Chapter 6

Renormalization Group

Let us return to the formula (5.42) relating the renormalized and the unrenormalized Green's functions:

$$\Gamma_R^{(n)}(p_1,\dots,p_n;\lambda,m,\mu,\epsilon) = Z^{n/2}\Gamma_0^{(n)}(p_1,\dots,p_n;\lambda_0,m_0,\epsilon), \tag{6.1}$$

where

$$\lambda \equiv \lambda_R(\lambda_0,m_0,\mu,\epsilon), \quad m \equiv m_R(\lambda_0,m_0,\mu,\epsilon). \tag{6.2}$$

Since the bare Green's function is independent of μ, we have

$$\begin{aligned} 0 &= \frac{d}{d\mu}\Gamma_0 = \frac{d}{d\mu}Z^{-n/2}\Gamma_R \\ &= -\frac{n}{2}Z^{-n/2-1}\left(\frac{d}{d\mu}Z\right)\Gamma_R + Z^{-n/2}\left(\frac{\partial}{\partial\mu}+\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m}+\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda}\right)\Gamma_R. \end{aligned} \tag{6.3}$$

Multiply this equation through by $Z^{n/2}\mu$ and we find the so-called *renormalization group* equation:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}-n\gamma+m\delta\frac{\partial}{\partial m}\right)\Gamma_R = 0. \tag{6.4}$$

Here the coefficients are

$$\beta\left(\lambda,\frac{m}{\mu},\epsilon\right) \equiv \mu\frac{\partial\lambda}{\partial\mu}, \tag{6.5a}$$

$$\gamma\left(\lambda,\frac{m}{\mu},\epsilon\right) \equiv \frac{1}{2}\mu\frac{\partial\ln Z}{\partial\mu}, \tag{6.5b}$$

$$\delta\left(\lambda,\frac{m}{\mu},\epsilon\right) \equiv \frac{1}{2}\mu\frac{\partial\ln m^2}{\partial\mu}, \tag{6.5c}$$

Clearly the values of the coefficients depend on the values chosen for F, G, and H, that is, on the renormalization scheme. A particularly simple choice is that of 't Hooft and Weinberg, the "mass-independent" renormalization. It amounts to setting

$$F = G = H = 0. \tag{6.6}$$

Then, through order g^2 we have from Eqs. (5.54) and (5.53)

$$\hat{g}_0 = \mu^\epsilon \left(\hat{g} + \hat{g}^2 \frac{3}{\epsilon} \right), \tag{6.7}$$

so if we regard $\hat{g}_0$ as fixed,

$$0 = \mu \frac{\partial \hat{g}_0}{\partial \mu} = \epsilon \hat{g}_0 + \mu^\epsilon \left(1 + \frac{6}{\epsilon} \hat{g} \right) \mu \frac{\partial \hat{g}}{\partial \mu}, \tag{6.8}$$

so the beta function is through order g^2

$$\begin{aligned} \beta(\hat{g}) &= \mu \frac{\partial}{\partial \mu} \hat{g} = -\epsilon \left(\hat{g} + \hat{g}^2 \frac{3}{\epsilon} \right) \left(1 - \frac{6}{\epsilon} \hat{g} \right) = -\epsilon \hat{g} + 3\hat{g}^2 \\ &\rightarrow 3\hat{g}^2 + \mathcal{O}(\hat{g}^3), \quad \text{as} \quad \epsilon \rightarrow 0. \end{aligned} \tag{6.9}$$

Next, in this scheme, through second order, the wavefunction renormalization constant is, from Eq. (5.49),

$$Z = 1 - \frac{\hat{g}^2}{12\epsilon}, \tag{6.10}$$

so the gamma function is through this order

$$\begin{aligned} \gamma(\hat{g}) &= \frac{1}{2} \mu \frac{\partial \ln Z}{\partial \mu} = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \left(-\frac{\hat{g}^2}{12\epsilon} \right) \\ &= -\frac{\hat{g}}{12\epsilon} \mu \frac{\partial}{\partial \mu} \hat{g} = -\frac{\hat{g}}{12\epsilon} (-\epsilon \hat{g}) \\ &= \frac{\hat{g}^2}{12} + \mathcal{O}(\hat{g}^3). \end{aligned} \tag{6.11}$$

