Chapter 5

Renormalization

What sense can we make of a divergent diagrammatic expansion? We will see that it is possible, to any finite order in λ, to absorb the infinities in the perturbation expansion into three infinite constants:

- m_0, the *bare mass*,
- λ_0, the *bare coupling constant*, and
- Z, the *wavefunction renormalization constant*.

We will see how these quantities are defined in the following.

5.1 Order λ

Let us first work to order λ. The amputated two-point Green's function is given by Eq. (3.122),

$$\Gamma^{(2)}(p) = -(p^2 + m^2) + \Sigma^{(1)}, \tag{5.1}$$

where the mass operator is given by Eq. (4.22), or

$$\Sigma^{(1)} = 12\hat{\lambda} m^2 \left(\frac{2}{\epsilon} + \psi(2) - \ln \hat{m}^2 \right), \tag{5.2}$$

where we have introduced the abbreviations

$$\hat{\lambda} = \frac{\lambda}{(4\pi)^2}, \quad \hat{m}^2 = \frac{m^2}{4\pi\mu^2}. \tag{5.3}$$

Recall, from Eq. (3.121), that the full propagator is

$$G^{(2)}(p) = -\frac{1}{\Gamma^{(2)}(p)}. \tag{5.4}$$

The only other diagram to this order, corresponding to the first graph in Fig. 3.1, is finite:

$$\Gamma^{(4)} = -4!\lambda. \tag{5.5}$$

How can we absorb the infinity, as $\epsilon \to 0$, in Σ? We note that it amounts to an infinite rescaling of m. Let us therefore write m_0 in place of m, and assume that m_0 can be written as a power series in the coupling λ:

$$m_0^2 = m_R^2 + \lambda a + \lambda^2 b + \dots, \tag{5.6}$$

where the first term in the series is called the renormalized mass-squared. It is considered to be the observed physical mass of the particle, and therefore must be defined by the location of the pole of $G^{(2)}$: to order λ

$$\Gamma^{(2)}(p) = -p^2 - m_R^2, \tag{5.7}$$

which says from Eq. (5.1) that

$$\lambda a = 12\hat{\lambda} m_R^2 \left(\frac{2}{\epsilon} + \psi(2) - \ln \hat{m}_R^2\right). \tag{5.8}$$

To this order, then, this is equivalent to using m_R^2 in the Lagrangian, but adding an extra *counterterm* to $\mathcal{L}$ to cancel Σ:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \lambda\phi^4 \\ &= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_R^2\phi^2 + \lambda\phi^4 + \mathcal{L}_{\text{ct}}, \end{aligned} \tag{5.9}$$

where

$$\mathcal{L}_{\text{ct}} = \frac{1}{2}\delta m^2 \phi^2, \tag{5.10}$$

with

$$\delta m^2 = 12\hat{\lambda} m_R^2 \left(\frac{2}{\epsilon} + \psi(2) - \ln \hat{m}_R^2\right). \tag{5.11}$$

The extra contribution to the mass operator from the counterterm is shown in Fig. 5.1. Evidently,

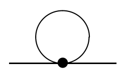
$$\Sigma^{(1)}\Big|_{m \to m_R} + \Sigma_{\text{ct}} = 0, \tag{5.12}$$

which is the content of Eq. (5.7).

5.2 Order λ^2

What happens in 2nd order? Now both two-and four-point functions are divergent.

$$\Sigma^{(1)}\Big|_{m\to m_R} =$$



$$= 12\hat{\lambda}m_R^2\left(\tfrac{2}{\epsilon} + \psi(2) - \ln \hat{m}_R^2\right)$$

$$\Sigma_{\rm ct} =$$

$$= -12\hat{\lambda}m_R^2\left(\tfrac{2}{\epsilon} + \psi(2) - \ln \hat{m}_R^2\right)$$

Figure 5.1: Diagram of the one-loop and the order λ counterterm contribution to the mass operator.

