Addendum to Solution to Problem 3.3

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A number of you imposed a condition on w. This is not necessary, the equations of motion are unchanged by the addition of a total derivative of an arbitrary function to the Lagrangian. The problem is that you assumed, erroneously, that the Euler-Lagrange equations hold, which is only true if $L(q, \dot{q}, t)$. The action principle makes no such restriction.

Let's suppose

$$L(q, \dot{q}, \ddot{q}, \dots, t).$$

Then the action principle gives for the equation of motion, upon varying with respect to q and integrating by parts suitably

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} + \dots = 0.$$

We know from the action principle that this equation of motion is unaltered when I replace

$$L \to L + \frac{dw}{dt}.$$

How does this work explicitly? That is, we want to show identically that

$$\frac{\partial \dot{w}}{\partial q} - \frac{d}{dt}\frac{\partial \dot{w}}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial \dot{w}}{\partial \ddot{q}} + \dots = 0. \tag{1}$$

By using

$$\dot{w} = \frac{d}{dt}w = \frac{\partial w}{\partial q}\dot{q} + \frac{\partial w}{\partial \dot{q}}\ddot{q} + \frac{\partial w}{\partial \ddot{q}}\dddot{q} + \dots + \frac{\partial w}{\partial t}$$

it is easy to see that

$$\begin{aligned}
\frac{\partial \dot{w}}{\partial q} &= \frac{d}{dt}\frac{\partial w}{\partial q} \\
\frac{\partial \dot{w}}{\partial \dot{q}} &= \frac{d}{dt}\frac{\partial w}{\partial \dot{q}} + \frac{\partial w}{\partial q}, \\
\frac{\partial \dot{w}}{\partial \ddot{q}} &= \frac{d}{dt}\frac{\partial w}{\partial \ddot{q}} + \frac{\partial w}{\partial \dot{q}},
\end{aligned}$$

etc. When this is inserted in Eq. (1) it is easy to see that every term cancels in pairs, so that equation is identically satisfied as required.