Physics 6433, Quantum Field Theory
Assignment #1
Due Monday, January 26, 2015

January 12, 2015

1. Consider the description of a single nonrelativistic particle in a central potential, $V(r)$. First introduce spherical polar coordinates r, θ, ϕ:

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

   Write the Lagrangian in polar coordinates. What are Lagrange's equations? Find the canonical momenta p_r, p_θ, p_ϕ. Write the Hamiltonian and the three Hamilton's equations. What is the physical significance of p_θ? Compute $\{p_\theta, H\}$.

2. Prove the following identities concerning Poisson brackets:

$$\{A, BC\} = \{A, B\}C + B\{A, C\}, \tag{1}$$

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{2}$$

3. A *vector* is an object which transforms in the same way as the position vector $\mathbf{r}$ under a rotation. Therefore, the infinitesimal change in a vector $\mathbf{V}$ under an infinitesimal rotation of the coordinate system is $\delta\mathbf{V} = \delta\boldsymbol{\omega} \times \mathbf{V}$. If $\mathbf{V}$ is constructed from $\mathbf{r}$ and $\mathbf{p}$, show that

$$\delta\mathbf{V} = \{\mathbf{L} \cdot \delta\boldsymbol{\omega}, \mathbf{V}\},$$

where we have used the Poisson bracket notation, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. [Hint: It may help to write $\mathbf{V}$ in the general form

$$V = a\mathbf{r} + b\mathbf{p} + c\mathbf{r} \times \mathbf{p},$$

where a, b, and c are functions of the scalars r, p, and $\mathbf{r} \cdot \mathbf{p}$.]

4. The generator of *dilatations* is defined by

$$D = \sum_i q_i p_i.$$

If A is a homogeneous function of the coordinates of degree n, i.e., if

$$A(\lambda q_i) = \lambda^n A(q_i),$$

show that

$$\{D, A\} = nA,$$

where again $\{\,,\}$ represents the Poisson bracket.

5. Verify that when we consider a rigid rotation

$$\delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}$$

in classical particle electrodynamics, the corresponding generator is

$$G = \delta\boldsymbol{\omega} \cdot \mathbf{J},$$

where the angular momentum is

$$\mathbf{J} = \sum_k \mathbf{r}_k \times m_k \mathbf{v}_k + \frac{1}{c} \int (d\mathbf{r})\, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}).$$

6. Verify that the coordinate transformation law for the field strength tensor,

$$\delta F_{\mu\nu} = -\delta x^\lambda \partial_\lambda F_{\mu\nu} - (\partial_\mu \delta x^\lambda) F_{\lambda\nu} - (\partial_\nu \delta x^\lambda) F_{\mu\lambda},$$

is consistent with that for the vector potential

$$\delta A_\mu = -\delta x^\lambda \partial_\lambda A_\mu - (\partial_\mu \delta x^\lambda) A_\lambda.$$

7. Using the free Maxwell equations, verify directly the energy-momentum conservation equation,
$$\partial_\mu t^{\mu\nu} = 0.$$
How is this equation modified if a charge density j_μ is present?

8. A scalar particle of mass m is described by a field $\phi(x)$ and is governed by the Lagrange density
$$\mathcal{L} = -\frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2\right).$$
Under an infinitesimal coordinate transformation, $\phi(x)$ transforms according to
$$\delta\phi(x) = -\delta x^\nu(x)\partial_\nu\phi(x).$$
As in the lecture, compute the corresponding energy-momentum tensor $t^{\mu\nu}$.

9. Scale invariance is violated by the energy-momentum tensor computed in the above problem, because, as $m \to 0$, $t = t^\mu{}_\mu \not\to 0$. To what limit does t converge? We can, however, introduce a new energy-momentum tensor, $\theta^{\mu\nu}$, by adding to $t^{\mu\nu}$ an identically conserved term,
$$\theta^{\mu\nu} = t^{\mu\nu} + a\left(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2\right)\phi^2.$$
Determine the number a so that $\theta = \theta^\mu{}_\mu \to 0$ as $m \to 0$. This is the so-called conformal stress tensor. Argue that $\theta^{\mu\nu}$ and $t^{\mu\nu}$ are equally as good at describing the energy and momentum of the system. [There is an inherent arbitrariness in defining the energy-momentum tensor.]