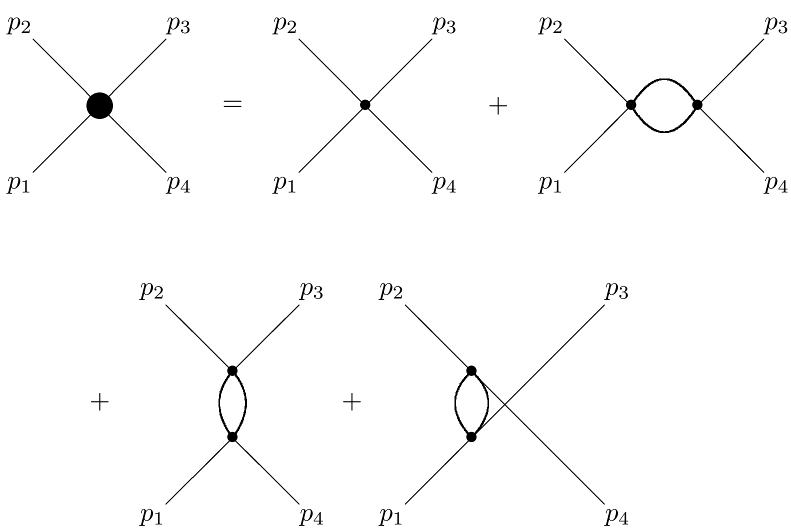


Figure 5.2: Feynman diagrams contibuting to the four-point function through order λ^2.

5.2.1 Four-point Function

The Feynman diagrams for the latter are given in Fig. 5.2. From Eq. (4.27), we have

$$\begin{aligned}\Gamma^{(4)}(p_1,p_2,p_3,p_4) &= -4!\lambda\mu^\epsilon\left\{1-\frac{3}{2}4!\hat{\lambda}\left[\frac{2}{\epsilon}+\psi(1)+2-\ln\hat{m}^2\right]\right.\\ &\qquad\left.+\frac{4!}{2}\hat{\lambda}A(s,t,u)\right\},\end{aligned} \tag{5.13}$$

where

$$A(s,t,u)=\sum_{z=s,t,u}\left(1+\frac{4m^2}{z}\right)^{1/2}\ln\frac{\sqrt{1+\frac{4m^2}{z}}+1}{\sqrt{1+\frac{4m^2}{z}}-1}, \tag{5.14}$$

in terms of the *Mandelstam variables*, which are defined by

$$s=(p_1+p_2)^2,\quad t=(p_1+p_3)^2,\quad u=(p_1+p_4)^2. \tag{5.15}$$

(Remember, all momenta are defined as ingoing.) Note that these variables satisfy

$$\begin{aligned}s+t+u &= 3p_1^2+p_2^2+p_3^2+p_4^2+2p_1\cdot p_2+2p_1\cdot p_3+2p_1\cdot(-p_1-p_2-p_3)\\ &= p_1^2+p_2^2+p_3^2+p_4^2=-\sum_{i=1}^4 m_i^2=-4m^2,\end{aligned} \tag{5.16}$$

on the "mass-shell," where $p_i^2=-m_i^2$. (We are now applying the result in the Minkowski region.)

We now can remove the divergence in $\Gamma^{(4)}$ by letting

$$\lambda\to\lambda_0=\lambda_R+A\lambda_R^2+B\lambda_R^3+\dots. \tag{5.17}$$

Here λ_0 is called the bare coupling constant. The renormalized coupling constant λ_R may be taken to be defined by the four-point function when all the external particles are on their mass shell and carry zero spatial momentum:

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4)\bigg|_{p_i^2=-m^2,\mathbf{p}_i=0}=-4!\lambda_R\mu^\epsilon. \tag{5.18}$$

The Minkowski point

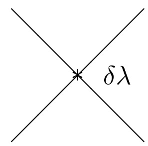
$$p_i^2=-m^2,\quad \mathbf{p}_i=0, \tag{5.19}$$

for example, corresponds to

$$p_1=p_2=-p_3=-p_4=(m,\mathbf{0}), \tag{5.20}$$

for which $t=u=0$, $s=-4m^2$ (the other possibilities are just a permutation of s, t, and u), and so we have from Eq. (5.14) the value

$$A(-4m^2,0,0)=4. \tag{5.21}$$



$$= -4!\delta\lambda\,\mu^\epsilon$$

Figure 5.3: Counterterm for the four-point function.

