Chapter 3

Scalar Field Theory

3.1 Canonical Formulation

The dispersion relation for a particle of mass m is

$$E^2 = \mathbf{p}^2 + m^2, \quad \mathbf{p}^2 = \mathbf{p} \cdot \mathbf{p}, \tag{3.1}$$

or, in relativistic notation, with $p^0 = E$,

$$0 = p_\mu p^\mu + m^2 \equiv p^2 + m^2. \tag{3.2}$$

A wave equation corresponding to this relation is

$$(-\partial^2 + m^2)\phi(x) = 0, \tag{3.3}$$

which is solved by a plane wave,

$$\phi(x) = e^{ip^\mu x_\mu} = e^{ip\cdot x}, \quad p^2 + m^2 = 0, \tag{3.4}$$

where we have introduced the scalar product

$$p \cdot x = p_\mu x^\mu = \mathbf{p} \cdot \mathbf{x} - p^0 t. \tag{3.5}$$

A Lagrange density which leads to this wave equation is

$$\begin{aligned} \mathcal{L}(x) &= -\frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + m^2\phi^2\right) \\ &= \frac{1}{2}\left(\dot{\phi}^2 - \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi - m^2\phi^2\right). \end{aligned} \tag{3.6}$$

The canonical momentum density is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x), \tag{3.7}$$

which means that the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{3.8}$$

As in the previous chapter, it is worthwhile to make a decomposition of ϕ into creation and annihilation operators,

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2\omega_k}\left[a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\ &= \int \frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2\omega_k}\left[a_{\mathbf{k}}e^{-i\omega_k t}e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}}e^{i\omega_k t}e^{-i\mathbf{k}\cdot\mathbf{x}}\right],\end{aligned} \tag{3.9}$$

where

$$a_{\mathbf{k}} = a_{\mathbf{k}}(0), \quad \omega_k = +\sqrt{\mathbf{k}^2 + m^2}. \tag{3.10}$$

Appearing here is a relativistically invariant momentum-space measure,

$$d\tilde{k} \equiv \frac{(d\mathbf{k})}{(2\pi)^3 2\omega_k} = \frac{(dk)}{(2\pi)^4}2\pi\delta(k^2+m^2)\eta(k^0), \tag{3.11}$$

since

$$\int_{-\infty}^{\infty} dk^0\,\delta((k^0)^2-\omega_k^2)\eta(k^0) = \frac{1}{2\omega_k}. \tag{3.12}$$

Thus we may write the plane wave expansion (3.9) of the scalar field as

$$\phi(x) = \int d\tilde{k}\left(a_k\, e^{ik\cdot x} + a^\dagger_k\, e^{-ik\cdot x}\right), \quad k^0 = \omega_k. \tag{3.13}$$

The canonical momentum density is

$$\pi(x) = \dot{\phi}(x) = -i\int d\tilde{k}\,\omega_k\left(a_k e^{ik\cdot x} - a^\dagger_k e^{-ik\cdot x}\right). \tag{3.14}$$

Since we must have the following *equal-time commutation relations*,

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\delta(\mathbf{x}-\mathbf{x}'), \tag{3.15a}$$
$$[\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] = 0, \tag{3.15b}$$
$$[\pi(\mathbf{x},t),\pi(\mathbf{x}',t)] = 0, \tag{3.15c}$$

the creation and annihilation operators satisfy the following commutation relations

$$[a_k, a^\dagger_{k'}] = 2\omega_k(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{3.16a}$$
$$[a_k, a_{k'}] = [a^\dagger_k, a^\dagger_{k'}] = 0. \tag{3.16b}$$

We verify this claim by the following straightforward computations for $x^0 = x'^0 = t$:

$$\begin{aligned}[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] &= -i\int d\tilde{k}\,d\tilde{k}'\,\omega_{k'}\left\{-[a_k,a^\dagger_{k'}]e^{ik\cdot x - ik'\cdot x'} + [a^\dagger_k, a_{k'}]e^{-ik\cdot x + ik'\cdot x'}\right\} \\ &= i\int\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2}\left\{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right\} \\ &= i\delta(\mathbf{x}-\mathbf{x}'),\end{aligned} \tag{3.17}$$

while

$$
\begin{aligned}
[\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] &= \int d\tilde{k}\, d\tilde{k}' \left\{[a_k, a^\dagger_{k'}]e^{ik\cdot x-ik'\cdot x'} + [a^\dagger_k, a_{k'}]e^{-ik\cdot x+ik'\cdot x'}\right\} \\
&= \int d\tilde{k} \left\{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} - e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\right\} \\
&= 0,
\end{aligned}
\tag{3.18}
$$

because $\int d\tilde{k} \to \int d\tilde{k}$ when $\mathbf{k} \to -\mathbf{k}$, and likewise for the π-π commutator.

Now the Hamiltonian, from Eq. (3.8), is

$$
\begin{aligned}
H = \int (d\mathbf{x})\,\mathcal{H} = \int (d\mathbf{x})\, d\tilde{k}\, d\tilde{k}' \Big\{ &-\frac{1}{2}\omega_k\omega_{k'} \left(a_k e^{ik\cdot x} - a^\dagger_k e^{-ik\cdot x}\right) \\
&\times \left(a_{k'} e^{ik'\cdot x} - a^\dagger_{k'} e^{-ik'\cdot x}\right) \\
&- \frac{1}{2}\mathbf{k}\cdot\mathbf{k}' \left(a_k e^{ik\cdot x} - a^\dagger_k e^{-ik\cdot x}\right)\left(a_{k'} e^{ik'\cdot x} - a^\dagger_{k'} e^{-ik'\cdot x}\right) \\
&+ \frac{1}{2}m^2 \left(a_k e^{ik\cdot x} + a^\dagger_k e^{-ik\cdot x}\right)\left(a_{k'} e^{ik'\cdot x} + a^\dagger_{k'} e^{-ik'\cdot x}\right)\Big\}.
\end{aligned}
\tag{3.19}
$$

We first note that the quantity

$$
\int (d\mathbf{x})\, e^{ik\cdot x+ik'\cdot x} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')e^{-2i\omega_k t} \tag{3.20}
$$

has a vanishing coefficient:

$$
\left(-\frac{1}{2}\omega_k^2 + \frac{1}{2}\mathbf{k}^2 + \frac{1}{2}m^2\right) a_{\mathbf{k}}a_{-\mathbf{k}} = 0. \tag{3.21}
$$

The same thing happens with the coefficient of

$$
\int (d\mathbf{x})\, e^{-ik\cdot x-ik'\cdot x} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')e^{2i\omega_k t}. \tag{3.22}
$$

But for the cross terms

$$
\int (d\mathbf{x})\, e^{ik\cdot x-ik'\cdot x} = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \tag{3.23}
$$

has the numerical factor

$$
\frac{1}{2}\omega_k^2 + \frac{1}{2}\mathbf{k}^2 + \frac{1}{2}m^2 = \omega_k^2, \tag{3.24}
$$

so the Hamiltonian is simply

$$
H = \frac{1}{2}\int d\tilde{k}\,\omega_k \left(a_k a^\dagger_k + a^\dagger_k a_k\right). \tag{3.25}
$$

This expresses the Hamiltonian of the system as a sum of harmonic oscillator energy operators.

