Chapter 7

A Synopsis of QED

We will here sketch the outlines of quantum electrodynamics, the theory of electrons and photons, and indicate how a calculation of an important physical quantity can be carried out in that theory.

7.1 Photons

The photon is a massless particle of helicity one, that is, the projection of its spin along its direction of motion (it necessarily travels at the speed of light because it is massless) is ± 1. It is described by the Maxwell Lagrange density,

$$\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu, \tag{7.1}$$

where the field strength $F^{\mu\nu}$ is related to the vector potential by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{7.2}$$

and J^μ is the electric current density, a prescribed source. If we require that the corresponding action

$$W[J, A] = \int (dx)\mathcal{L}_A \tag{7.3}$$

be stationary under variations in A, we obtain Maxwell's equations in the form

$$\partial_\nu F^{\mu\nu} = J^\nu. \tag{7.4}$$

Due to the antisymmetry of the field-strength tensor, $F^{\mu\nu} = -F^{\nu\mu}$, we learn that the electric current must be conserved,

$$\partial_\nu J^\nu = 0. \tag{7.5}$$

As a consequence, we further see that the vector potential is not unique: we can make a *gauge transformation*

$$A^\mu \to A^\mu + \partial^\mu \lambda \tag{7.6}$$

with an arbitrary function $\lambda(x)$ without changing the action or the field strength.

If, quantum-mechanically, we wish to describe the exchange of a photon between two specified currents J_μ, we can effectively eliminate the vector potential by the action-at-a-distance statement (up to a gauge transformation)

$$A_\mu(x) = \int (dx') D_+(x - x') J_\mu(x'). \tag{7.7}$$

Here D_+ is just the massless propagator,

$$D_+(x - x') = \Delta_+(x - x'; m^2 = 0) = \int \frac{(dk)}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{k^2 - i\epsilon}. \tag{7.8}$$

Then is is easily seen that

$$\begin{aligned} W[J] &= \int (dx)(J^\mu A_\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}) \\ &= \frac{1}{2} \int (dx) J^\mu A_\mu \\ &= \frac{1}{2} \int (dx)(dx') J^\mu(x) D_+(x - x') J_\mu(x). \end{aligned} \tag{7.9}$$

7.2 Electrons

Dirac discovered spin, and antiparticles, by trying to take the square root of the Klein-Gordon equation. The latter is a second-order equation, but is relativistically invariant. Dirac sought a first-order equation which was also Lorentz invariant. This was only possible by introducing matrices, the famous Dirac matrices. For massive particles, it turns out that the simplest possibility for these matrices is that they be four by four. In modern notation, the Dirac equation is

$$\left(\gamma^\mu \frac{1}{i} \partial_\mu + m\right) \psi(s) = \eta(x), \tag{7.10}$$

where η is a source for the electron field ψ. This is a "square root" of the Klein-Gordon equation in that sense that

$$\begin{aligned} \left(\gamma^\mu \frac{1}{i} \partial_\mu - m\right) \left(\gamma^\nu \frac{1}{i} \partial_\nu + m\right) &= -\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2 \\ &= \partial^2 - m^2, \end{aligned} \tag{7.11}$$

just the operator appearing in the Klein-Gordon equation (3.3) provided the gamma matrices satisfy the following anticommutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}. \tag{7.12}$$

The relation between the effective field ψ and the source η can be written in terms of the Green's function

$$\psi(x) = \int (dx') G_+(x - x') \eta(x'), \tag{7.13}$$

where from Eq. (7.10) G_+ must satisfy

$$\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right) G_+(x-x') = \delta(x-x'), \tag{7.14}$$

which from Eq. (7.11) is immediately solved in terms of the scalar propagator Δ_+:

$$\begin{aligned} G_+(x-x') &= \left(m - \gamma^\mu \frac{1}{i}\partial_\mu\right) \Delta_+(x-x') \\ &= \int \frac{(dp)}{(2\pi)^4} \frac{m - \gamma\cdot p}{m^2+p^2-i\epsilon} e^{ip\cdot(x-x')}, \end{aligned} \tag{7.15}$$

or in momentum space

$$G_+(p) = \frac{1}{m+\gamma\cdot p - i\epsilon}. \tag{7.16}$$

The exchange of an electron between electron sources is described by the vacuum persistence amplitude

$$\langle 0_+|0_-\rangle^\eta = \exp\left[\frac{i}{2}\int (dx)(dx')\eta(x)\gamma^0 G_+(x-x')\eta(x')\right]. \tag{7.17}$$

