Chapter 4

Divergences

It is apparent that the perturbative expansion given in the previous chapter is *divergent*, term by term. (An entirely separate issue is that of the convergence of the perturbation series.) For example, consider the order λ one-particle irreducible graph in Fig. 3.9, which corresponds to the mass operator (3.109), or in four-dimensional spherical coordinates, $(dl) = d\Omega\, l^3\, dl$, and $\Omega = 2\pi^2$ being the area of a unit four-sphere,

$$
\begin{aligned}
-12\lambda \int \frac{(dl)}{(2\pi)^4} \frac{1}{l^2+m^2} &= -\frac{12\lambda\Omega}{(2\pi)^4} \frac{1}{2} \int_0^{\Lambda^2} \frac{l^2\, dl^2}{l^2+m^2} \\
&= -\frac{6\lambda\Omega}{(2\pi)^4} \left(\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right), \qquad (4.1)
\end{aligned}
$$

where we have introduced an ultraviolet momentum cutoff $\Lambda \gg m$. This diagram is *quadratically* divergent. Another example is the contribution to the four-point function given in Fig. 3.6, or in momentum space given in Fig. 4.1. The corresponding momentum-space integral is

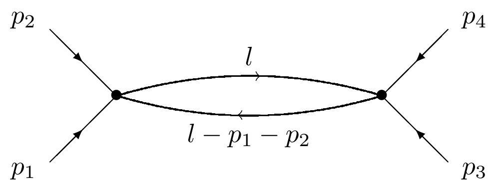

Figure 4.1: A second-order Feynman diagram for the one-particle irreducible four-point function $G^{(4)}(p_1, p_2, p_3, p_4)$ corresponding to Eq. (4.2). All permutations of the momentum labels have to be considered. The arrows indicate the sense of momentum flow.

$$
\begin{aligned}
&\frac{(4!\lambda)^2}{2}\int\frac{(dl)}{(2\pi)^4}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2} \\
&= \frac{288\lambda^2}{(2\pi)^4}\frac{1}{2}\int_0^\Lambda l^2\,dl^2\int d\Omega\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}. \qquad (4.2)
\end{aligned}
$$

This is clearly *logarithmically divergent.* However, let us make it more explicit by using a trick due to Schwinger. Write

$$
\frac{1}{l^2+m^2} = \int_0^\infty ds\, e^{-s(l^2+m^2)}, \qquad (4.3)
$$

where the parameter s is called the Euclidean proper time. Then we can write the product of the two denominators in Eq. (4.2) as

$$
\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2} = \int_0^\infty ds\,ds'\,e^{-s'(l^2+m^2)-s[(l-p_1-p_2)^2+m^2]}. \qquad (4.4)
$$

Rewrite this by introducing a parameter u by a change of variables,

$$
s\to su, \quad s'\to s(1-u), \quad ds\,ds'\to s\,ds\,du, \qquad (4.5)
$$

where u ranges from 0 to 1. (The integration variable u is usually called a Feynman parameter, but Feynman acknowledged that he borrowed the idea from Schwinger, just as Schwinger adopted Feynman's propagator.) Then the exponent in Eq. (4.4) is

$$
\begin{aligned}
&s'(l^2+m^2)+s[(l-p_1-p_2)^2+m^2] \\
&= s\{l^2[u+(1-u)]-2u\,l\cdot(p_1+p_2)+u(p_1+p_2)^2+m^2[u+(1-u)]\} \\
&= s[l^2-2u\,l\cdot(p_1+p_2)+u(p_1+p_2)^2+m^2] \\
&= s\{[l-u(p_1+p_2)]^2+u(1-u)(p_1+p_2)^2+m^2\}, \qquad (4.6)
\end{aligned}
$$

where in the last line we have completed the square. Therefore, the diagram 4.1 represented by Eq. (4.2) becomes

$$
\frac{288\lambda^2}{(2\pi)^4}\int_0^\infty ds\,s\int_0^1 du\int(dl)e^{-s[l-u(p_1+p_2)]^2-s[u(1-u)(p_1+p_2)^2+m^2]}. \qquad (4.7)
$$