The *vacuum* state $|0\rangle$ is the state of lowest energy; it is the no-particle state, annihilated by all a_k:

$$a_k|0\rangle = 0, \quad \forall\,\mathbf{k}. \tag{3.26}$$

It will be a *zero-energy* state if we *normal-order* H:

$$:H: = \int d\tilde{k}\,\frac{1}{2}\omega_k :(a_k a_k^\dagger + a_k^\dagger a_k): \equiv \int d\tilde{k}\,\omega_k\, a_k^\dagger a_k, \tag{3.27}$$

where $:\dots:$ means that we put all annihilation operators to the *right* of all creation operators (i.e., forget the commutators). The zero-point energy is thus suppressed. (Yet the zero-point energy has been experimentally observed in the Casimir effect, which is the basis of the van der Waals forces which hold the world together!)

3.2 Path Integral Formulation

Let us now write down, in analogy to the formula (2.123), the path-integral formulation of the vacuum persistence amplitude for a free scalar field, described by the Lagrangian (3.6), in the presence of a source K:

$$\begin{aligned}\langle 0,+\infty|0,-\infty\rangle^K &= \int[d\phi]\exp\left\{i\int(dx)\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + K\phi\right]\right\}\\ &= Z_0[K],\end{aligned} \tag{3.28}$$

where the notation suggests an analogy (it is deeper than that) with the partition function of statistical mechanics. As before, we can actually do the ϕ integral here. As in Sec. 2.3.1, this is most easily done by introducing a Fourier transform:

$$\phi(x) = \int\frac{(dp)}{(2\pi)^4}e^{ip\cdot x}\phi(p), \tag{3.29}$$

so that the action is

$$\begin{aligned}&\int(dx)\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + K\phi\right]\\ &= \int\frac{(dp)}{(2\pi)^4}\phi(-p)\left[-\frac{1}{2}(p^2+m^2)\right]\phi(p) + K(p)\phi(-p)\\ &= -\frac{1}{2}\int\frac{(dp)}{(2\pi)^4}\left\{\left[\phi(-p)(p^2+m^2) - K(-p)\right]\frac{1}{p^2+m^2}\left[\phi(p)(p^2+m^2) - K(p)\right]\right.\\ &\qquad\left. - K(-p)\frac{1}{p^2+m^2}K(p)\right\}.\end{aligned} \tag{3.30}$$

Again we can shift the integration variable so we can write

$$\begin{aligned}Z_0[K] &= \int[d\phi]e^{-\frac{i}{2}\int\frac{(dp)}{(2\pi)^4}\phi(-p)(p^2+m^2)\phi(p)}e^{\frac{i}{2}\int\frac{(dp)}{(2\pi)^4}K(-p)(p^2+m^2)^{-1}K(p)}\\ &= Z_0[0]\exp\left[\frac{i}{2}\int\frac{(dp)}{(2\pi)^4}K(-p)\frac{1}{p^2+m^2}K(p)\right],\end{aligned} \tag{3.31}$$

where

$$Z_0[0] = \langle 0, +\infty | 0, -\infty \rangle^{K=0} = 1. \tag{3.32}$$

To give meaning to the remaining momentum-space integral, quadratic in K, we note that we should have, once again, inserted a convergence factor:

$$e^{-\epsilon \int (dx) \frac{1}{2}\phi^2} = e^{i \int (dx) \frac{i\epsilon}{2}\phi^2}, \tag{3.33}$$

which is equivalent to adding an imaginary part to m^2:

$$m^2 \to m^2 - i\epsilon. \tag{3.34}$$

Thus we have, analogously to Eq. (2.112),

$$\begin{aligned} Z_0[K] &= \exp\left[\frac{i}{2}\int \frac{(dp)}{(2\pi)^4} K(-p) \frac{1}{p^2 + m^2 - i\epsilon} K(p)\right] \\ &= \exp\left[\frac{i}{2}\int (dx)(dx') K(x) \Delta_+(x - x') K(x')\right], \end{aligned} \tag{3.35}$$

where the *propagation function* (or propagator or Green's function) is

$$\Delta_+(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip\cdot(x-x')}}{p^2 + m^2 - i\epsilon}. \tag{3.36}$$

This is often called the Feynman propagator. Note that Δ_+ satisfies an inhomogeneous Klein-Gordon equation,

$$(-\partial^2 + m^2)\Delta_+(x - x') = \delta(x - x'). \tag{3.37}$$

3.3 Euclidean Space

Another way to define this propagator is to go into four-dimensional Euclidean space,

$$i|x^0 - x'^0| \to |x_4 - x'_4|. \tag{3.38}$$

Then, in momentum space, we perform a complex frequency rotation,

$$-ip_0 \to p_4, \quad (dp) = dp_0(d\mathbf{p}) \to i(dp)_E. \tag{3.39}$$

Then the Minkowski space propagator is related to the corresponding Euclidean one by

$$\Delta_+(x - x') \to i \int \frac{(dp)_E}{(2\pi)^4} \frac{e^{ip_\mu (x-x')_\mu}}{p_\mu p_\mu + m^2} \equiv i\Delta_E(x - x'), \tag{3.40}$$

where the Euclidean scalar product is, for example,

$$p_\mu p_\mu = p_1^2 + p_2^2 + p_3^2 + p_4^2 \equiv p_E^2. \tag{3.41}$$

The corresponding transformation on the configuration-space quantities,

$$(dx) = dx^0\,(d\mathbf{x}) \to -i(dx)_E, \quad (\partial\phi)^2 \to (\partial\phi)^2_E, \tag{3.42}$$

so the corresponding path integral for the generating function becomes

$$Z_0[K] \to \int [d\phi] \exp\left\{-\int (dx)_E \left[\frac{1}{2}(\partial\phi)^2_E + \frac{1}{2}m^2\phi^2 - K\phi\right]\right\}, \tag{3.43}$$

or

$$Z_{0,E}[K] = \exp\left[\frac{1}{2}\int (dx)_E (dx')_E K(x)\Delta_E(x-x')K(x')\right]. \tag{3.44}$$