The bare mass is related to the renormalized mass by Eqs. (5.52) and (5.51), or

$$m_0^2 = m^2 \left(1 + \frac{\hat{g}^2}{12\epsilon} + \frac{\hat{g}}{\epsilon} + \frac{2\hat{g}^2}{\epsilon^2} - \frac{\hat{g}^2}{2\epsilon} \right), \tag{6.12}$$

so because the bare mass does not depend upon μ,

$$\begin{aligned} 0 = \mu \frac{\partial m_0^2}{\partial \mu} &= \mu \frac{\partial m^2}{\partial \mu} \left(1 + \frac{\hat{g}}{\epsilon} - \frac{5\hat{g}^2}{12\epsilon} + \frac{2\hat{g}^2}{\epsilon^2} \right) \\ &\quad + m^2 \left(\frac{1}{\epsilon} - \frac{5\hat{g}}{6\epsilon} + \frac{4\hat{g}}{\epsilon^2} \right) (-\epsilon \hat{g} + 3\hat{g}^2), \end{aligned} \tag{6.13}$$

again using the first line of Eq. (6.9), so the third coefficient is

$$\begin{aligned}\delta &= \frac{1}{2}\frac{\mu}{m^2}\frac{\partial m^2}{\partial \mu} \\ &= -\frac{1}{2}(-\epsilon\hat{g}+3\hat{g}^2)\left(\frac{1}{\epsilon}-\frac{5\hat{g}}{6\epsilon}+\frac{4\hat{g}}{\epsilon^2}-\frac{\hat{g}}{\epsilon^2}\right) \\ &= -\frac{1}{2}\left[-\hat{g}+\hat{g}^2\left(\frac{3}{\epsilon}+\frac{5}{6}-\frac{3}{\epsilon}\right)\right]+\mathcal{O}(\hat{g}^3) \\ &= \frac{1}{2}\hat{g}-\frac{5}{12}\hat{g}^2+\mathcal{O}(\hat{g}^3). \end{aligned} \tag{6.14}$$

Let us see how the renormalization group equation is satisfied. In lowest order it reads

$$\left(\mu\frac{\partial}{\partial\mu}+m\frac{\hat{g}}{2}\frac{\partial}{\partial m}\right)\Gamma_R=\mathcal{O}(\hat{g}^2). \tag{6.15}$$

The only Green's function which has nontrivial dependence on μ and m in lowest order in the mass-independent renormalization scheme is

$$\Gamma^{(2)}=-(p^2+m^2)+\frac{1}{2}\hat{g}m^2\left[\psi(2)-\ln\hat{m}^2\right], \tag{6.16}$$

where we have merely cancelled the $1/\epsilon$ divergence in Eq. (5.2). The derivatives appearing in Eq. (6.15) are, because $\hat{m}^2=m^2/(4\pi\mu^2)$,

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(2)}=\hat{g}m^2, \quad m\frac{\hat{g}}{2}\frac{\partial}{\partial m}m^2=\hat{g}m^2, \tag{6.17}$$

so the renormalization group equation is satisfied through this order. Note therefore that the renormalization group equation determines the coefficient of the logarithm in $\Gamma^{(2)}$.

