Chapter 7

Spectral Representations

We have formulated perturbation theory in four-dimensional Euclidean space—we now have to translate back into Minkowski space. There we use the metric $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = +1$. Recall our previous discussion of the propagation function in Sec. 3.3. We pass from the Euclidean to the Minkowskian propagator by

$$\Delta_E(x - x') \to \frac{1}{i}\Delta_+(x - x'), \tag{7.1}$$

or in momentum space

$$\frac{1}{p^2 + m^2} \to \frac{-i}{p^2 + m^2 - i\epsilon}. \tag{7.2}$$

The corresponding Minkowski-space Feynman rule for a line is shown in Fig. 7.1.

What about vertices? Because $dx_4 \to i\,dx^0$, the interaction term in the action becomes

$$-\int (dx)_E \lambda\phi^4 \to -i\int (dx)_M \lambda\phi^4. \tag{7.3}$$

Thus the vertex rule is as shown in Fig. 7.1. And, of course, loop integrations are over $(dk)_M/(2\pi)^4$, $(dk)_M = dk_0(d\mathbf{k})$. So, one can do perturbation theory

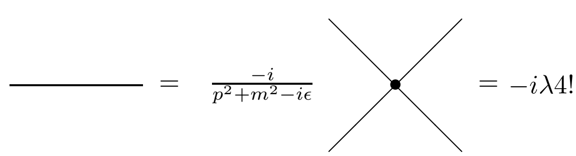

Figure 7.1: Minkowski-space Feynman rules.

directly in Minkowski space. But it may be simpler to calculate the Feynman diagrams in Euclidean space and then translate the result to Minkowski space by analytic continuation. For a diagram with L loops, V vertices, and I internal lines, the Minkowski and Euclidean Green's functions are related by

$$\Gamma_M^{(n)}(p_1,\dots,p_n;m^2) = i^{I+V+L}(-1)^I\Gamma_E^{(n)}(p_1,\dots,p_n;m^2-i\epsilon), \tag{7.4}$$

where the Euclidean Green's function is evaluated at a complex Euclidean momentum,

$$p = (-ip_0,\mathbf{p}). \tag{7.5}$$

In the relation (7.4) the factor $(-i)^I i^V$ arises from the Minkowski Feynman rules, while the i^L comes from the correspondence $i(dk)_E = (dk)_M$. Now from Eq. (5.55), $L = I - V + 1$, so the Minkowski and Euclidean Green's functions are related by

$$\Gamma_M^{(n)}(p_1,\dots,p_n;m^2) = i\Gamma_E^{(n)}(p_1,\dots,p_n;m^2-i\epsilon). \tag{7.6}$$

Consider the four-point function, described in Sec. 5.2.1, and Fig. 5.2. Using the zero-momentum renormalization condition (5.26), or

$$\Gamma^{(4)}\bigg|_{p_i=0} = -4!\,\lambda = -(4\pi)^2 g, \tag{7.7}$$

we have (although we have now dropped the caret over g, its appearance signifying division by $(4\pi)^2$ is retained for the Green's function)

$$\hat\Gamma^{(4)}(p_1,p_2,p_3) = -g + g^2\left[3 - \frac{1}{2}A(s,t,u)\right] \tag{7.8}$$

in Euclidean space. Here

$$A(s,t,u) = \sum_{z=s,t,u}\sqrt{1+\frac{4m^2}{z}}\ln\frac{\sqrt{1+\frac{4m^2}{z}}+1}{\sqrt{1+\frac{4m^2}{z}}-1}. \tag{7.9}$$

The translation of this into Minkowski space is immediate:

$$\hat\Gamma_M^{(4)}(p_1,p_2,p_3) = -ig + ig^2\left[3 - \frac{1}{2}A(s,t,u)\right], \tag{7.10}$$

but now

$$s = (p_1+p_2)^2 = (\mathbf{p}_1+\mathbf{p}_2)^2 - (p_1^0+p_2^0)^2, \tag{7.11}$$

etc. Thus $A(s,t,u)$ has singularities for physical values of the momenta.