Let us be more precise. Starting from Eq. (3.36) we carry out the p_0 integration:

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{ip_0(x-x')^0}}{-(p_0-\omega_p+i\epsilon)(p_0+\omega_p-i\epsilon)} = \frac{i}{2\omega_p} e^{-i\omega_p|x^0-x'^0|}, \tag{3.45}$$

where $\omega_p = +\sqrt{\mathbf{p}^2+m^2}$. Here, for $(x-x')^0 > 0$, we have closed the p_0 contour with an infinite semicircle in the upper half plane, encircling the pole at $p_0 = -\omega_p + i\epsilon$ in a positive sense, while for $(x-x')^0 < 0$ we close the contour in the lower half plane, and encircle the pole at $p_0 = +\omega_p - i\epsilon$ in a negative sense. Thus we find

$$\Delta_+(x-x') = \Delta_+(x'-x) = \begin{cases} i\int d\tilde{p}\, e^{ip\cdot(x-x')}, & (x-x')^0 > 0, \\ i\int d\tilde{p}\, e^{-ip\cdot(x-x')}, & (x-x')^0 < 0, \end{cases} \tag{3.46}$$

where now $p^0 = -p_0 = \omega_p$, and we have noted that

$$\int (d\mathbf{p})\, f(\mathbf{p}^2) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} = \int (d\mathbf{p})\, f(\mathbf{p}^2) e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{3.47}$$

The form of the propagation function exhibited in Eq. (3.46) shows that positive frequency excitations move forward in time.

We can now perform the Euclidean transformation (3.38), which gives

$$\Delta_+(x-x') \to i\Delta_E(x-x') = i\int \frac{(d\mathbf{p})}{(2\pi)^3} \frac{1}{2\omega_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-\omega_p|x_4-x_4'|}. \tag{3.48}$$

But now we use the identity

$$\frac{1}{2\omega_p} e^{-\omega_p|x_4-x_4'|} = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{e^{ip_4(x_4-x_4')}}{p_4^2+\omega_p^2}, \tag{3.49}$$

which again is easily verified by the residue theorem. Thus we have

$$\begin{aligned} \Delta_E(x-x') &= \int \frac{(d\mathbf{p})\, dp_4}{(2\pi)^4} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+ip_4(x_4-x_4')}}{p_4^2+\mathbf{p}^2+m^2} \\ &= \int \frac{(dp)_E}{(2\pi)^4} \frac{e^{ip\cdot(x-x')_E}}{p_E^2+m^2}, \end{aligned} \tag{3.50}$$

as stated in Eq. (3.40).

Of course, we can rotate back,

$$|x_4 - x_4'| \to i|x^0 - x'^0|, \tag{3.51}$$

to recover Δ_+, but we can also keep rotating in the same sense:

$$|x_4 - x_4'| \to -i|x^0 - x'^0|, \tag{3.52}$$

so the Euclidean propagator becomes

$$\begin{aligned}\Delta_E(x-x') \to& \int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+i\omega_p|x^0-x'^0|}\\ =& \int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2\omega_p}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+i\omega_p|x^0-x'^0|}\\ =& \; i\Delta_-(x-x'),\end{aligned} \tag{3.53}$$

that is,

$$\Delta_-(x-x') = \begin{cases} -i\int d\tilde{p}\, e^{-ip\cdot(x-x')}, & (x-x')^0 > 0,\\ -i\int d\tilde{p}\, e^{ip\cdot(x-x')}, & (x-x')^0 < 0.\end{cases} \tag{3.54}$$

Since, once again an elementary contour integration yields

$$\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{e^{ip_0(x-x')^0}}{-p_0^2+\omega_p^2+i\epsilon} = -\frac{i}{2\omega_p}e^{i\omega_p|x^0-x'^0|}, \tag{3.55}$$

we have

$$\Delta_-(x-x') = \Delta_-(x'-x) = \int \frac{(dp)}{(2\pi)^4}\frac{e^{ip\cdot(x-x')}}{p^2+m^2+i\epsilon} \tag{3.56}$$

in Minkowski space.

It is particularly easy to get explicit limiting forms for the Euclidean propagation function. Let $R = |x-x'|$, and choose the four-dimensional coordinate system so that $\mathbf{x}-\mathbf{x}' = \mathbf{0}$, $x_4 - x_4' = R$. Then from Eq. (3.48)

$$\Delta_E(R) = \int d\tilde{p}\, e^{-\omega_p R} \tag{3.57}$$

has an integrand which is rotationally invariant in three-dimensions, so

$$(d\mathbf{p}) = 4\pi p^2\, dp = 4\pi p\, \omega_p\, d\omega_p = 4\pi\sqrt{\omega_p^2 - m^2}\,\omega_p\, d\omega_p, \tag{3.58}$$

and hence

$$\Delta_E(R) = \frac{1}{4\pi^2}\int_m^\infty d\omega_p\sqrt{\omega_p^2-m^2}\, e^{-\omega_p R} = \frac{m}{4\pi^2 R}K_1(mR), \tag{3.59}$$

where K_1 is the modified Bessel function of order 1. At very small distances, much smaller than the Compton wavelength of the particle, $mR \ll 1$, we can drop the mass altogether and get

$$mR \ll 1: \quad \Delta_E(R) \sim \frac{1}{4\pi^2 R^2}\int_0^\infty dx\, x\, e^{-x} = \frac{1}{4\pi^2 R^2}. \tag{3.60}$$

In the opposite limit of large distances, the Euclidean propagator is exponentially small, and most of the contribution to the integral comes from values of ω_p near m:

$$\begin{aligned}\Delta_E(R) &= \frac{1}{4\pi^2 R^2}\int_{mR}^\infty dx\sqrt{x^2-(mR)^2}\,e^{-x}\\ &\sim \frac{1}{4\pi^2 R^2}\sqrt{2mR}\int_0^\infty du\,u^{1/2}e^{-u-mR}\\ &= \frac{\sqrt{2m}}{4\pi^2 R^{3/2}}e^{-mR}\,\Gamma\left(\frac{3}{2}\right),\end{aligned} \tag{3.61}$$

where $u = x - mR$, or, because $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$,

$$mR \gg 1: \quad \Delta_E(R) \sim \frac{\sqrt{2m}}{(4\pi R)^{3/2}}e^{-mR}. \tag{3.62}$$

In homework you will derive the limiting behaviors of Δ_+.

