Physics 6433, Quantum Field Theory
Assignment #4
Due Wednesday, February 20, 2002

February 13, 2002

1. How is the calculation of the transformation function for the harmonic oscillator, $\langle a^{\dagger\prime}, t_1 | a'', t_2\rangle$, given in class, modified when the zero-point energy, $H_0 = \frac{1}{2}\omega$, is included in the Hamiltonian? What is the effect on the energy eigenvalues and the energy eigenfunctions?

2. From the relation between annihilation eigenvectors and energy eigenvectors,

$$
\begin{aligned}
|a'\rangle &= \sum_{n=0}^{\infty} \frac{(a')^n}{\sqrt{n!}} |n\rangle, \\
\langle a^{\dagger\prime}| &= \sum_{n=0}^{\infty} \langle n | \frac{(a^{\dagger\prime})^n}{\sqrt{n!}},
\end{aligned}
$$

   deduce a closed form for $\langle a^{\dagger\prime} | a''\rangle$ analogous to

$$
\langle p' | q' \rangle = \frac{1}{\sqrt{2\pi}} e^{-ip'q'}
$$

   .

3. Using

$$
G_q - G_p = \delta\left(\sum_a q_a \cdot p_a\right),
$$

   write $\delta\langle q' | p'\rangle$, and evaluate $\langle q' | p' \rangle$.

4. Show that the result derived in class for the forced harmonic oscillator can be written as

$$\frac{\langle a^{\dagger\prime}, t_1 | a'', t_2\rangle^K}{\langle 0, t_1 | 0, t_2\rangle^K} = \exp\left\{a^{\dagger\prime} e^{-i\omega(t_1 - t_2)} a'' - i a^{\dagger\prime} e^{-i\omega t_1}\gamma - i\gamma^* e^{i\omega t_2} a''\right\},$$

where $\gamma = K(\omega)$. Inserting a complete set of energy eigenstates gives

$$\langle a^{\dagger\prime}, t_1 | a'', t_2\rangle^K = \sum_{n,n'} \langle a^{\dagger\prime} | n\rangle \langle n, t_1 | n', t_2\rangle^K \langle n' | a''\rangle.$$

Define the *scattering matrix* $S_{n,n'}$ by

$$\langle n, t_1 | n', t_2\rangle^K = e^{-in\omega t_1} S_{n,n'} e^{in'\omega t_2}.$$

Define

$$u = -i a^{\dagger\prime} e^{-i\omega t_1}, \quad v = -i a'' e^{i\omega t_2},$$

and therefore obtain

$$e^{-uv + u\gamma + v\gamma^*} = \sum_{n,n'} \frac{u^n}{\sqrt{n!}} i^n \frac{S_{n,n'}}{S_{0,0}} i^{n'} \frac{v^{n'}}{\sqrt{n'!}}. \tag{1}$$

Note that the left side of this equation is invariant under the substitution

$$u \to \frac{\gamma^*}{\gamma} v, \quad v \to \frac{\gamma}{\gamma^*} u,$$

and thereby deduce

$$\left(\frac{\gamma^*}{\gamma}\right)^{n-n'} S_{n,n'} = S_{n',n},$$

so

$$|S_{n,n'}|^2 = |S_{n',n}|^2.$$

Now carry out a double expansion of Eq. (1) in u and v as follows. The coefficient of $v^{n'}/\sqrt{n'!}$ is

$$\frac{(\gamma^* - u)^{n'}}{\sqrt{n'!}} e^{u\gamma} = \sum_n \frac{u^n}{\sqrt{n!}} i^{n+n'} \frac{S_{n,n'}}{S_{0,0}},$$

and therefore

$$i^{n+n'} \frac{S_{n,n'}}{S_{0,0}} = \frac{1}{\sqrt{n!\, n'!}} \left(\frac{d}{du}\right)^n (\gamma^* - u)^{n'} e^{u\gamma}\Big|_{u=0}.$$

Show that this can be written in terms of the Laguerre polynomials, defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \left(\frac{d}{dx}\right)^n x^{n+\alpha} e^{-x},$$

as follows

$$i^{n+n'} \frac{S_{n,n'}}{S_{0,0}} = \sqrt{\frac{n!}{n'!}} (-1)^n (\gamma^*)^{n'-n} L_n^{(n'-n)}(|\gamma|^2).$$

Check this result for $n = 0$. [We have then $L_0^{(n')} = 1$; and we already know $|S_{0,0}|^2 = e^{-\gamma^2}$.] You should obtain the known result for $|S_{n,0}|^2$. Also check the completeness relation

$$\sum_{n'} |S_{n,n'}|^2 = 1;$$

and in this way learn something about Laguerre polynomials.