The appearance of γ^0 is required by Lorentz invariance; it plays the role of a metric tensor in forming the scalar product of the spinors. But there is something somewhat peculiar here: it is easily shown that $\gamma^0 G_+(x-x')$ is totally antisymmetrical,

$$[\gamma^0 G_+(x-x')]^T = -\gamma^0 G_+(x'-x), \tag{7.18}$$

where the T superscript signifies transposition. [We use the so-called Majorana representation, where $\gamma^0\gamma^\mu$ is symmetrical, and γ^0 is antisymmetrical,

$$(\gamma^0\gamma^\mu)^T = \gamma^0\gamma^\mu, \quad (\gamma^0)^T = -\gamma^0.] \tag{7.19}$$

If the sources were ordinary numbers, this would mean that the vacuum persistence amplitude is identically unity, indicating the absence of any such particle. We are forced to conclude that the fermionic sources are *anticommuting* numbers, that is, elements of a Grassmann or exterior algebra:

$$\eta_\zeta(x)\eta_{\zeta'}(x') = -\eta_{\zeta'}(x')\eta_\zeta(x), \tag{7.20}$$

where the subscripts indicate the four spinorial components. From this, one may easily show that the unitarity (probability conservation) property is satisfied,

$$|\langle 0_+|0_-\rangle|^2 \le 1. \tag{7.21}$$

The action for the free electron inferred from Eq. (7.17) is

$$W[\eta,\psi] = \int (dx)\left[\eta(x)\gamma^0\psi(x) + \mathcal{L}(x)\right], \quad \mathcal{L} = -\frac{1}{2}\psi\gamma^0\left(\gamma^\mu\frac{1}{i}\partial_\mu + m\right)\psi, \tag{7.22}$$

which when varied with respect to ψ yields the Dirac equation (7.10).

7.3 Quantum Electrodynamics

The route to spinor electrodynamics is through gauge transformations. Implicit in such is the notion that the electron possesses electric charge. We can describe the electron with a real field and an imaginary charge matrix q (see homework), or, more conventionally, by letting ψ be complex. In the latter formulation, we write the action as

$$W[\eta, J; \psi, A] = \int (dx)(\bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu + \mathcal{L}), \tag{7.23}$$

with the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\psi}\left(\gamma^\mu \frac{1}{i}D_\mu + m\right)\psi, \tag{7.24}$$

in terms of the field strength (7.2), the covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu, \tag{7.25}$$

and where the overbar means

$$\bar{\eta} = \eta^\dagger \gamma^0, \quad \bar{\psi} = \psi^\dagger \gamma^0. \tag{7.26}$$

This action is invariant under gauge transformations, where the covariant derivative term has a cancellation between the transformation of the photon field,

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{7.27}$$

and the phase transformation of the electron field,

$$\psi \to e^{ie\lambda}\psi. \tag{7.28}$$

Variation of this action under $\bar{\psi}$, ψ, and A variations lead to the expected field equations

$$\delta\bar{\psi}: \quad \left(\gamma^\mu \frac{1}{i}D_\mu + m\right)\psi = \eta, \tag{7.29a}$$

$$\delta\psi: \quad \bar{\psi}\left(-\gamma^\mu \frac{1}{i}\overleftarrow{D}_\mu + m\right) = \bar{\eta}, \quad \overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu + ieA_\mu, \tag{7.29b}$$

$$\delta A_\mu: \quad \partial_\nu F^{\mu\nu} = j^\mu + J^\mu, \quad j^\mu = e\bar{\psi}\gamma^\mu\psi. \tag{7.29c}$$

(It is easy to verify that the two forms of the Dirac equation are equivalent.) From this action, we can read off the Feynman rules for QED:

1. An electron line (represented in our figures by a solid line) carrying momentum p corresponds to the amplitude

$$\frac{-i}{m + \gamma \cdot p - i\epsilon} = -i\frac{m - \gamma \cdot p}{m^2 + p^2 - i\epsilon}. \tag{7.30}$$

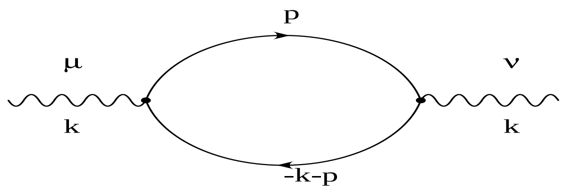

Figure 7.1: Order α vacuum polarization.