3.3.1 Unitarity

The vacuum-to-vacuum probability amplitude must have modulus less than or equal to one,

$$|\langle 0_+|0_-\rangle^K|^2 \le 1. \tag{3.63}$$

But from Eq. (3.35)

$$\begin{aligned}|\langle 0_+|0_-\rangle^K|^2 &= \exp\left[-\int(dx)(dx')K(x)\mathrm{Im}\,\Delta_+(x-x')K(x')\right]\\ &= \exp\left[-\int d\tilde{p}\,|K(p)|^2\right],\end{aligned} \tag{3.64}$$

which follows from Eq. (3.46), so this is less than unity unless $K = 0$. Note that this requirement would not be satisfied if we used Δ_- in place of Δ_+, because the sign of the imaginary part is reversed.

3.4 Interactions

A self-interacting scalar field is described by a Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) - V(\phi). \tag{3.65}$$

An important, although apparently trivial, example is $V(\phi) = \lambda\phi^4$, where the coupling constant is dimensionless. (The dimension of ϕ is that of mass or of length^{-1}.) Now we can no longer integrate the vacuum persistence amplitude,

$$\langle 0, +\infty|0, -\infty\rangle^K = Z[K] = \int[d\phi]\exp\left[i\int(dx)(\mathcal{L}+K\phi)\right]. \tag{3.66}$$

Only approximate techniques exist for finding $Z[K]$, such as perturbation theory, which is based on expanding in powers of the coupling constant λ, which is then treated as small.

3.5 Green's Functions

These are the coefficients of $K(x_1)\dots K(x_n)$ when $Z[K]$ is expanded in a Taylor series in powers of the source. It is in this sense that $Z[K]$ is a generating function:

$$Z[K]=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int(dx_1)\dots(dx_n)K(x_1)\dots K(x_n)G^{(n)}(x_1,\dots,x_n), \tag{3.67}$$

or in terms of functional derivatives,

$$G^{(n)}(x_1,\dots,x_n)=(-i)^n\frac{\delta^n}{\delta K(x_1)\dots\delta K(x_n)}Z(K)\bigg|_{K=0}. \tag{3.68}$$

For the free theory, with $V=0$, we have from Eq. (3.35)

$$\begin{aligned}Z_0[K] &= \exp\left[\frac{i}{2}\int(dx)(dx')K(x)\Delta_+(x-x')K(x')\right]\\ &= \sum_{m=0}^{\infty}\frac{i^m}{2^m}\frac{1}{m!}\int(dx_1)\dots(dx_{2m})K(x_1)\dots K(x_{2m})\Delta_+(x_1-x_2)\\ &\qquad\times\Delta_+(x_3-x_4)\dots\Delta_+(x_{2m-1}-x_{2m}).\end{aligned} \tag{3.69}$$

Thus the odd Green's functions all vanish,

$$G_0^{(2k+1)}=0, \tag{3.70a}$$

while the first few even Green's functions are

$$G^{(2)}(x_1,x_2) = -i\Delta_+(x_1-x_2), \tag{3.70b}$$

$$\begin{aligned}G^{(4)}(x_1,x_2,x_3,x_4) &= -\Delta_+(x_1-x_2)\Delta_+(x_3-x_4)-\Delta_+(x_1-x_3)\Delta_+(x_2-x_4)\\ &\quad-\Delta_+(x_1-x_4)\Delta_+(x_2-x_3),\end{aligned} \tag{3.70c}$$

etc. The factorization of terms suggests that we introduce *connected* Green's functions by writing

$$Z[K]=e^{iW[K]}, \tag{3.71}$$

where

$$iW[K]=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int(dx_1)\dots(dx_n)K(x_1)\dots K(x_n)G_c^{(n)}(x_1,\dots,x_n). \tag{3.72}$$

For the free theory the only nonvanishing G_c is

$$G_{c,0}^{(2)}(x_1,x_2)=-i\Delta_+(x_1-x_2). \tag{3.73}$$

In the following we will use the connected Green's function exclusively and correspondingly will drop the c subscript.

3.6 Feynman Rules for Euclidean $\lambda\phi^4$ Theory

We will start from the generating function in Euclidean space, see Eq. (3.43),

$$Z_E[K] = \int [d\phi] \exp\left\{ -\int (dx)_E \left[\frac{1}{2}(\partial\phi)^2_E + \frac{1}{2}m^2\phi^2 + \lambda\phi^4 - K\phi \right] \right\}, \quad (3.74)$$

in terms of the dimensionless coupling constant λ. To get the connected Euclidean Green's function we first write

$$Z_E[K] = e^{W_E[K]}, \quad (3.75)$$

in terms of which the connected Green's functions are obtained by differentiation:

$$G_E^{(n)}(x_1, \dots, x_n) = \frac{\delta^n}{\delta K(x_1) \dots \delta K(x_n)} W_E[K] \bigg|_{K=0}. \quad (3.76)$$

We can, of course, not calculate $G_E^{(n)}$ exactly; we can, however, obtain a series expansion of $G_E^{(n)}$ in powers of λ. The (diagrammatic) technique for calculating $G_E^{(n)}$ is called perturbation theory.

When $\lambda = 0$ we have a free theory, for which from Eq. (3.44)

$$W_{E,0}[K] = \frac{1}{2} \int (dx)_E (dx')_E K(x) \Delta_E(x - x') K(x'), \quad (3.77)$$

where [Eq. (3.50)]

$$\Delta_E(x - x') = \int \frac{(dp)_E}{(2\pi)^4} \frac{e^{ip\cdot(x-x')_E}}{p_E^2 + m^2}. \quad (3.78)$$

so

$$G_{E,0}^{(2)}(x - x') = \frac{\delta^2}{\delta K(x) \delta K(x')} W_{E,0}[K] = \Delta_E(x - x'), \quad (3.79)$$

which is just what is obtained from the result (3.73) by the replacement $\Delta_+ \to i\Delta_E$, as stated in Eq. (3.40).

The perturbative expansion in λ may be generated by the functional derivative form (henceforward, in this section, we will understand and therefore drop the E subscript)

$$Z[K] = \exp\left[-\int (dx) \lambda \left(\frac{\delta}{\delta K(x)} \right)^4 \right] e^{W_0[K]}, \quad (3.80)$$

which follows because

$$\frac{\delta}{\delta K(x)} e^{W_0[K]} = \int [d\phi]\, \phi(x) \exp\left[-\int (dx) \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 - K\phi \right) \right]. \quad (3.81)$$

Now write

$$Z[K] = e^{W_0} e^{-W_0} \exp\left[-\int (dx)\lambda \left(\frac{\delta}{\delta K(x)}\right)^4\right] e^{W_0}$$
$$= e^{W_0}\left\{1 + e^{-W_0}\left(\exp\left[-\int (dx)\lambda\left(\frac{\delta}{\delta K(x)}\right)^4\right] - 1\right) e^{W_0}\right\}, \quad (3.82)$$

or, taking the logarithm,

$$W[K] = W_0[K] + \ln\left\{1 + e^{-W_0}\left(\exp\left[-\int (dx)\lambda\left(\frac{\delta}{\delta K(x)}\right)^4\right] - 1\right) e^{W_0}\right\}. \quad (3.83)$$

Because λ is now regarded as small, so is

$$\delta = e^{-W_0}\left(\exp\left[-\int (dx)\lambda\left(\frac{\delta}{\delta K(x)}\right)^4\right] - 1\right) e^{W_0}$$
$$= \lambda\delta_1 + \lambda^2\delta_2 + \ldots, \quad (3.84)$$

if we expand in powers of λ.