Thus the condition (5.18) implies for the first term in Eq. (5.17)

$$A\lambda_R^2 = 4!\frac{3}{2}\lambda_R\hat{\lambda}_R\left[\frac{2}{\epsilon} + \psi(1) + \frac{2}{3} - \ln\hat{m}_R^2\right]. \tag{5.22}$$

In effect, we introduce another counterterm in $\mathcal{L}$:

$$\lambda_0\mu^\epsilon\phi^4 = \lambda_R\mu^\epsilon\phi^4 + \mathcal{L}'_{\rm ct}, \tag{5.23}$$

where

$$\mathcal{L}'_{\rm ct} = \delta\lambda\,\mu^\epsilon\phi^4, \tag{5.24}$$

with

$$\delta\lambda = \frac{9}{4}\frac{\lambda_R^2}{\pi^2}\left[\frac{2}{\epsilon} + \psi(1) + \frac{2}{3} - \ln\hat{m}_R^2\right]\phi^4. \tag{5.25}$$

The counterterm may be represented by an additional four-point graph, shown in Fig. 5.3.

Although the above choice of the renormalized coupling constant is rather physical, it is not unique. For example, we could also define it as the value of the four-point function at all momenta equal to zero:

$$\Gamma^{(4)}\Big|_{p_i=0} = -4!\lambda_R\mu^\epsilon, \tag{5.26}$$

so since then $A(0,0,0) = 6$,

$$\delta\lambda = \frac{9}{4}\frac{\lambda_R^2}{\pi^2}\left[\frac{2}{\epsilon} + \psi(1) - \ln\hat{m}_R^2\right]\phi^4. \tag{5.27}$$

The finite part of the counterterm has been changed by this alternative prescription. In general let us write

$$\delta\lambda = \frac{9}{2\pi^2}\lambda_R^2\frac{1}{\epsilon}\left(1 + \frac{\epsilon}{2}\beta\right), \tag{5.28}$$

where β is a function of ϵ and $\hat{m}^2$; $\beta = \psi(1) - \ln\hat{m}_R^2$ in the scheme given by Eq. (5.26).

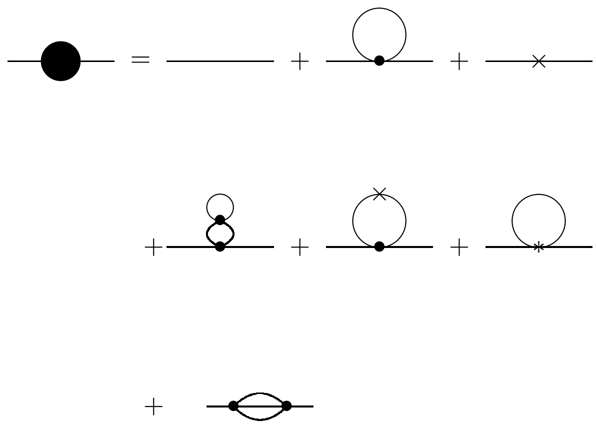


Figure 5.4: Feynman diagrams, including counterterms, contributing to the mass operator through two-loop order.

5.2.2 Two-point Function

We now turn to the two-point function in order λ^2. The graphs through order λ^2 including the counter terms are shown in Fig. 5.4. According to Eqs. (3.126), (4.22), (4.23), and (4.56), as well as the expressions for the counterterms (we have written λ for λ_R),