Consider the function which appears in Eq. (7.9),

$$f(p^2) = \sqrt{1+\frac{4m^2}{p^2}}\ln\frac{\sqrt{1+\frac{4m^2}{p^2}}+1}{\sqrt{1+\frac{4m^2}{p^2}}-1}. \tag{7.12}$$

When $-p^2 = 4m^2$ a branch point is encountered, and if $-p^2 > 4m^2$, f develops an imaginary part.

Let $x = \sqrt{1 + 4m^2/p^2}$. If $-p^2 < 4m^2$, $x = iy$ is imaginary, and

$$f = iy \ln \frac{iy+1}{iy-1} = 2y \operatorname{arccot} y \tag{7.13}$$

is real. On the other hand, if $-p^2 > 4m^2$, $0 < x < 1$, and

$$f = x \ln \frac{x+1}{x-1} = -i\pi x + x \ln \frac{1+x}{1-x} = -i\pi x + 2x \operatorname{arctanh} x, \tag{7.14}$$

so

$$\Im f = -i\pi \sqrt{1 + \frac{4m^2}{p^2}}, \quad -p^2 > 4m^2. \tag{7.15}$$

This suggests a *spectral representation* (or a *dispersion relation*) for f,

$$f(p^2) = \int_{4m^2}^{\infty} dM^2 \frac{a(M^2)}{p^2 + M^2 - i\epsilon}, \tag{7.16}$$

where $a(M^2)$ is called the spectral function. From the property

$$\frac{1}{x - i\epsilon} = P\frac{1}{x} + i\pi\delta(x) \tag{7.17}$$

(P signifies the Cauchy principal value), we deduce from Eq. (7.15) that

$$a(M^2) = -\sqrt{1 - \frac{4m^2}{M^2}}. \tag{7.18}$$

Inserting this spectral function into the spectral representation (7.16) does not result in a convergent integral. However, we note that we will obtain the same imaginary part at $p^2 = -M^2$ if we insert the factor $-p^2/M^2$, so let us try the "subtracted" relation

$$f(p^2) = p^2 \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \frac{\sqrt{1 - \frac{4m^2}{M^2}}}{p^2 + M^2 - i\epsilon}. \tag{7.19}$$

[We call this a "subtracted" dispersion relation because

$$\frac{p^2}{M^2} \frac{1}{p^2 + M^2 - i\epsilon} = \frac{1}{M^2} - \frac{1}{p^2 + M^2 - i\epsilon}, \tag{7.20}$$

so the two representations (7.16) and (7.19) differ by an infinite constant.]

Now let us do the spectral integration in Eq. (7.19) to see if correctly reproduces the real part. Let

$$v = \sqrt{1 - \frac{4m^2}{M^2}}, \quad 1 - v^2 = \frac{4m^2}{M^2}, \tag{7.21}$$

so

$$2v\,dv = \frac{4m^2}{M^4}dM^2 = \frac{1}{4m^2}(1-v^2)^2\,dM^2, \tag{7.22}$$

so we have

$$\begin{aligned} f(p^2) &= 2p^2\int_0^1 \frac{v^2\,dv}{4m^2+p^2(1-v^2)} \\ &= \int_0^1 dv\left[\frac{\sqrt{4m^2+p^2}}{\sqrt{4m^2+p^2}+pv}+\frac{\sqrt{4m^2+p^2}}{\sqrt{4m^2+p^2}-pv}-2\right] \\ &= \sqrt{1+\frac{4m^2}{p^2}}\ln\frac{\sqrt{1+\frac{4m^2}{p^2}}+1}{\sqrt{1+\frac{4m^2}{p^2}}-1}-2, \end{aligned} \tag{7.23}$$

where the added constant assures that $f(p^2)\to 0$ as $p^2\to 0$. Thus, up to that constant, we have reproduced the function (7.12).