We will consider the lowest order in perturbation theory, namely the first term in this series:

$$\delta_1 = -e^{-W_0}\int (dx)\left(\frac{\delta}{\delta K(x)}\right)^4 e^{W_0}. \quad (3.85)$$

We work out the derivatives successively:

$$\delta_1 = -e^{-W_0}\int (dx)\left(\frac{\delta}{\delta K(x)}\right)^3\left[e^{W_0}\int (dy)\Delta(x-y)K(y)\right]$$
$$= -e^{-W_0}\int (dx)\left(\frac{\delta}{\delta K(x)}\right)^2 e^{W_0}\left[\int (dy)\,(dz)\,\Delta(x-y)K(y)\Delta(x-z)K(z) + \Delta(0)\right]$$
$$= -e^{-W_0}\int (dx)\frac{\delta}{\delta K(x)} e^{W_0}\left[\int (dy)\,(dz)\,(dw)\,\Delta(x-y)K(y)\Delta(x-z)K(z) \times\Delta(x-w)K(w) + \Delta(0)(2+1)\int (dy)\Delta(x-y)K(y)\right]$$
$$= -\int (dx)\left[\int (dy)(dz)(dw)(dv)\Delta(x-y)\Delta(x-z)\Delta(x-w)\Delta(x-v) \times K(y)K(z)K(w)K(v) + (3+3)\Delta(0)\int (dy)(dz)\Delta(x-y)\Delta(x-z)K(y)K(z) + 3\Delta(0)^2\right]. \quad (3.86)$$

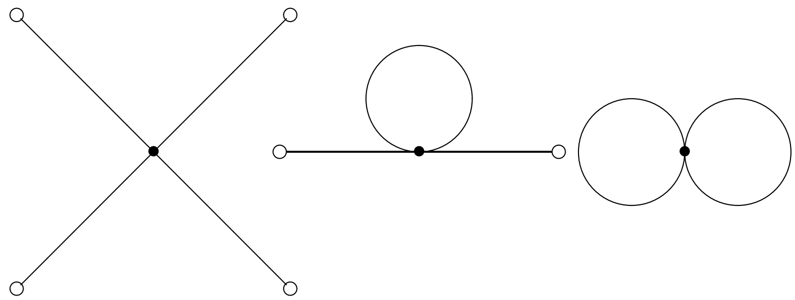

Figure 3.1: Diagrammatic representation of the three terms in Eq. (3.86). The lines represent the propagators, the small open circles the sources, and the heavy dots the interactions.

Here $\Delta(0) = \Delta(x-x)$. The three terms in this result can be represented by the diagrams shown in Fig. 3.1. From this we can calculate the two- and three-point functions (connected Green's functions) through order λ:

$$\begin{aligned} G^{(2)}(x-y) &= \Delta(x-y) + \frac{\delta^2}{\delta K(x)\delta K(y)}\lambda\delta_1\bigg|_{K=0} \\ &= \Delta(x-y) - 12\lambda\int(dz)\Delta(x-z)\Delta(z-z)\Delta(z-y), \end{aligned} \tag{3.87a}$$

$$\begin{aligned} G^{(4)}(x_1,x_2,x_3,x_4) &= \frac{\delta^4}{\delta K(x_1)\dots\delta K(x_4)}\lambda\delta_1 \\ &= -24\lambda\int(dy)\Delta(x_1-y)\Delta(x_2-y)\Delta(x_3-y)\Delta(x_4-y). \end{aligned} \tag{3.87b}$$

Next, we Fourier transform these results, according to the definition:

$$\begin{aligned} &\int(dx_1)\cdots(dx_n)e^{-ip_1\cdot x_1-\ldots-ip_n\cdot x_n}G^{(n)}(x_1,\ldots,x_n) \\ &= G^{(n)}(p_1,\ldots,p_n)(2\pi)^4\delta(p_1+\ldots+p_n), \end{aligned} \tag{3.88}$$

where the momentum-conserving delta function arises from the translational invariance of G. Then it is easily seen that

$$G^{(2)}(p) \equiv G^{(2)}(p,-p) = \frac{1}{p^2+m^2} - 12\lambda\frac{1}{(p^2+m^2)^2}\int\frac{(dq)}{(2\pi)^4}\frac{1}{q^2+m^2}, \tag{3.89}$$

and

$$G^{(4)}(p_1,p_2,p_3,p_4) = -24\lambda\frac{1}{(p_1^2+m^2)(p_2^2+m^2)(p_3^2+m^2)(p_4^2+m^2)}. \tag{3.90}$$

3.7 Second-Order Perturbation Theory

Now let us proceed to second order in λ. The next term in δ, Eq. (3.84), is

$$\begin{aligned}\lambda^2\delta_2[K] &= e^{-W_0}\frac{1}{2}\int(dx)\lambda\left(\frac{\delta}{\delta K(x)}\right)^4\int(dy)\lambda\left(\frac{\delta}{\delta K(y)}\right)^4 e^{W_0}\\ &= e^{-W_0}\frac{1}{2}\int(dx)\lambda\left(\frac{\delta}{\delta K(x)}\right)^4 e^{W_0}\left(-\lambda\delta_1[K]\right),\end{aligned}\tag{3.91}$$

which uses Eq. (3.85). Next we use Leibnitz' rule,

$$\begin{aligned}&\left(\frac{\delta}{\delta K(x)}\right)^4 e^{W_0}\delta_1[K] = \left[\left(\frac{\delta}{\delta K(x)}\right)^4 e^{W_0}\right]\delta_1[K]\\ &+4\left[\left(\frac{\delta}{\delta K(x)}\right)^3 e^{W_0}\right]\frac{\delta}{\delta K(x)}\delta_1[K]+6\left[\left(\frac{\delta}{\delta K(x)}\right)^2 e^{W_0}\right]\left(\frac{\delta}{\delta K(x)}\right)^2\delta_1[K]\\ &+4\left[\frac{\delta}{\delta K(x)}e^{W_0}\right]\left(\frac{\delta}{\delta K(x)}\right)^3\delta_1[K]+e^{W_0}\left(\frac{\delta}{\delta K(x)}\right)^4\delta_1[K].\end{aligned}\tag{3.92}$$