$$
\begin{aligned}
\Gamma^{(2)}(p) &= -(p^2+m^2) + 12\hat\lambda m^2\left(\frac{2}{\epsilon} + \psi(2) - \ln \hat m^2\right) - \delta m^2 \\
&\quad + 144\hat\lambda^2 m^2 \Big\{ -\frac{4}{\epsilon^2} - \frac{2}{\epsilon}\left[1 + 2\psi(1) - 2\ln\hat m^2\right] + 2\ln^2\hat m^2 \\
&\qquad + 2\left[1+2\psi(1)\right]\ln\hat m^2 - \frac{1}{2}\left[1+2\psi(1)\right]^2 - \frac{1}{2}\left[1+2\psi'(1)\right] \Big\} \\
&\quad + 576\hat\lambda^2 m^2\left\{\frac{1}{\epsilon^2} + \frac{1}{2\epsilon}\left[\psi(1) - \ln\hat m^2 + \alpha\right] + \mathcal{O}(1)\right\} \\
&\quad + 432\hat\lambda^2 m^2\left\{\frac{4}{\epsilon^2} + \frac{2}{\epsilon}\left[\psi(1) + 1 - \ln\hat m^2 + \beta\right] + \mathcal{O}(1)\right\} \\
&\quad - 96\hat\lambda^2\left\{\frac{6m^2}{\epsilon^2}\left[1 + \epsilon\left(\frac{3}{2} + \psi(1) - \ln\hat m^2\right)\right] + \frac{p^2}{2\epsilon} + \mathcal{O}(1)\right\} \\
&= -p^2 - m^2 - \delta m^2 + 12\hat\lambda m^2\left(\frac{2}{\epsilon} + \psi(2) - \ln\hat m^2\right)
\end{aligned}
$$

$$+ (4!)^2\hat{\lambda}^2 m^2 \left\{ \frac{2}{\epsilon^2} + \frac{1}{2\epsilon}(\alpha + 3\beta - 1) - \frac{p^2}{12\epsilon m^2} + \mathcal{O}(1) \right\}. \quad (5.29)$$

Here we have evaluated the two one-loop graphs with counterterm insertions as follows. The fifth graph in Fig. 5.4 is

$$\begin{aligned}
\Sigma_5 &= -\frac{\lambda 4!}{2}(-\delta m^2)\mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + m^2)^2} \\
&= -\frac{\lambda 4!}{2}(-\delta m^2)\mu^{4-d} \frac{(m^2)^{d/2-2}}{(4\pi)^{d/2}} \Gamma(2 - d/2) \\
&= \frac{4!\lambda}{2}\delta m^2 \frac{1}{(4\pi)^2} \left(\frac{m^2}{4\pi\mu^2} \right)^{-\epsilon/2} \frac{2}{\epsilon} \left[1 + \frac{\epsilon}{2}\psi(1) \right] \\
&= \frac{4!\hat{\lambda}}{\epsilon} 12\hat{\lambda} m^2 \frac{2}{\epsilon} \left(1 + \frac{\epsilon}{2}\alpha + \dots \right) \left[1 + \frac{\epsilon}{2} \left(\psi(1) - \ln \hat{m}^2 \right) + \dots \right] \\
&= 288\hat{\lambda}^2 m^2 \frac{2}{\epsilon^2} \left\{ 1 + \frac{\epsilon}{2} \left[\psi(1) - \ln \hat{m}^2 + \alpha \right] + \dots \right\}, \qquad (5.30)
\end{aligned}$$

where in the scheme given in Sec. 5.1 [see Eq. (5.11)],

$$\alpha = \psi(2) - \ln \hat{m}^2, \qquad (5.31)$$

but in other renormalization schemes, α may have a different value. The sixth graph in Fig. 5.4 is from Eq. (4.22) or (5.2)

$$\begin{aligned}
\Sigma_6 &= 12\delta\hat{\lambda} m^2 \left[\frac{2}{\epsilon} + \psi(2) - \ln \hat{m}^2 \right] \\
&= 12 m^2 \hat{\lambda}^2 \frac{3}{2} 4! \left(\frac{2}{\epsilon} + \beta \right) \left[\frac{2}{\epsilon} + \psi(2) - \ln \hat{m}^2 \right] \\
&= 432\hat{\lambda}^2 m^2 \left\{ \frac{4}{\epsilon^2} + \frac{2}{\epsilon} \left[\psi(1) - \ln \hat{m}^2 + \beta + 1 \right] + \dots \right\}. \qquad (5.32)
\end{aligned}$$

In the second line we used the result from Eq. (5.28). We have only displayed the divergent parts of these graphs.