2. A photon line (represented in our figures by a wiggly line) carrying momentum k is represented by the amplitude

$$\frac{-i}{k^2 - i\epsilon}. \tag{7.31}$$

3. A vertex coupling a photon with incoming momentum k, an electron with incoming momentum p_1, and a electron with outgoing momentum p_2 is represented by the amplitude

$$-ie\gamma_\mu (2\pi)^4 \delta(k + p_1 - p_2). \tag{7.32}$$

4. For external on-shell lines supply an appropriate wavefunction: a polarization vector $e^\mu_{p\lambda}$ for a photon, and spinors $u_{p\sigma}$, $u^*_{p\sigma}\gamma^0$ for electrons.

5. A factor of -1 must be supplied for each closed electron loop.

6. There also are $-$ signs coming from the permutation of external fermion lines.

In Fig. 7.1 we show the Feynman diagram corresponding to the $\mathcal{O}(e^2)$ correction to the photon propagator, the so-called vacuum polarization. According to the above rules, it corresponds to the amplitude

$$-\int \frac{(dp)}{(2\pi)^4} \mathrm{Tr}\,(-ie\gamma^\mu) \frac{-i}{m + \gamma \cdot p - i\epsilon} (-ie\gamma^\nu) \frac{-i}{m - \gamma \cdot (p + k) - i\epsilon}, \tag{7.33}$$

where Tr stands for the trace over the Dirac matrices.

A few more words about the electron spinors. In the rest frame of the electron, we have the spinors v_σ, which are eigenvectors of the third component of spin and of γ^0:

$$\sigma_3 v_\sigma = \sigma v_\sigma, \quad \gamma^0 v_\sigma = v_\sigma, \tag{7.34}$$

and are normalized according to

$$v^\dagger_\sigma v_{\sigma'} = \delta_{\sigma\sigma'}. \tag{7.35}$$

The spinors $u_{p\sigma}$ are obtained from these by a boost:

$$u_{p\sigma} = \sqrt{\frac{p^0 + m}{2m}} v_\sigma + \sqrt{\frac{p^0 - m}{2m}} v^*_{-\sigma}. \tag{7.36}$$

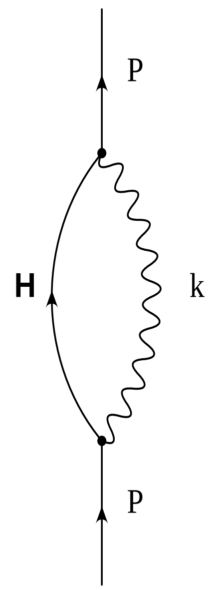

Figure 7.2: Radiative correction to the propagation of an electron in an external magnetic field H.

These satisfy the momentum-space Dirac equation

$$(m + \gamma p)u_{p\sigma} = 0, \tag{7.37}$$

and the projection operator that annihilates such solutions has the resolution

$$\frac{m - \gamma p}{2m} = \sum_\sigma u_{p\sigma} u_{p\sigma}^\dagger \gamma_0. \tag{7.38}$$

7.4 The Anomalous Magnetic Moment of the Electron

Here we offer a derivation of the electron's $g - 2$ anomaly based on a correction the the electron propagator in an external magnetic field $\mathbf{H}$. Consider the process shown in Fig. 7.2. When $\mathbf{H} = \mathbf{0}$ the vacuum persistence amplitude for this process is given by

$$\frac{(ie)^2}{2} \int \frac{(dP)}{(2\pi)^4} \psi(-P)\gamma^0\gamma^\mu \int \frac{(dk)}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{m + \gamma \cdot (P - k)} \gamma_\mu \psi(P). \tag{7.39}$$