We recognize in the first term here the square of the lowest order term:

$$\delta_2[K]=\frac{1}{2}(\delta_1[K])^2+\delta_2^c[K],\tag{3.93}$$

and so this first term cancels the δ_1^2 term in the expansion of the logarithm in W, Eq. (3.83):

$$W[K]=W_0[K]+\lambda\delta_1[K]-\frac{\lambda^2}{2}(\delta_1[K])^2+\lambda^2\delta_2[K]+\dots.\tag{3.94}$$

We call $\delta_2^c[K]$ the "connected part." Referring to Eq. (3.86) we now easily work it out:

$$\begin{aligned}\delta_2^c[K] = \frac{1}{2}\int(dx)\Big\{&4\Big[\int(dy)(dz)(dw)\Delta(x-y)K(y)\Delta(x-z)K(z)\\ &\quad\times\Delta(x-w)K(w)+3\Delta(0)\int(dy)\Delta(x-y)K(y)\Big]\\ &\times\Big[4\int(dy)(dz)(dw)(du)\Delta(u-x)\Delta(u-y)\Delta(u-z)\Delta(u-w)\\ &\quad\times K(y)K(z)K(w)+12\Delta(0)\int(du)(dy)\Delta(u-x)\Delta(u-y)K(y)\Big]\\ &+6\Big[\int(dy)(dz)\Delta(x-y)\Delta(x-z)K(y)K(z)+\Delta(0)\Big]\\ &\times\Big[12\int(du)(dy)(dz)\Delta(u-x)\Delta(u-x)\Delta(u-y)\Delta(u-z)K(y)K(z)\end{aligned}$$

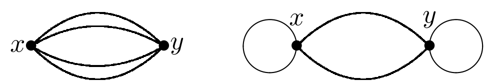

Figure 3.2: Vacuum diagrams giving the second-order contribution to the zero-point function.

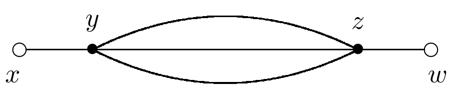

Figure 3.3: A second-order contribution to the two-point function corresponding to Eq. (3.97a).

$$
\begin{aligned}
&\qquad + 12\Delta(0)\int(du)\Delta(u-x)^2\Big] \\
&+ 4\int(dy)\Delta(x-y)K(y)\cdot 24\int(du)(dz)\Delta(u-x)^3\Delta(u-z)K(z) \\
&\qquad + 24\int(du)\Delta(u-x)^4\Big\}.
\end{aligned}
\tag{3.95}
$$

We organize this by the number of sources in each term. There are two terms that have no sources:

$$
\delta_2^c[0] = 12\int(dx)(dy)\Delta(x-y)^4 + 36\Delta(0)^2\int(dx)(dy)\Delta(x-y)^2. \tag{3.96}
$$

These "vacuum diagrams" are indicated in Fig. 3.2. There are three terms corresponding to contributions to the two-point function: The first comes from the 8th line of Eq. (3.95):

$$
48\int(dx)(dy)(dz)(dw)K(x)\Delta(x-y)\Delta(y-z)^3\Delta(z-w)K(w), \tag{3.97a}
$$

which is illustrated in Fig. 3.3. The second contribution to the two-point func-

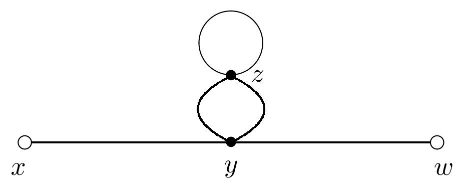

Figure 3.4: Another second-order contribution to the two-point function corresponding to Eq. (3.97b).

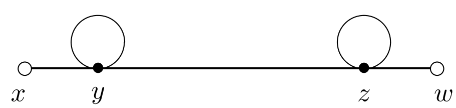

Figure 3.5: The third second-order contribution to the two-point function corresponding to Eq. (3.97c).

tion comes from the 5th through 7th lines of Eq. (3.95):

$$72\int(dx)(dy)(dz)(dw)K(x)\Delta(x-y)\Delta(y-z)^2\Delta(z-z)\Delta(y-w)K(w), \quad (3.97b)$$

is illustrated in Fig. 3.4. The third contribution to the two-point function comes from the 2nd and 4th lines of Eq. (3.95):

$$72\int(dx)(dy)(dz)(dw)K(x)\Delta(x-y)\Delta(y-y)\Delta(y-z)\Delta(z-z)\Delta(z-w)K(w), \quad (3.97c)$$

which is shown in Fig. 3.5.

There are two contributions to the four-point function. The first comes from the 5th and 6th lines of Eq. (3.95):

$$36\int(dx)(dy)(dz)(dw)(du)(dv)K(x)\Delta(x-y)K(z)\Delta(z-y)\Delta(y-w)^2 \times\Delta(w-u)K(u)\Delta(w-v)K(v), \quad (3.98a)$$

which is diagramed in Fig. 3.6. The second contribution to the four-point function arises from the first 4 lines of Eq. (3.95):

$$48\int(dx)(dy)(dz)(dw)(du)(dv)K(x)\Delta(x-y)\Delta(y-y)\Delta(y-z)$$

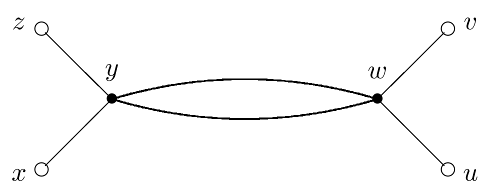

Figure 3.6: A second-order contribution to the four-point function corresponding to Eq. (3.98a).

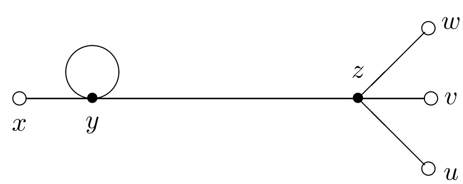

Figure 3.7: A second-order contribution to the four-point function corresponding to Eq. (3.98b).

$$\times\Delta(z-w)K(w)\Delta(z-v)K(v)\Delta(z-u)K(u), \tag{3.98b}$$

which is graphed in Fig. 3.7.

