Physics 6433, Quantum Field Theory
Assignment #2
Due Monday, February 9, 2015

February 3, 2015

1. The point of these first two problems is to explore operator variations in the Schwinger action principle. First, note that for one continuous degree of freedom (primes denote eigenvalues of the corresponding operators)
$$U(q',p') = e^{iq'p-ip'q}$$
forms a complete operator basis. That is, for an arbitrary operator function X of q and p,
$$X = \int \frac{dq'\,dp'}{2\pi} U(q',p')\mathrm{Tr}(U(q',p')^\dagger X),$$
which is a kind of operator Fourier integral. From this follows the orthogonality statement:
$$\mathrm{Tr}\left[U(q'',p'')^\dagger U(q',p')\right] = 2\pi\delta(q'-q'')\delta(p'-p'').$$
Introduce the Fourier coefficients
$$F(q,p) = \int \frac{dq'\,dp'}{2\pi} U(q',p') f(q',p'),$$
and show that if the operator F is Hermitian, the Fourier coefficients satisfy
$$f(q',p') = f(-q',-p')^*,$$

and for two Hermitian operators F and G compute, in terms of their Fourier coefficients, their commutator and anticommutator, or

$$\frac{1}{i}[F, G], \quad \{F, G\} \equiv 2F.G.$$

Show that

$$\frac{1}{i}[F, G] = -\tan \frac{1}{2}\left[\frac{\partial}{\partial p_F}\frac{\partial}{\partial q_G} - \frac{\partial}{\partial p_G}\frac{\partial}{\partial q_F}\right]\{F, G\},$$

where the subscripts indicate the function to be differentiated.

2. Now consider an operator variation of the action, that is, make variations in q_a and p_a, δq_a, δp_a, that are operators. Deduce that it is still fair to conclude

$$\frac{\partial H}{\partial p_a} = \frac{dq_a}{dt}, \quad -\frac{\partial H}{\partial q_a} = \frac{dp_a}{dt}.$$

For a general variation with t fixed,

$$\delta H = \sum_a \left[\delta q_a . \frac{\partial H}{\partial q_a} + \delta p_a . \frac{\partial H}{\partial p_a}\right],$$

so with

$$q_a - \delta q_a = U^{-1} q_a U,$$
$$p_a - \delta p_a = U^{-1} p_a U,$$

where

$$U = 1 + iG,$$

show that

$$\begin{aligned}\frac{1}{i}[H, G] &= \sum_a \left[\frac{\partial H}{\partial q_a} . \frac{\partial G}{\partial p_a} - \frac{\partial H}{\partial p_a} . \frac{\partial G}{\partial q_a}\right] \\ &= 2 \tan \left[\sum_a \frac{1}{2}\left(\frac{\partial}{\partial q_{a,H}}\frac{\partial}{\partial p_{a,G}} - \frac{\partial}{\partial p_{a,H}}\frac{\partial}{\partial q_{a,G}}\right)\right] H.G\end{aligned}$$

from the previous problem. This equality will hold true, for arbitrary H, if G is less than cubic in q's and p's. [It will also be true for arbitrary G if H is quadratic (simple harmonic oscillator).] All the familiar geometric transformations are included in this class.

3. Show that the effect of changing the Lagrangian by a total time derivative,

$$\bar{L} = L - \frac{d}{dt}w,$$

is only to change the generators, and not the equations of motion. How do the generators change?

4. From the Schwinger action principle, by considering endpoint variations, derive the Schrödinger equation,

$$-\frac{1}{i}\frac{\delta\langle t_1|t_2\rangle}{\delta t_1} = \langle t_1|H|t_2\rangle,$$

and the coördinate-space representation of the momentum operator,

$$\frac{1}{i}\frac{\delta\langle t_1|t_2\rangle}{\delta q_1'} = \langle t_1|p_1|t_2\rangle.$$