Now we require that this two-point function (5.29) be finite. In part this can be achieved by a cancellation between the mass counterterm and the divergent parts:

$$\delta_1 m^2 = m^2 \left\{ 2\frac{(4!)^2\hat{\lambda}^2}{\epsilon^2} + \frac{4!\hat{\lambda}}{\epsilon} \left[1 + 4!\hat{\lambda}\frac{1}{2}(\alpha + 3\beta - 1) \right] \right\}, \qquad (5.33)$$

and then regarding $m^2 = m_R^2$ as finite. However, there is also a divergence proportional to p^2. Write the remainder of $\Gamma^{(2)}$ as

$$\begin{aligned}
-p^2 \left[1 + \frac{(4!)^2\hat{\lambda}^2}{12\epsilon} \right] - m_R^2 - \delta_2 m^2 &= -\left[1 + \frac{(4!)^2\hat{\lambda}^2}{12\epsilon} \right] (p^2 + m_R^2) \\
&\quad - \delta_2 m^2 + \frac{(4!)^2\hat{\lambda}^2}{12\epsilon} m_R^2 \\
&= -Z^{-1}(p^2 + m_R^2), \qquad (5.34)
\end{aligned}$$

where, up to finite terms,

$$Z = 1 - \frac{(4!)^2\hat\lambda^2}{12\epsilon}, \tag{5.35a}$$

$$\delta_2 m^2 = \frac{(4!)^2\hat\lambda^2}{12\epsilon} m_R^2. \tag{5.35b}$$

Here we have written the mass counterterm as

$$\delta m^2 = \delta_1 m^2 + \delta_2 m^2, \tag{5.36}$$

and the bare mass as

$$m_0^2 = m_R^2 + \delta m^2 = (m_R^2 + \delta_1 m^2) Z^{-1} \tag{5.37}$$

5.3 Renormalized Green's Functions

The Lagrangian, then, may be expressed either in terms of bare quantities,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 + \frac{1}{2}m_0^2\phi_0^2 + \lambda_0\phi_0^4, \tag{5.38}$$

or in terms of renormalized ones,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m_R^2\phi^2 + \lambda_R\mu^\epsilon\phi^4 + \mathcal{L}_{\rm ct}, \tag{5.39}$$

where the counterterm Lagrangian is

$$\mathcal{L}_{\rm ct} = \frac{1}{2}A\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\delta m^2\phi^2 + \delta\lambda\,\mu^\epsilon\phi^4, \tag{5.40}$$

where $\delta\lambda$, δm^2, and A may be expressed as power series in the renormalized coupling constant λ_R, with divergent coefficients. *Note that only structures present in the original Lagrangian appear in* $\mathcal{L}_{\rm ct}$. This is what we mean by a *renormalizable theory*—a nonrenormalizable theory would require the introduction of an infinite number of counterterms, with structures that do not appear in $\mathcal{L}$. [Einstein's gravity theory, general relativity, is a prime example of a nonrenormalizable theory.] By comparing the two forms of $\mathcal{L}$ above we see

$$\phi_0 = (1+A)^{1/2}\phi \equiv Z^{1/2}\phi, \tag{5.41a}$$

$$m_0^2 Z = m_R^2 + \delta m^2, \quad m_0^2 = (m_R^2 + \delta m^2)Z^{-1}, \tag{5.41b}$$

$$\lambda_0 Z^2 = (\lambda_R + \delta\lambda)\mu^\epsilon, \quad \lambda_0 = (\lambda_R + \delta\lambda)Z^{-2}\mu^\epsilon. \tag{5.41c}$$

[Since $Z = 1 + \mathcal{O}(\lambda^2)$, the Z correction to the coupling appear only in order λ^3.] Note that we are now absorbing μ^ϵ into the definition of λ_0, so that the bare theory contains no reference to the arbitrary mass scale μ, which is an artifact of renormalization. In the above ϕ_0 is called the bare field, and Z the wavefunction renormalization constant.