Finally, there is exactly one contribution to the six-point function, coming from the 1st and 3rd lines of Eq. (3.95):

$$8\int(dx)(dy)(dz)(dw)(du)(dv)(ds)(dt)K(x)\Delta(x-y)K(z)\Delta(z-y)$$
$$\times K(w)\Delta(w-y)\Delta(y-u)\Delta(u-v)K(v)\Delta(u-s)K(s)\Delta(u-t)K(t), \tag{3.99}$$

as represented in Fig. 3.8.

We have now associated *coordinate-space* Feynman diagrams with each amplitude given. These diagrams have a precise meaning, given by the following rules:

1. With each small open circle labeled x, associate a source $K(x)$.
2. With each line from x to y, associate a propagator $\Delta(x-y)$.
3. With each vertex, associate the interaction strength $-4!\lambda$.
4. Integrate over all coordinates.

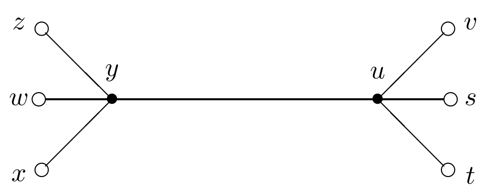

Figure 3.8: The second-order contribution to the six-point function corresponding to Eq. (3.99).

5. Divide by a symmetry number, which is the number of rearrangements of the graph which leave it unchanged.

It is evident that these rules reproduce the functional structure given in the analytic expressions. We only have to verify that the precise numerical factors are reproduced. In order λ we start with the vacuum graph shown in Fig. 3.1. There the vertex factor divided by the symmetry number is

$$\frac{-4!\lambda}{2\cdot 2\cdot 2} = -3\lambda, \tag{3.100}$$

where two factors of 2 come from flipping each closed loop, and the third 2 comes from flipping the whole graph. The two-point function in that figure has a factor of

$$\frac{-4!\lambda}{2\cdot 2} = -6\lambda, \tag{3.101}$$

and the four-point function in order λ has a factor of

$$\frac{-4!\lambda}{4!} = -\lambda, \tag{3.102}$$

all of which factors are seen in Eq. (3.86).

In order λ^2 the two graphs for the vacuum amplitude in Fig. 3.2 have the factors

$$\frac{(4!\lambda)^2}{2\cdot 4!} = 12\lambda^2 \quad \text{and} \quad \frac{(4!\lambda)^2}{2\cdot 2\cdot 2\cdot 2} = 36\lambda^2, \tag{3.103}$$

which agree with those seen in Eq. (3.96). As for the two-point function, the graph in Fig. 3.3 has the factor

$$\frac{(4!\lambda)^2}{2\cdot 3!} = 48\lambda^2, \tag{3.104}$$

that in Fig. 3.4 has the factor

$$\frac{(4!\lambda)^2}{2\cdot 2\cdot 2} = 72\lambda^2, \tag{3.105}$$

and that in Fig. 3.5 has

$$\frac{(4!\lambda)^2}{2\cdot 2\cdot 2} = 72\lambda^2, \tag{3.106}$$

coinciding with the amplitudes given in Eqs. (3.97a), (3.97b), and (3.97c), respectively. The four-point functions in Figs. 3.6 and 3.7 have the factors

$$\frac{(4!\lambda)^2}{2\cdot 2\cdot 2\cdot 2} = 36\lambda^2 \quad \text{and} \quad \frac{(4!\lambda)^2}{2\cdot 3!} = 48\lambda^2, \tag{3.107}$$

respectively, coinciding to what we had in Eqs. (3.98a) and (3.98b). Finally, the six-point function in Fig. 3.8 has the factor

$$\frac{(4!\lambda)^2}{(3!)^2 2} = 8\lambda^2, \tag{3.108}$$

which agrees with Eq. (3.99).

From these results we can easily obtain the Green's functions through order λ^2. For example, from Eqs. (3.87a), (3.97a), (3.97b), and (3.97c), the two-point function is

$$\begin{aligned} G^{(2)}(x,y) \;=\; & \Delta(x-y) - 12\lambda \int (dz)\Delta(x-z)\Delta(z-z)\Delta(z-y) \\ & + 96\lambda^2 \int (dz)(dw)\Delta(x-z)\Delta(z-w)^3\Delta(w-y) \\ & + 144\lambda^2 \int (dz)(dw)\Delta(x-z)\Delta(z-w)^2\Delta(w-w)\Delta(z-y) \\ & + 144\lambda^2 \int (dz)(dw)\Delta(x-z)\Delta(z-z)\Delta(z-w)\Delta(w-w) \\ & \quad \times\Delta(w-y) + \mathcal{O}(\lambda^3). \end{aligned} \tag{3.109}$$

By Fourier-transforming, as in Eq. (3.88), we find immediately

$$\begin{aligned} G^{(2)}(p) \;=\; & \frac{1}{p^2+m^2} - \frac{12\lambda}{(p^2+m^2)^2}\int \frac{(dl)}{(2\pi)^2}\frac{1}{l^2+m^2} \\ & + \frac{96\lambda^2}{(p^2+m^2)^2}\int \frac{(dq_1)}{(2\pi)^4}\frac{(dq_2)}{(2\pi)^4}\frac{(dq_3)}{(2\pi)^4}\frac{(2\pi)^4\delta(q_1+q_2+q_3-p)}{(q_1^2+m^2)(q_2^2+m^2)(q_3^2+m^2)} \\ & + \frac{144\lambda^2}{(p^2+m^2)^2}\int \frac{(dq_1)}{(2\pi)^4}\frac{(dq_2)}{(2\pi)^4}\frac{(2\pi)^4\delta(q_1+q_2)}{(q_1^2+m^2)(q_2^2+m^2)}\int \frac{(dl)}{(2\pi)^4}\frac{1}{l^2+m^2} \\ & + \frac{144\lambda^2}{(p^2+m^2)^3}\int \frac{(dl)}{(2\pi)^4}\frac{1}{l^2+m^2}\int \frac{(dl')}{(2\pi)^4}\frac{1}{l'^2+m^2} + \mathcal{O}(\lambda^3). \end{aligned} \tag{3.110}$$

We represent these momentum-space Green's functions by Feynman diagrams or graphs, where a line carrying momentum p is represented by $1/(p^2+m^2)$, and a vertex corresponds to the factor $-4!\lambda$. To obtain $G^{(n)}$ to order λ^k, draw all topologically distinct graphs with n external lines having up to k four-point

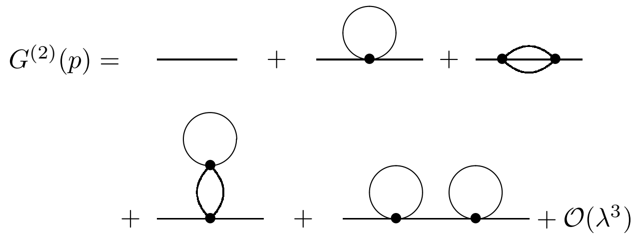

Figure 3.9: Feynman diagrams corresponding to the two-point Green's function.

vertices. Conserve momentum at a vertex joining lines carrying momenta p_1, p_2, p_3, p_4 by supplying the factor $(2\pi)^4\delta(p_1 + p_2 + p_3 + p_4)$, where we adopt the convention that all momenta are flowing into the vertex. Integrate over all loop momentum, with the four-dimensional element $(dl)/(2\pi)^4$. Divide by the symmetry number.