In terms of this spectral representation, the second-order contribution to the four-point function (7.10) has the remarkably simple form

$$\begin{aligned} 3-\frac{1}{2}A(s,t,u) &= \frac{1}{2}\int_{4m^2}^\infty \frac{dM^2}{M^2}\sqrt{1-\frac{4m^2}{M^2}}\left[-\frac{s}{s+M^2-i\epsilon}\right. \\ &\qquad \left. -\frac{t}{t+M^2-i\epsilon}-\frac{u}{u+M^2-i\epsilon}\right]. \end{aligned} \tag{7.24}$$

It is important to note that the factor of $-p^2/M^2$ inserted into the spectral integral does not change the imaginary part, but

1. makes the integral converge, and
2. guarantees that the normalization condition that the order g^2 contribution vanishes at $s=t=u=0$ is satisfied.

7.1 Source Theory

Now we ask the question: Can we derive this very convenient and simple representation directly?

Consider the *causal* arrangement shown in Fig. 7.2. Because of the casual arrangement, where a real particle propagates from an earlier point x' to a later point x, we may use the propagator in the form (3.46), or

$$x^0 > x'^0: \quad \Delta_+(x-x') = i\int d\tilde p\, e^{ip\cdot(x-x')}, \quad d\tilde p = \frac{(d\mathbf{p})}{(2\pi)^3 2\omega_p}, \tag{7.25}$$

where $p^0=\omega_p=\sqrt{p^2+m^2}$. Thus, in place of $(dp)/[(2\pi)^4(p^2+m^2)]$ for the internal lines we use merely $id\tilde p$, so for the causal loop in Fig. 7.2

$$(-i)^2\int\frac{(dp_1)}{(2\pi)^4}\frac{(dp_2)}{(2\pi)^4}\frac{(2\pi)^4\delta(p_1+p_2-p)}{(p_1+m^2-i\epsilon)(p_2+m^2-i\epsilon)}$$

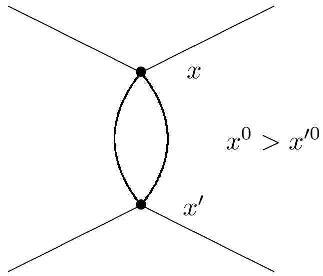

Figure 7.2: A causal arrangement in which two real particles are exchanged between interaction points x' and x, where the latter point is later in time than the earlier point.

$$\to \int d\tilde{p}_1\, d\tilde{p}_2\, (2\pi)^4 \delta(p_1 + p_2 - p). \tag{7.26}$$

Now we carry out the latter phase-space integral,

$$\begin{aligned} I &= \int \frac{(d\mathbf{p}_1)}{(2\pi)^3 2\omega_{p_1}} \frac{(d\mathbf{p}_2)}{(2\pi)^3 2\omega_{p_2}} (2\pi)^4 \delta(p_1 + p_2 - p) \\ &= \frac{1}{(2\pi)^2} \int \frac{(d\mathbf{p}_1)}{2\omega_{p_1}} \frac{1}{2\omega_{p_2}} \delta(p_1^0 + p_2^0 - p^0), \quad \mathbf{p}_2 = \mathbf{p} - \mathbf{p}_1. \end{aligned} \tag{7.27}$$

Evaluate this by working in the rest frame of p, which for the situation envisaged in Fig. 7.2 is assumed to be timelike: $\mathbf{p} = \mathbf{0}$, $p^0 = M \geq 2m$, so that

$$\mathbf{p}_1 + \mathbf{p}_2 = 0, \quad p_1^0 + p_2^0 = M, \quad p_i^0 = \sqrt{\mathbf{p}_1^2 + m^2}. \tag{7.28}$$