All the infinities in the theory can be absorbed into the three infinite constants, λ_0, m_0, and Z, and finite, renormalized Green's functions are given by

$$\Gamma_R^{(n)}(p_1,\dots,p_n;\lambda_R,m_R,\mu,\epsilon) = Z^{n/2}\Gamma_0^{(n)}(p_1,\dots,p_n;\lambda_0,m_0,\epsilon). \tag{5.42}$$

For example, for the two-point function,

$$p^2 + m_R^2 = Z[p^2 + m_0^2 - \Sigma(p,\lambda_0,m_0,\epsilon)]. \tag{5.43}$$

The corresponding formula for the full Green's functions is

$$G_R^{(n)}(p_1,\dots,p_n;\lambda_R,m_R,\mu,\epsilon) = Z^{-n/2}G_0^{(n)}(p_1,\dots,p_n;\lambda_0,m_0,\epsilon). \tag{5.44}$$

This is proved as follows. The bare generating function is given by

$$Z_0[K] = \int [d\phi_0] e^{-\int (dx)\left[\frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \lambda_0\phi_0^4 - K\phi_0\right]}. \tag{5.45}$$

The renormalized generating function is given by

$$Z[K] = \int [d\phi] e^{-\int (dx)\left[\frac{Z}{2}(\partial\phi)^2 + \frac{Z}{2}m_0^2\phi^2 + Z^2\lambda_0\phi^4 - K\phi\right]} = Z_0[KZ^{-1/2}]. \tag{5.46}$$

where we have rescaled the functional integral:

$$\phi_0 = Z^{1/2}\phi \to \phi. \tag{5.47}$$

(Remember that the functional integral is undefined up to an overall constant.) Then the renormalized Green's functions, which are finite when expressed in terms of λ_R and m_R, are defined by

$$\begin{aligned} G^{(n)}(p_1,\dots,p_n) &= \frac{\delta^n}{\delta K(p_1)\cdots\delta K(p_n)} \ln Z[K]\bigg|_{K=0} \\ &= Z^{-n/2}\frac{\delta^n}{\delta K(p_1)\cdots\delta K(p_n)} \ln Z_0[K]\bigg|_{K=0} \\ &= Z^{-n/2} G_0^{(n)}(p_1,\dots,p_n), \end{aligned} \tag{5.48}$$

where the unrenormalized Green's function is taken to depend on λ_0, m_0, and ϵ. For the corresponding proof of the statement (5.42) about the $\Gamma^{(n)}$'s, see the homework.

Let us summarize our results for the relations between bare and renormalized parameters, and introduce a bit more general notation. Here we will drop the R subscript on renormalized quantities, and let $g = 4!\lambda$. Through order g^2 the wavefunction renormalization constant has the form [see Eq. (5.35a)]

$$Z = 1 - \frac{\hat{g}^2}{12\epsilon} - \hat{g}^2 H_2(\epsilon,\hat{m}^2), \tag{5.49}$$

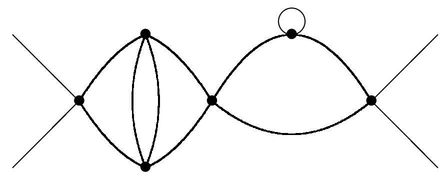


Figure 5.5: Typical graph contributing to the four-point function.

where $\hat{g} = g/(4\pi)^2$ and $\hat{m}^2 = m^2/(4\pi\mu^2)$. Here H_2 is a function which is finite as $\epsilon \to 0$. The bare field is related to the renormalized field by

$$\phi_0 = Z^{1/2}\phi. \tag{5.50}$$

The mass counterterm is given by [see Eq. (5.33)]

$$\begin{aligned}\delta m^2 \; = m^2 \bigg\{ & \hat{g}\left[\frac{1}{\epsilon} + \frac{1}{2}F_1(\epsilon, \hat{m}^2)\right] + \frac{2\hat{g}^2}{\epsilon^2} \\ & + \frac{\hat{g}^2}{2\epsilon}[F_1(\epsilon, \hat{m}^2) + 3G_1(\epsilon, \hat{m}^2) - 1] + \frac{1}{2}\hat{g}^2 F_2(\epsilon, \hat{m}^2)\bigg\}. \end{aligned} \tag{5.51}$$