Let's go to the next order and examine the equation satisfied by the four-point function. From Eqs. (6.9), (6.11), and (6.14), the renormalization group equation is

$$\left[\mu\frac{\partial}{\partial\mu}+3\hat{g}^2\frac{\partial}{\partial\hat{g}}-4\frac{\hat{g}^2}{12}+m\left(\frac{1}{2}\hat{g}-\frac{5}{12}\hat{g}^2\right)\frac{\partial}{\partial m}\right]\Gamma^{(4)}=0, \tag{6.18}$$

where, simply removing the divergent term from Eq. (5.13), we have at $\epsilon=0$

$$\hat{\Gamma}^{(4)}=\frac{\Gamma^{(4)}}{(4\pi)^2}=-\hat{g}\left\{1-\frac{3}{2}\hat{g}\left[\psi(1)+2-\ln\hat{m}^2\right]+\frac{\hat{g}}{2}A(s,t,u)\right\}. \tag{6.19}$$

Again, the order $\hat{g}^2$ terms cancel because

$$\mu\frac{\partial}{\partial\mu}\hat{\Gamma}^{(4)}=3\hat{g}^2, \quad 3\hat{g}^2\frac{\partial}{\partial\hat{g}}\hat{\Gamma}^{(4)}=-3\hat{g}^2+\mathcal{O}(\hat{g}^3); \tag{6.20}$$

again, the coefficient of the $\ln\hat{m}^2$ term is determined.

Now let us use the renormalization group to learn something we don't know. Consider $\Gamma^{(2)}$ in second order. Write

$$\Gamma^{(2)} = -(p^2 + m^2) + \frac{1}{2}\hat{g}m^2 \left[\psi(2) - \ln\hat{m}^2\right] + \hat{g}^2 m^2 f\left(\frac{m^2}{\mu^2}, \frac{p^2}{m^2}\right), \tag{6.21}$$

Then the $\mathcal{O}(\hat{g}^2)$ terms in the renormalization group equations are

$$\begin{aligned} 0 &= \hat{g}^2 m^2 \left(-2\hat{m}^2\right) \frac{\partial}{\partial \hat{m}^2} f + \frac{3}{2}\hat{g}^2 m^2 \left[\psi(2) - \ln\hat{m}^2\right] \\ &\quad - \frac{1}{6}\hat{g}^2(-p^2 - m^2) + \frac{5}{6}\hat{g}^2 m^2 + \frac{1}{2}\hat{g}^2 m^2 \left[\psi(2) - \ln\hat{m}^2\right] - \frac{1}{2}\hat{g}^2 m^2 \\ &= \hat{g}^2 m^2 \left\{-2\hat{m}^2 \frac{\partial}{\partial \hat{m}^2} f + 2\left[\psi(2) - \ln\hat{m}^2\right] + \frac{1}{6}\frac{p^2}{m^2} + \frac{1}{2}\right\}. \end{aligned} \tag{6.22}$$

This allows us to determine the $\hat{m}^2$ dependence of f:

$$f(\hat{m}^2, p^2/m^2) = \left[\psi(2) + \frac{1}{4}\right]\ln\hat{m}^2 - \frac{1}{2}\ln^2\hat{m}^2 + \frac{p^2}{12m^2}\ln\hat{m}^2 + g(p^2/m^2). \tag{6.23}$$

A direct proof of this $\hat{m}^2$ dependence will be given in the homework.

We may also obtain the p^2 dependence of the two-point function in the massless case. There we have the form

$$\Gamma^{(2)} = -p^2 + \hat{g}^2 p^2 h\left(\frac{p^2}{\mu^2}\right), \tag{6.24}$$

and the renormalization group equation is through $\mathcal{O}(\hat{g}^2)$

$$-2\frac{p^2}{\mu^2}\hat{g}^2 p^2 h' + \frac{1}{6}\hat{g}^2 p^2 = 0, \tag{6.25}$$

or

$$h' = \frac{1}{12}\frac{\mu^2}{p^2}, \tag{6.26}$$

or

$$h = \frac{1}{12}\ln\frac{p^2}{\mu^2} + \text{constant}, \tag{6.27}$$

which is consistent with the result of Problem 9.2.

