Physics 6433, Quantum Field Theory
Assignment #4
Due Monday, February 23, 2015

February 17, 2015

1. Show that the result derived in class for the forced harmonic oscillator can be written as

$$\frac{\langle a^{\dagger\prime}, t_1 | a'', t_2\rangle^K}{\langle 0, t_1 | 0, t_2\rangle^K} = \exp\left\{ a^{\dagger\prime} e^{-i\omega(t_1 - t_2)} a'' - i a^{\dagger\prime} e^{-i\omega t_1}\gamma - i\gamma^* e^{i\omega t_2} a''\right\},$$

where $\gamma = K(\omega)$. Inserting a complete set of energy eigenstates gives

$$\langle a^{\dagger\prime}, t_1 | a'', t_2\rangle^K = \sum_{n,n'} \langle a^{\dagger\prime} | n\rangle \langle n, t_1 | n', t_2\rangle^K \langle n' | a''\rangle.$$

Define the *scattering matrix* $S_{n,n'}$ by

$$\langle n, t_1 | n', t_2\rangle^K = e^{-in\omega t_1} S_{n,n'} e^{in'\omega t_2}.$$

Define

$$u = -i a^{\dagger\prime} e^{-i\omega t_1}, \quad v = -i a'' e^{i\omega t_2},$$

and therefore obtain

$$e^{-uv + u\gamma + v\gamma^*} = \sum_{n,n'} \frac{u^n}{\sqrt{n!}} i^n \frac{S_{n,n'}}{S_{0,0}} i^{n'} \frac{v^{n'}}{\sqrt{n'!}}. \tag{1}$$

Note that the left side of this equation is invariant under the substitution

$$u \to \frac{\gamma^*}{\gamma} v, \quad v \to \frac{\gamma}{\gamma^*} u,$$

and thereby deduce

$$\left(\frac{\gamma^*}{\gamma}\right)^{n-n'} S_{n,n'} = S_{n',n},$$

so

$$|S_{n,n'}|^2 = |S_{n',n}|^2.$$

Now carry out a double expansion of Eq. (1) in u and v as follows. The coefficient of $v^{n'}/\sqrt{n'!}$ is

$$\frac{(\gamma^* - u)^{n'}}{\sqrt{n'!}} e^{u\gamma} = \sum_n \frac{u^n}{\sqrt{n!}} i^{n+n'} \frac{S_{n,n'}}{S_{0,0}},$$

and therefore

$$i^{n+n'} \frac{S_{n,n'}}{S_{0,0}} = \frac{1}{\sqrt{n!\,n'!}} \left(\frac{d}{du}\right)^n (\gamma^* - u)^{n'} e^{u\gamma}\bigg|_{u=0}.$$

Show that this can be written in terms of the Laguerre polynomials, defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \left(\frac{d}{dx}\right)^n x^{n+\alpha} e^{-x},$$

as follows

$$i^{n+n'} \frac{S_{n,n'}}{S_{0,0}} = \sqrt{\frac{n!}{n'!}} (-1)^n (\gamma^*)^{n'-n} L_n^{(n'-n)}(|\gamma|^2).$$

Check this result for $n = 0$. [We have then $L_0^{(n')} = 1$; and we already know $|S_{0,0}|^2 = e^{-\gamma^2}$.] You should obtain the known result for $|S_{n,0}|^2$. Also check the completeness relation

$$\sum_{n'} |S_{n,n'}|^2 = 1;$$

and in this way learn something about Laguerre polynomials.

2. Show that for a system with a single degree of freedom, if $t_2 < t', t'' < t_1$,

$$\int [dq][dp]\, q(t')q(t'') \exp\left[i \int_{t_2}^{t_1} dt\, (p\dot{q} - H)\right] = \langle q', t_1 | T(q(t')q(t'')) | q'', t_2 \rangle,$$

where T means the time-ordered product of the two position operators,

$$T(q(t')q(t'')) = \begin{cases} q(t')q(t'') & \text{if } t' > t'', \\ q(t'')q(t) & \text{if } t'' > t'. \end{cases}$$

Here $\langle q', t_1|q(t_1) = q'\langle q', t_1|$, $q(t_2)|q'', t_2\rangle = q''|q'', t_2\rangle$.

3. By transforming to Euclidean time, $\tau = it$, show that we obtain the following form for the ground-state persistence amplitude for the harmonic oscillator:

$$Z_E[F] = \int [dq] \exp\left\{-\int d\tau \left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{1}{2}\omega^2 q^2 - Fq\right]\right\}.$$

As in class, carry out the functional integral, and find

$$Z_E[F] = Z_E[0] \exp\left[-\frac{1}{2}\int d\tau\, d\sigma\, F(\tau) G_E(\tau - \sigma) F(\sigma)\right],$$

and give the form of the Euclidean Green's function G_E. Then use analytic continuation to obtain the G found in class.