Again the functions F_1, F_2, G_1, and G_2 are finite as $\epsilon \to 0$. (In the above we called F_1, α, and G_1, β.) The bare mass is related to the renormalized mass by

$$m_0^2 = (m^2 + \delta m^2)Z^{-1}. \tag{5.52}$$

(Here δm^2 is what we called $\delta_1 m^2$ above.) The coupling counterterm is given by [see Eq. (5.28)]

$$\delta\hat{g} = \hat{g}^2 \frac{3}{\epsilon}\left[1 + \frac{\epsilon}{2}G_1(\epsilon, \hat{m}^2)\right], \tag{5.53}$$

and the bare coupling is given in terms of the renormalized coupling by

$$\hat{g}_0 = \mu^\epsilon(\hat{g} + \delta\hat{g})Z^{-2}. \tag{5.54}$$

5.4 Divergences of Feynman Diagrams

Have we really got them all? Consider a general graph, with I internal momentum, L loops, and V vertices, such as that shown in Fig. 5.5. A moment's reflection show that these numbers are related by the following formula:

$$L = I - V + 1. \tag{5.55}$$

For the example in Fig. 5.5, $I = 10$, $V = 6$, and indeed $L = 5$. Here the V arises because there is one momentum restriction per vertex, and the 1 because

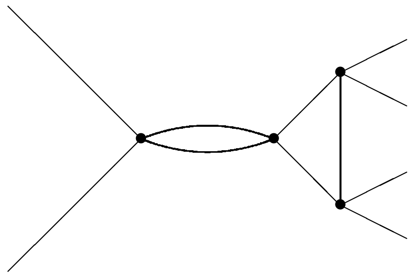


Figure 5.6: Superficially convergent graph contributing to $\Gamma^{(6)}$ which contains the divergent subgraph in Fig. 4.1.

of overall momentum conservation. Now the *superficial degree of divergence* of a Feynman diagram is given by

$$D = dL - 2I, \tag{5.56}$$

where d is the number of dimensions of spacetime, because for each loop integration one integrates the volume element $d^d l$, and for each internal line there is a propagator $1/[(l-p)^2+m^2]$. For the example given in Fig. 5.5, $D = 0$, signifying a logarithmic divergence. (Internal subgraphs are more divergent.) These first two relations hold for any scalar theory. Now for the $\lambda\phi^4$ theory, each vertex has four lines coming from it, so because each internal line connects to two vertices,

$$4V = E + 2I, \tag{5.57}$$

both sides of which are 24 in the above example. Combining these relations, we find for the superficial degree of divergence

$$\begin{aligned} D &= d(I - V + 1) - 2I = d - dV + (d-2)(2V - E/2) \\ &= d + (d-4)V - \frac{d-2}{2}E = 4 - E, \end{aligned} \tag{5.58}$$

where the last form holds in four dimensions.

If $D \geq 0$ the diagram is divergent:

- $\Gamma^{(2)}$ has $D = 2$, so the two-point function possesses a quadratic divergence.
- $\Gamma^{(4)}$ has $D = 0$ so the four-point function possesses a logarithmic divergence.

If $E > 4$, or $D < 0$, the diagram is superficially convergent, but it may contain divergent subintegrations, as shown in Fig. 5.6. *Weinberg's theorem* states that a Feynman diagram is convergent if its superficial degree of divergence, and that of all its subgraphs, is negative. The only *primitive* divergences are in $\Gamma^{(2)}$ and $\Gamma^{(4)}$—these divergences can be absorbed by renormalization of parameters in $\mathcal{L}$—this is the essence of a renormalizable theory. (The difficult point is establishing renormalizability is in proving that overlapping divergence do not generate terms like $\frac{1}{\epsilon}\ln p^2$.)