To incorporate the effects of the magnetic field, we make the minimal substitution, with q being the charge matrix,

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{7.40}$$

$$P \to \Pi = P - eqA, \tag{7.41}$$

so the gauge-covariant momentum satisfies

$$[\Pi^\mu, \Pi^\nu] = ieqF^{\mu\nu}, \tag{7.42}$$

in terms of the field strength tensor, assumed here constant. Further, we compute

$$\begin{aligned}(\gamma \cdot \Pi)^2 &= \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\Pi_\mu\Pi_\nu + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\Pi_\mu\Pi_\nu \\ &= -\Pi^2 - i\sigma^{\mu\nu}\frac{i}{2}eqF_{\mu\nu} \\ &= -\Pi^2 + eq\sigma F, \quad \sigma F = \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = \boldsymbol{\sigma} \cdot \mathbf{H},\end{aligned} \tag{7.43}$$

for the case of an external magnetic field. The electron propagator then is

$$\frac{1}{m + \gamma \cdot (\Pi - k)} = \frac{m - \gamma \cdot (\Pi - k)}{m^2 + (\Pi - k)^2 - eq\sigma F}. \tag{7.44}$$

As we have seen before, it is useful to combine the denominators in an exponential representation. Write

$$\begin{aligned}\frac{1}{k^2}\frac{1}{(\Pi - k)^2 - eq\sigma F + m^2} &= -\int_0^\infty ds_1\, ds_2\, e^{-is_1k^2 - is_2[(\Pi-k)^2 - eq\sigma F + m^2]} \\ &= -\int_0^\infty ds\, s \int_0^1 du\, e^{-is\chi(u)},\end{aligned} \tag{7.45}$$

where we have introduced

$$s_1 = s(1-u), \quad s_2 = su, \tag{7.46}$$

and

$$\begin{aligned}\chi(u) &= (1-u)k^2 + u[(\Pi - k)^2 - eq\sigma F + m^2] \\ &= (k - u\Pi)^2 + u(1-u)\Pi^2 + u(m^2 - eq\sigma F).\end{aligned} \tag{7.47}$$

Now we carry out the k integration by a Euclidean rotation,

$$\int \frac{(dk)}{(2\pi)^4}e^{-isk^2} = i\int \frac{(dk)_E}{(2\pi)^4}e^{-isk_E^2} = -\frac{i}{16\pi^2 s^2}. \tag{7.48}$$

so then we have here for the basic integral

$$\int \frac{(dk)}{(2\pi)^4}e^{-is\chi(u)} = -\frac{i}{16\pi^2}\frac{1}{s^2}e^{-isu^2(m^2 - eq\sigma F)}e^{-is\mathcal{H}}, \tag{7.49}$$

where

$$\mathcal{H} = u(1-u)(\Pi^2 + m^2 - eq\sigma F) = u(1-u)[m^2 - (\gamma \cdot \Pi)^2]. \tag{7.50}$$

Here, in doing the k integration, we have ignored the noncommutativity of Π, because this would give rise to a term proportional to $[\Pi^\mu,\Pi^\nu]F_{\mu\nu}\propto F^2$, which is irrelevant for the magnetic moment term, which is linear in F.

What actually appears in the $P\to\Pi$ generalization of Eq. (7.39) is

$$\begin{aligned} &e^2\int\frac{(dk)}{(2\pi)^2}\gamma^\mu[m-\gamma\cdot(\Pi-k)]e^{-is\chi}\gamma_\mu \\ &\quad = e^2\int\frac{(dk)}{(2\pi)^4}\{[m+\gamma\cdot(\Pi-k)]\gamma^\mu+2(\Pi-k)^\mu\}e^{-is\chi}\gamma_\mu. \end{aligned} \tag{7.51}$$