We illustrate these rules for the two-point function, or modified propagation function, $G^{(2)}$, in Fig. 3.9. It is evident that the above rules reproduce the results given in Eq. (3.110),where the symmetry numbers of the five graphs shown are 1, 2, 3!, 4, and 4, respectively, since the external lines carry a definite sense of momentum and therefore cannot be reversed.

It is convenient to introduce some further definitions.

- An *amputated* or *truncated* Green's function has the propagators on the external legs removed. Thus, in order λ the amputated two-point function is
$$-12\lambda \int \frac{(dl)}{(2\pi)^4} \frac{1}{l^2 + m^2}. \tag{3.111}$$

- A *one-particle irreducible* or *proper* Green's function is an amputated Green's function which remains connected when any one internal line is cut. Thus, the first two $\mathcal{O}(\lambda^2)$ contributions to $G^{(2)}$ in Fig. 3.9 are one-particle irreducible, while the third is not.

3.8 Mass Operator

We illustrate the above definitions by defining $\Sigma(p)$, represented by a filled-in circle, as the sum of all amputated one-particle irreducible two-point graphs, the first few of which are shown in Fig. 3.10. Here the slashes through the external lines signifies that the external propagators are to be removed. Put these together by adding internal and external lines, so that we obtain the full connected Green's function as shown in Fig. 3.11. Evidently, when the one-particle irreducible graphs in Fig. 3.10 are inserted for Σ in Fig. 3.11 we recover

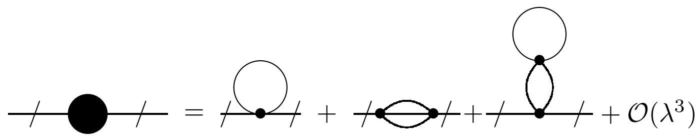

Figure 3.10: Mass operator $\Sigma(p)$ contributions through order λ^2.

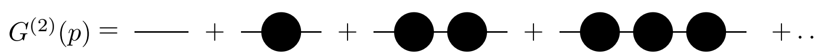

Figure 3.11: Full two-point function formed from iteration of the mass operator.

all the graphs shown in Fig. 3.9. Analytically, this sum is just a geometric series,

$$\begin{aligned} G^{(2)}(p) &= \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Sigma(p)\frac{1}{p^2+m^2} \\ &\quad + \frac{1}{p^2+m^2}\Sigma(p)\frac{1}{p^2+m^2}\Sigma(p)\frac{1}{p^2+m^2} + \dots \\ &= \frac{1}{p^2+m^2-\Sigma(p)}, \end{aligned} \tag{3.112}$$

which reveals why $-\Sigma(p)$ is called the mass operator.

We can see this result directly using functional methods, introduced originally by Schwinger. Define an *effective field* $\tilde{\phi}(x)$ by

$$\tilde{\phi}(x) = \frac{\delta W[K]}{\delta K(x)}, \tag{3.113}$$

which is a "classical" object, at least in the sense that it is not an operator. Then we define an effective action by a Legendre transformation,

$$\Gamma[\tilde{\phi}] = W[K] - \int (dx) K(x)\tilde{\phi}(x). \tag{3.114}$$

From this definition we have the equations

$$\frac{\delta \Gamma}{\delta \tilde{\phi}(x)} = -K(x), \quad \frac{\delta \Gamma}{\delta K(x)} = 0, \tag{3.115}$$

so the implicit $\tilde{\phi}$ dependence drops out. The one-particle irreducible graphs (Green's functions) are defined by variations of the effective action with respect to the effective field:

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n}{\delta\tilde{\phi}(x_1)\ldots\delta\tilde{\phi}(x_n)}\Gamma[\tilde{\phi}]\Big|_{\tilde{\phi}=0}. \tag{3.116}$$

For example, the irreducible two-point function is

$$\Gamma^{(2)}(x, y) = \frac{\delta^2\Gamma}{\delta\tilde{\phi}(x)\delta\tilde{\phi}(y)}\Big|_{\tilde{\phi}=0} = -\frac{\delta K(x)}{\delta\tilde{\phi}(y)}\Big|_{\tilde{\phi}=0}, \tag{3.117}$$

while the corresponding connected Green's function is

$$G^{(2)}(x, y) = \frac{\delta^2 W}{\delta K(x)\delta K(y)}\Big|_{K=0} = \frac{\delta\tilde{\phi}(y)}{\delta K(x)}\Big|_{K=0}, \tag{3.118}$$

which says that the two functions are inverses of each other,

$$G^{(2)\,-1} = -\Gamma^{(2)}, \tag{3.119}$$

which is to be interpreted in the matrix sense, or

$$\int (dy)\Gamma^{(2)}(x, y)G^{(2)}(y, z) = -\delta(x - z). \tag{3.120}$$

In momentum space, this says

$$\Gamma^{(2)}(p)G^{(2)}(p) = -1, \tag{3.121}$$

so according to Eq. (3.112),

$$\Gamma^{(2)}(p) = -p^2 - m^2 + \Sigma(p), \tag{3.122}$$

in which we see the expected appearance of the mass operator.

It is worth recalling that for a free theory

$$W_0[K] = \frac{1}{2}\int \frac{(dp)}{(2\pi)^4}K(-p)\frac{1}{p^2 + m^2}K(p), \tag{3.123}$$

so then the effective field is

$$\tilde{\phi}(p) = \frac{1}{p^2 + m^2}K(p), \tag{3.124}$$

and so, substituting this into Eq. (3.114), we see that the effective action is

$$\Gamma_0[\tilde{\phi}] = -\frac{1}{2}\int \frac{(dp)}{(2\pi)^4}\tilde{\phi}(-p)(p^2 + m^2)\tilde{\phi}(p), \tag{3.125}$$

and thus the free one-particle irreducible two-point function is

$$\Gamma_0^{(2)}(p) = -(p^2 + m^2), \tag{3.126}$$

as seen in Eq. (3.122).