6.1 Running Coupling

The β function tells us how the coupling $\hat{g}$ evolves with the mass scale μ. To leading order

$$\mu\frac{\partial\hat{g}}{\partial\mu} = \beta(\hat{g}) = 3\hat{g}^2, \tag{6.28}$$

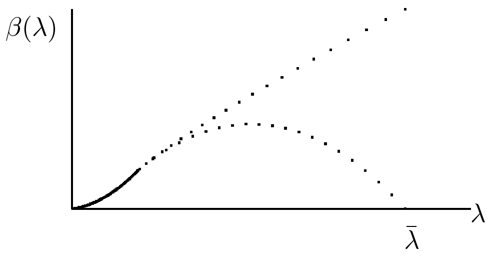

Figure 6.1: Beta function for $\lambda\phi^4$ theory. Shown is the infrared stable behavior at $\lambda = 0$. Different possible evolutions are shown for large λ.

or

$$\frac{1}{3}\frac{d\hat{g}}{\hat{g}^2} = \frac{d\mu}{\mu}, \tag{6.29}$$

which is integrated easily,

$$-\frac{1}{3}\left(\frac{1}{\hat{g}(\mu)} - \frac{1}{\hat{g}(\mu_0)}\right) = \ln\frac{\mu}{\mu_0}. \tag{6.30}$$

Thais is

$$\hat{g}(\mu) = \frac{\hat{g}(\mu_0)}{1 - 3\hat{g}(\mu_0)\ln\mu/\mu_0}. \tag{6.31}$$

As μ increases, $\hat{g}$ increases; perturbation theory can only be valid when

$$3\hat{g}(\mu_0)\ln\mu/\mu_0 \ll 1. \tag{6.32}$$

However, if we took this formula literally, it would say that $\hat{g} = \infty$ when

$$\frac{\mu}{\mu_0} = \exp\left(\frac{1}{3\hat{g}(\mu_0)}\right). \tag{6.33}$$

This is the famous *Landau singularity*, or *ghost pole*. It is of no importance in a weakly coupled theory like QED, but could be very important in strongly coupled QCD.

A point at which $\beta(\lambda) = 0$ is called a *fixed point*. Here we have $\lambda = 0$ as a fixed point, where $\beta'(0) = 0$, but $\beta''(0) > 0$. The infrared behavior of β is sketched in Fig. 6.1. The fixed point at zero coupling is an infrared stable point. It means that the coupling λ increases as the distance decreases, or as the energy increases.

Could there be another fixed point at $\lambda = \bar{\lambda}$ as shown? If so, in the vicinity of that point

$$\mu\frac{\partial\lambda}{\partial\mu} = (\lambda - \bar{\lambda})\beta'(\bar{\lambda}), \tag{6.34}$$

so if $\beta'(\bar{\lambda}) < 0$ as shown, λ is driven toward $\bar{\lambda}$:

$$\frac{1}{\beta'}\frac{d\lambda}{\lambda - \bar{\lambda}} = \frac{d\mu}{\mu}, \tag{6.35}$$

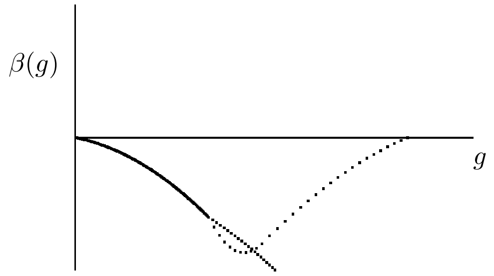

Figure 6.2: Beta function for quantum chromodynamics. Shown is the ultraviolet fixed point at $g = 0$. Different possible evolutions are shown for large g.

or

$$\frac{1}{\beta'}\ln(\lambda - \bar{\lambda}) = \ln \mu/\mu_0 + \text{constant}, \tag{6.36}$$

or

$$\beta' < 0: \quad \lambda - \bar{\lambda} = \text{constant}\left(\frac{\mu}{\mu_0}\right)^{\beta'} \to 0 \quad \text{as} \quad \mu \to \infty. \tag{6.37}$$

Thus $\bar{\lambda}$ is an ultraviolet stable fixed point.