Then

$$\begin{aligned} I &= \frac{1}{(2\pi)^2} \int \frac{4\pi p_1^2\, dp_1}{4(p_1^0)^2} \delta(2p_1^0 - M) = \frac{1}{4\pi} \int dp_1^0\, \frac{p_1}{p_1^0}\, \delta(2p_1^0 - M) \\ &= \frac{1}{4\pi} \frac{1}{2} \frac{\sqrt{(p_1^0)^2 - m^2}}{p_1^0} \bigg|_{p_1^0 = \frac{M}{2}} = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{M^2}}. \end{aligned} \tag{7.29}$$

For the general case when the masses are unequal, see Eq. (7.43). In coordinate space, the causal contribution to the four-point function is

$$\begin{aligned} &-\lambda^2 (4!)^2 \int \frac{(dp)}{(2\pi)^4} 2\pi\delta(p^2 + M^2) \frac{dM^2}{2\pi} \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{M^2}} e^{ip\cdot(x-x')} \\ &= -(4\pi)^2 g^2 \int d\tilde{p}|_M dM^2 \sqrt{1 - \frac{4m^2}{M^2}} e^{ip\cdot(x-x')}, \end{aligned} \tag{7.30}$$

which uses Eq. (3.11), with $p^0 = \sqrt{\mathbf{p}^2 + M^2}$. We recognize here the casual form of the propagator (7.25),

$$x^0 > x'^0 : \quad i \int d\tilde{p}|_M e^{ip\cdot(x-x')} = \Delta_+(x, x'; M^2), \tag{7.31}$$

which causal condition can be relaxed by using the general form of Δ_+:

$$\Delta_+(x-x';M^2)=\int\frac{(dp)}{(2\pi)^4}\frac{e^{ip\cdot(x-x')}}{p^2+M^2-i\epsilon}. \tag{7.32}$$

Then, back in momentum space we have for the second-order contribution to the four-point function (a factor of 1/2 arise because the causal arrangement could have either $x^0>x'^0$ or the other way around)

$$\hat\Gamma^{(4),2}=i\sum_{p^2=s,t,u}\frac{1}{2}g^2\int_{4m^2}^\infty dM^2\sqrt{1-\frac{4m^2}{M^2}}\frac{1}{p^2+M^2-i\epsilon}+\text{ c.t.}, \tag{7.33}$$

where c.t. (contact term) is a constant, or a delta-function in coordinate space, chosen to enforce the normalization condition that $\Gamma^{(4),2}=0$ at $s=t=u=0$, that is, in effect,

$$\frac{1}{p^2+M^2}\to\frac{1}{p^2+M^2}-\frac{1}{M^2}=-\frac{p^2}{M^2}\frac{1}{p^2+M^2}, \tag{7.34}$$

and so the result (7.10), (7.24) is reproduced:

$$\hat\Gamma^{(4),2}=-i\frac{g^2}{2}\sum_{p^2=s,t,u}p^2\int_{4m^2}^\infty\frac{dM^2}{M^2}\sqrt{1-\frac{4m^2}{M^2}}\frac{1}{p^2+M^2-i\epsilon}. \tag{7.35}$$

This is the essence of Schwinger's *source theory*, in which no infinities are ever encountered.

7.2 Cross Sections

We finally must address the question of how to proceed from the Green's functions we have calculated, which represent generalized scattering amplitudes, to compute cross sections which can be measured in the laboratory. We first recognize that for a weak source, the probability amplitude of producing (absorbing) a particle of momentum p, within an element of momentum $d\tilde p$, is

$$\langle p|0_-\rangle^K=i\sqrt{d\tilde p}K(p),\quad \langle 0_+|p\rangle^K=i\sqrt{d\tilde p}K(-p), \tag{7.36}$$

which may be deduced by considering a source composed of two parts, $K=K_1+K_2$, where the support of K_1 is entirely later than that of K_2. Then

$$\begin{aligned}\langle 0_+|0_-\rangle^{K_1+K_2}&\approx 1+i\int(dx)(dy)K_1(x)\Delta_+(x-y)K_2(y)+\dots\\&=1-\int d\tilde p\,K_1(-p)K_2(p)+\dots\\&=\sum_{n_p}\langle 0_+|n_p\rangle^{K_1}\langle n_p|0_-\rangle^{K_2},\end{aligned} \tag{7.37}$$

where in the last line we are summing over all multiparticle states transmitted between the earlier source and the later source. Here we have used the causal form of the propagator

$$\Delta_+(x-y) = i\int d\tilde{p}\, e^{ip(x-y)}, \quad x^0 - y^0 > 0. \tag{7.38}$$

The weak-source result (7.36) now follows.

