_\perp^{d-2}dk_\perp}{e^{2\kappa a} - 1} = \frac{1}{n}\int_0^{\pi/2} \cos^{d-2}\theta d\theta \int_0^\infty \frac{\kappa^d d\kappa}{e^{2\kappa a} - 1}. \tag{6.6}$$

We now make use of the formulas

$$\int_0^{\pi/2} \cos^{d-2}\theta d\theta = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}, \tag{6.7}$$

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$$\int_0^\infty \frac{\kappa^d d\kappa}{e^{2\kappa a} - 1} = \frac{\Gamma(D)\zeta(D)}{(2a)^D}, \tag{6.8}$$

$$\Gamma(D) = \frac{2^{D-1/2}}{\sqrt{2\pi}} \Gamma\left(\frac{D}{2}\right) \Gamma\left(\frac{D+1}{2}\right), \tag{6.9}$$

and insert the expression for Ω_{d-2}, to get

$$P = -\frac{(D-2)(D-1)}{n} \frac{\Gamma(D/2)\zeta(D)}{(4\pi)^{D/2} a^D}. \tag{6.10}$$

This expression is quite simple; no regularization procedure was required to achieve it. The presence of the medium is apparent only in the factor n in the denominator. If $n = 1$, including the case of a vacuum as well as the case of a "relativistic" medium satisfying $\varepsilon = 1/\mu$, the expression reduces to that derived earlier [44]. This result parallels that obtained in the $T = 0$ parts of the energy, cf. Eqs. (3.17) and (3.18). It is nearly identical to the result found in Ref. [40] for the scalar case in D dimensions [Eq. (2.35) there], differing only in the evident factor $(D-1)/n$.

Considering now the electromagnetic field energy density w in the cavity, we find that the situation becomes more delicate. The natural way to calculate w is via the energy-momentum tensor. This procedure - carried out by Alnes *et al.* in the case of a vacuum cavity [36, 46] - led in the case of metallic boundary conditions to the result

$$w = -\frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2} a^D} \left[\zeta(D) + \left(\frac{D}{2} - 2\right) f_D\left(\frac{z}{a}\right)\right] \equiv w_1 + w_2, \tag{6.11}$$

where

$$f_D\left(\frac{z}{a}\right) = \zeta_H\left(D, \frac{z}{a}\right) + \zeta_H\left(D, 1 - \frac{z}{a}\right), \tag{6.12}$$

ζ_H being the Hurwitz zeta function. Note that the first term yields the pressure (6.10),

$$-\frac{\partial}{\partial a} a w_1 = P, \tag{6.13}$$

so that the second term in the energy density, w_2, which diverges like z^{-D} close to the surface when $D > 4$, does not contribute to the force between the plates. It can explicitly seen that written in physical variables this term is independent of the separation between the plates and hence does not contribute to the force. This anomaly can actually be seen to manifest itself in another way if we go back to the expression (6.1) for the energy-momentum tensor: its trace $S_{\mu\mu}^M$ is *nonvanishing* when $D > 4$. Physically, as emphasized in Ref. [36], the divergent self energy of a single surface is related to the lack of conformal invariance of the electromagnetic Lagrangian for $D > 4$. All of this is exactly as seen in Ref. [40], Chap. 11, for the scalar field.

It turns out that the anomaly can be regularized away by subtracting off the self energy for both plates. Then, the second term in Eq. (6.11) is absent, and only the first, finite, terms in w remains.

As mentioned, these calculations of w were made for the case of a vacuum cavity. The result was found via a combination of dimensional and zeta function regularizations [50]. The result could be recalculated for a medium cavity, but such a detailed calculation is hardly justified in view of the simple occurrence of n in the expression (6.10). In fact, since

in physical units, $w = \hbar c/a^D$ times a function of D, it is clear (for example, Ref. [42], Eq. (36.12)) that all we have to do to insert a uniform medium between the plates is replace c by c/n; this shows that the same factor n will appear in the denominator of the expression for w as it did in W or P. Thus, after regularization, we obtain the relationship

$$P = (D - 1)w_1, \tag{6.14}$$

which is the same connection as for a vacuum.

VII.   CONCLUSIONS

The purpose of this paper is to review the conventional theory of the interaction between metallic and dielectric plates, with the purpose of illuminating those points where anomalies might be present. Thus, a question might arise about the incorporation of dispersion into the Lifshitz formula, which we address in Sec. V. The interplay between local surface energy divergences and the breaking of conformal symmetry is revisited in Sec. VI, with the new element in our analysis being the inclusion of a refractive index between the planes. The clarity brought to bear by these analyses will now allow us to understand more fully the physical controversies surrounding both temperature dependence in metals and semiconductors, and in the question of surface energies and their significance.

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