Most theories, like this one, have an infrared stable fixed point at $\lambda = 0$. QCD is the exception. There $\beta(g) < 0$ for small g, as sketched in Fig. 6.2. Then $g = 0$ is an ultraviolet fixed point, and presumably $g \to 0$ as $\mu \to \infty$ (high energies, or short distances). This phenomenon is known as *asymptotic freedom*.

6.2 Scaling

We can translate the renormalization group equation into one which tells us how Green's functions scale with momenta. We do this by combining Eq. (6.4) with the fact that the "engineering" dimension of Γ_R is $4-n+\frac{\epsilon}{2}(n-2)$ (homework), which means we can write

$$\Gamma_R^{(n)}(p_i; m, \mu) = \mu^{4-n+(\epsilon/2)(n-2)}\Gamma\left(\frac{p_i}{m}; \frac{\mu}{m}\right), \tag{6.38}$$

so we deduce that

$$\mu\frac{\partial}{\partial\mu}\Gamma_R^{(n)} + \left[4 - n + \frac{\epsilon}{2}(n-2)\right]\Gamma_R^{(n)} + \mu^{4-n+(\epsilon/2)(n-2)}\frac{\mu}{m}\frac{\partial}{\partial(\mu/m)}\Gamma. \tag{6.39}$$

On the other hand,

$$m\frac{\partial}{\partial m}\Gamma_R^{(n)} = \mu^{4-n+(\epsilon/2)(n-2)}\left[-\sum_i \frac{p_i}{m}\frac{\partial}{\partial(p_i/m)} - \frac{\mu}{m}\frac{\partial}{\partial(\mu/m)}\right]\Gamma. \tag{6.40}$$

Therefore,

$$\mu\frac{\partial}{\partial\mu}\Gamma_R^{(n)} = \left\{\left[4-n+\frac{\epsilon}{2}(n-2)\right] - m\frac{\partial}{\partial m} - \sum_i p_i\frac{\partial}{\partial p_i}\right\}\Gamma_R^{(n)}, \tag{6.41}$$

or, if we scale all the momenta together,

$$p_i \to sp_i, \quad \sum_i p_i\frac{\partial}{\partial p_i} = s\frac{\partial}{\partial s}, \tag{6.42}$$

we have

$$\mu\frac{\partial}{\partial\mu}\Gamma_R^{(n)} = \left[4-n+\frac{\epsilon}{2}(n-2) - m\frac{\partial}{\partial m} - s\frac{\partial}{\partial s}\right]\Gamma_R^{(n)}. \tag{6.43}$$

Using this result, the renormalization group for $\Gamma_R^{(n)} = \Gamma_R^{(n)}(sp_i; m, \lambda, \mu)$ is

$$\left[-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial\lambda} + (\delta-1)m\frac{\partial}{\partial m} + 4 - n - n\gamma\right]\Gamma_R^{(n)} = 0, \tag{6.44}$$

where we have set $\epsilon = 0$. You see now why γ is called the *anomalous dimension.* In the 't Hooft-Weinberg scheme, the coefficients β, δ, γ do not depend on m, but only on λ, so the above equation may be integrated. Define scale-dependent couplings and masses by

$$s\frac{\partial\bar\lambda(s)}{\partial s} = \beta(\bar\lambda(s)), \tag{6.45a}$$

$$s\frac{\partial\bar m(s)}{\partial s} = \bar m(s)[\delta(\bar\lambda(s)) - 1], \tag{6.45b}$$

subject to the initial conditions

$$\bar\lambda(s=1) = \lambda, \quad \bar m(s=1) = m. \tag{6.46}$$

Then the solution to the renormalization group equation is

$$\Gamma_R^{(n)}(sp_i; m, \lambda, \mu) = s^{4-n}e^{-n\int_1^s \frac{ds'}{s'}\gamma(\bar\lambda(s'))}\Gamma_R^{(n)}(p_i; \bar m(s), \bar\lambda(s), \mu). \tag{6.47}$$