By virtue of the external Dirac field, we can set (on the outside) $\gamma\cdot\Pi+m\to 0$; then we can do the k integration by writing it in terms of

$$\int\frac{(dk)}{(2\pi)^4}(k-u\Pi)^\mu e^{-is\chi}. \tag{7.52}$$

This would be zero if the Πs were commuting variables. Although they are not, we get here something proportional to $F^{\mu\nu}\Pi_\nu$, which is contracted with γ^μ:

$$\gamma_\mu F^{\mu\nu}\Pi_\nu=\frac{i}{2}[\sigma F,\gamma\cdot\Pi+m]\to 0, \tag{7.53}$$

where again we have ignored the F dependence in χ. So, in the numerator we may replace k^μ by $u\Pi^\mu$. The expression (7.51) is then, to $\mathcal{O}(F)$ is ($\alpha=e^2/4\pi$)

$$\begin{aligned} &-\frac{ie^2}{16\pi^2}\frac{1}{s^2}e^{-isu^2m^2}\left[-\gamma\cdot u\Pi\gamma^\mu+2(1-u)\Pi^\mu\right]e^{-is\mathcal{H}}\left(1+isu^2eq\sigma F\right)\gamma_\mu \\ &\to -\frac{i\alpha}{4\pi}\frac{1}{s^2}e^{-ism^2u^2}m\big[u\gamma^\mu e^{-is\mathcal{H}}(1+isu^2eq\sigma F)\gamma_\mu \\ &\qquad -2(1-u)e^{-is\mathcal{H}}(1+isu^2\sigma F)\big], \end{aligned} \tag{7.54}$$

where we have again used Eq. (7.53). Now we evaluate this by putting the $isu^2eq\sigma F$ term in the exponent:

$$\begin{aligned} \gamma^\mu e^{-is(\mathcal{H}-u^2eq\sigma F)}\gamma_\mu &= \gamma^\mu\left[e^{-is[u(1-u)(\Pi^2+m^2)]}(1+isueq\sigma F)\right]\gamma_\mu \\ &= -4e^{-isu(1-u)(\Pi^2+m^2)}=-4e^{-isu(1-u)[m^2-(\gamma\cdot\Pi)^2+eq\sigma F]} \\ &= -4e^{-is\mathcal{H}}\left[1-isu(1-u)eq\sigma F\right] \\ &\to -4\left[1-isu(1-u)eq\sigma F\right], \end{aligned} \tag{7.55}$$

where in the second line we have used the fact that $\gamma^\lambda\sigma_{\alpha\beta}\gamma_\lambda=0$. Thus we have from Eq. (7.54),

$$\begin{aligned} &-\frac{i\alpha}{4\pi}\frac{1}{s^2}e^{-ism^2u^2}m\left\{-4u[1-isu(1-u)eq\sigma F]-2(1-u)(1+isu^2eq\sigma F)\right\} \\ &= -\frac{i\alpha}{4\pi}\frac{1}{s^2}me^{-ism^2u^2}\left[-2(1+u)+2isu^2(1-u)eq\sigma F\right]. \end{aligned} \tag{7.56}$$

The first term here describes a modification of the electron propagator, which is involved in renormalization of the mass of the electron. The second term is what is of interest here:

$$\frac{\alpha}{2\pi}\frac{m}{s}u^2(1-u)e^{-ism^2u^2}eq\sigma F. \tag{7.57}$$

The integrals over the parameters s and u are as follows:

$$\int_0^\infty \frac{ds\,s}{s}\int_0^1 du\,u^2(1-u)e^{-isu^2(m^2-i\epsilon)} = -\frac{1}{im^2}\int_0^1 du\,(1-u) = \frac{i}{2m^2}, \tag{7.58}$$

and then the vacuum amplitude (7.39) is

$$\frac{i}{2}\int (dx)\psi(x)\gamma^0\frac{eq}{2m}\sigma F\frac{\alpha}{2\pi}\psi(x). \tag{7.59}$$

This is interpreted as a correction to the g-factor of the electron, where $g = 2$ for a particle described by the Dirac equation:

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} = \frac{1}{2\pi}\frac{1}{137.036} = 0.0011614, \tag{7.60}$$

which is to be compared to the latest experimental value

$$\left(\frac{g-2}{2}\right)_{\text{exp}} = 0.00115965218073(28); \tag{7.61}$$

the discrepancy is entirely due to higher order QED effects, which have been computed out to 10th order! Using these calculations, we can infer an incredibly accurate value of the fine structure constant,

$$\alpha^{-1} = 137.035999084(51). \tag{7.62}$$

For the latest experimental results, see Gabrielse et al., Phys. Rev. Lett. **100**, 120801 (2008).