Now we recall the definition of the 1PI Green's functions. The vacuum persistence amplitude corresponding to two particles coming in and two particle going out of a collision process is

$$\int (dx_1)(dx_2)(dx_2)(dx_4)\tilde{\phi}_1(x_1)\tilde{\phi}_2(x_2)\tilde{\phi}_3(x_3)\tilde{\phi}_4(x_4)\Gamma^{(4)}(x_1,x_2,x_3,x_4), \tag{7.39}$$

where for $x^0 > y^0$

$$\phi_1(x) = \int (dy)\Delta_+(x-y)K_1(y) = i\int d\tilde{p}\int (dy)e^{ip(x-y)}K_1(y), \tag{7.40}$$

or in momentum space

$$\phi_{1p} = i d\tilde{p}\, K_1(p). \tag{7.41}$$

Thus we define the scattering amplitude as

$$\langle p_1 p_2|T|p_3 p_4\rangle = \sqrt{d\tilde{p}_1 d\tilde{p}_2 d\tilde{p}_3 d\tilde{p}_4}\Gamma^{(4)}. \tag{7.42}$$

To get the probability for scattering, we take the absolute square of this amplitude, and integrate over the two outgoing (final state) momentum constrained by energy-momentum conservation. In homework you will show

$$\int d\tilde{p}_a\, d\tilde{p}_b\, \delta^{(4)}(p_a+p_b-P) = \frac{d\Omega}{32\pi^2 M^2}\left[M^2-(m_a+m_b)^2\right]^{1/2}\left[M^2-(m_a-m_b)^2\right]^{1/2}, \tag{7.43}$$

where $p_a^2 = -m_a^2$, $p_b^2 = -m_b^2$, $P^2 = -M^2$, and $d\Omega$ is the element of solid angle for the relative momentum.

To define a differential cross section, we divide the above by the invariant flux. The particle flux associated with a single particle in a small momentum cell is

$$s^\mu = 2p^\mu d\tilde{p}; \tag{7.44}$$

for colliding beams, such as occur with the LHC, we have to combine two such fluxes in an invariant way, such that it vanishes if the single particle fluxes are proportional. The invariant flux is

$$F = \left[(s_a s_b)^2 - s_a^2 s_b^2\right]^{1/2}. \tag{7.45}$$

In homework you will show

$$F = d\tilde{p}_a\, d\tilde{p}_b 2\left[M^2-(m_a+m_b)^2\right]^{1/2}\left[M^2-(m_a-m_b)^2\right]^{1/2}. \tag{7.46}$$

The common square root factors cancel for a purely elastic scattering process where the initial and final particles are the same. Thus we find the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{|\Gamma^{(4)}|^2}{64\pi^2 M^2}. \tag{7.47}$$

Here $M^2 = -s = 4E^2$, where E is the energy of either beam in the center-of-mass frame. The other variables upon which the scattering amplitude depends upon are (here p_1 and p_2 are incoming, p_3 and p_4 are outgoing)

$$t = (p_1 - p_3)^2 = 4p^2 \sin^2 \theta/2, \quad u = (p_1 - p_4)^2 = 4p^2 \cos^2 \theta/2, \tag{7.48}$$

where $p = \sqrt{E^2 - m^2}$ is the momentum of the particles in the center-of-mass frame, and θ is the scattering angle. For further details, see homework.