The proof of this result is quite immediate: Since the renormalization group equation is true for any λ, m, it is also true that

$$\left\{-\sum_i p_i\frac{\partial}{\partial p_i} + \beta(\bar\lambda(s))\frac{\partial}{\partial\bar\lambda(s)} + [\delta(\bar\lambda(s)) - 1]\bar m(s)\frac{\partial}{\partial\bar m(s)} + 4 - n - n\gamma(\bar\lambda(s))\right\}$$
$$\times\Gamma_R^{(n)}(p_i; \bar m(s), \bar\lambda(s), \mu) = 0, \tag{6.48}$$

which implies

$$0 = s\frac{d}{ds}\left[s^{4-n}e^{-n\int_1^s \frac{ds'}{s'}\gamma(\bar\lambda(s'))}\right]\Gamma_R^{(n)}(s^{-1}p_i; \bar m(s), \bar\lambda(s), \mu), \tag{6.49}$$

from which the result (6.47) follows.

Apart from the anomalous dimension effect, the Green's functions are governed by $\bar{\lambda}(s)$, $\bar{m}(s)$, as we scale the momenta, $p_i \to p_i s$.

How does all this work for our $\lambda\phi^4$ theory if we merely use our lowest-order perturbative results? We have already studied the behavior of the running coupling [we have dropped the caret, but $1/(4\pi)^2$ is still subsumed into g],

$$s\frac{\partial \bar{g}(s)}{\partial s} = 3\bar{g}^2, \quad \bar{g}(1) = g, \tag{6.50a}$$

which has solution (6.31), or

$$\bar{g}(s) = \frac{g}{1 - 3g\ln s}. \tag{6.50b}$$

The running mass is governed by [see Eq. (6.14)]

$$\frac{s}{\bar{m}}\frac{\partial \bar{m}}{\partial s} = \delta(\bar{\lambda}(s)) - 1 = -1 + \frac{1}{2}\bar{g}(s) - \frac{5}{12}\bar{g}^2(s), \tag{6.51}$$

which implies, if we use Eqs. (6.50a) and (6.50b)

$$\begin{aligned}
\ln \bar{m}(s) &= -\ln s - \frac{5}{36}\bar{g}(s) + \frac{1}{2}\int \frac{ds}{s}\frac{g}{1-3g\ln s} \\
&= -\ln s - \frac{5}{36}\bar{g}(s) - \frac{1}{6}\ln(1 - 3g\ln s) + \text{ constant} \\
&= -\ln s - \frac{5}{36}\bar{g}(s) - \frac{1}{6}\ln[g/\bar{g}(s)] + \text{ constant}.
\end{aligned} \tag{6.52}$$

The solution of this equation is

$$\bar{m}(s) = \frac{m}{s}\left(\frac{g(s)}{g}\right)^{1/6} e^{-\frac{5}{36}[\bar{g}(s)-g]} \tag{6.53}$$

which tends to zero as $s \to \infty$, which is to say that the mass becomes irrelevant at high energy, as we would expect.

As for the anomalous dimension, we know from Eq. (6.11) that

$$\gamma(\bar{g}(s)) = \frac{\bar{g}^2(s)}{12} = \frac{1}{36}s\frac{\partial \bar{g}}{\partial s}. \tag{6.54}$$

Putting all this together, we learn the scaling behavior of the proper Green's functions,

$$\Gamma_R^{(n)}(sp_i; m, g, \mu) = s^{4-n-\frac{n}{36}[\bar{g}(s)-g]}\Gamma_R^{(n)}(p_i; \bar{m}(s), \bar{g}(s), \mu). \tag{6.55}$$

Of course, these results break down for large s, where the Landau singularity in $\bar{g}(s)$ is approached.