

Physics 6433, Quantum Field Theory
Assignment #9
Due Monday, April 20

April 13, 2015

1. Prove that the unrenormalized, but regulated, proper (one-particle irreducible) Green's functions and the renormalized proper Green's functions are related by

$$\Gamma_R^{(n)}(p_1, \dots, p_n; \lambda_R, m_R, \mu, \epsilon) = Z^{n/2} \Gamma_0^{(n)}(p_1, \dots, p_n; \lambda_0, m_0, \epsilon).$$

2. Derive the formula for the superficial degree of divergence of a ϕ^N theory in d dimensions. Show therefore:
 - (a) In $d = 4$ dimensions, that if $N > 4$ there are an infinite number of primitively divergent diagrams. (The more vertices, the more divergent.) Such a theory is not renormalizable.
 - (b) In $d = 2$ dimensions, that the degree of divergence does not depend on N , and that the more vertices the more *convergent*!
 - (c) In $d = 6$ dimensions, $\lambda\phi^3$ possesses only a finite number of primitively divergent graphs (they are the one-, two-, and three-point functions).
 - (d) In $d \geq 7$ dimensions, there are no theories with a finite number of primitively divergent diagrams.
3. In four dimensions, find all primitively divergent diagrams for the ϕ^3 theory. For each give an example in lowest order of perturbation theory.

4. Contrary to naive expectation, vacuum energy is observable and important. The purpose of these two problems is to illustrate the point, and calculate what is called the Casimir energy for a scalar field confined between parallel plates.

Begin by recalling the expression for the energy-momentum tensor of a scalar field, Problem 1.8,

$$t^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + g^{\mu\nu} \mathcal{L}.$$

Show that the vacuum expectation value of t^{00} , the energy density, may then be written as

$$\langle t^{00} \rangle = \lim_{x \rightarrow x'} \partial^0 \partial'^0 \frac{1}{i} G(x, x'),$$

by recalling Problem 5.2. Now obtain a Fourier decomposition representation for G appropriate to the region between two parallel plates perpendicular to the z -axis, where the boundary condition at the plates is that ϕ is zero there. For a massless field, by carrying out the frequency integration, show that the energy/area is given by the expression

$$u = \frac{1}{2} \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} \sum_n \sqrt{k_\perp^2 + n^2 \pi^2 / a^2}, \quad (1)$$

if the distance between the plates is a . Equation (1) is just a way of writing the usual zero point energy, $\frac{1}{2} \sum \hbar \omega$.

5. Although the expression (1) is very divergent, it may be evaluated by
- (a) carrying out the k_\perp integration by use of the dimensional regularization integral (see Problem 7.2), and then
 - (b) identifying the divergent n sum by using the definition of the Riemann zeta function,

$$\sum_{n=1}^{\infty} n^m = \zeta(-m).$$

Evaluate u , and the force per unit area, $\mathcal{F} = -du/da$. [Use the fact that $\zeta(-3) = 1/120$.] This is exactly one-half the Casimir force experimentally measured to exist between parallel, uncharged, perfectly conducting plates. The extra factor of two comes from the two polarization states of the photon.