If it were legitimate to shift the integration variable

$$
[l-u(p_1+p_2)]^2\to l^2, \qquad (4.8)
$$

we would get

$$
\begin{aligned}
\int(dl)e^{-sl^2} &= \frac{1}{2}\Omega\int_0^\infty l^2\,dl^2\,e^{-sl^2} = \frac{\Omega}{2s^2} \\
&= \left[\int_{-\infty}^\infty dl\,e^{-sl^2}\right]^4 = \frac{\pi^2}{s^2}, \qquad (4.9)
\end{aligned}
$$

(which proves $\Omega = 2\pi^2$), and the divergence now appears as a singularity at small s:

$$\frac{144\lambda^2}{(2\pi)^4}\Omega \int_{s_0\to 0}^{\infty} \frac{ds}{s} \int_0^1 du\, e^{-s[u(1-u)(p_1+p_2)^2+m^2]}. \tag{4.10}$$

These divergences are a real artifact of field theory—we cannot eliminate them, but only sweep them away with a process called *renormalization.* To deal with them, we must *regulate* the theory, which we can do with cutoffs in large momentum (Λ) or small s (s_0). There are a number of formal techniques (Pauli-Villars, zeta-function regularization), but the most common nowadays is *dimensional regularization*, invented by 't Hooft and Veltman.

4.1 Dimensional Regularization

We use the proper-time representation encountered above, so that the generic momentum (loop) integral we encounter is

$$\int \frac{(dl)}{(2\pi)^4} e^{-s[(l-q)^2+M^2]}. \tag{4.11}$$

What we do is let the number of dimensions go from 4 to d, and then

$$\begin{aligned}\int \frac{d^d l}{(2\pi)^d} e^{-s(l^2+M^2)} &= \left[\frac{1}{\sqrt{s}}\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-p^2}\right]^d e^{-sM^2} \\ &= \frac{1}{(4\pi s)^{d/2}} e^{-sM^2}.\end{aligned} \tag{4.12}$$

Then the loop integral is, according to Eq. (4.3), provided d is chosen so that the integral converges, $d < 2$,

$$\begin{aligned}\int \frac{d^d l}{(2\pi)^d}\frac{1}{(l-q)^2+M^2} &= \int_0^\infty \frac{ds}{2^d \pi^{d/2} s^{d/2}} e^{-sM^2} \\ &= \frac{1}{(4\pi)^{d/2}}(M^2)^{d/2-1}\int_0^\infty \frac{dt}{t} t^{1-d/2} e^{-t} \\ &= \frac{M^{d-2}}{(4\pi)^{d/2}}\Gamma(1-d/2),\end{aligned} \tag{4.13}$$

and more generally, if we use Eq. (4.12) again,

$$\begin{aligned}\int \frac{d^d l}{(2\pi)^d}\frac{1}{(l-q)^2+M^2)^{n+1}} &= \int \frac{d^d l}{(2\pi)^d}\frac{1}{\Gamma(n+1)}\int_0^\infty ds\, s^n\, e^{-s(l^2+M^2)} \\ &= \frac{1}{\Gamma(n+1)}\int_0^\infty \frac{ds\, s^{n-d/2}}{(4\pi)^{d/2}} e^{-sM^2} \\ &= \frac{M^{d-2n-2}}{(4\pi)^{d/2}}\frac{\Gamma(n+1-d/2)}{\Gamma(n+1)}.\end{aligned} \tag{4.14}$$

The essence of dimensional regularization is that these results, derived under the assumption that $d < 2$ so that the integral converge, are taken to be true for all d. This is a type of analytic continuation. We see that a quadratically divergent diagram ($n = 0$) has simple poles at $d = 2, 4, \ldots$, while a logarithmically divergent diagram ($n = 1$) has poles at $d = 4, 6, \ldots$. Loop integrals with $n > 1$ have no divergences in four dimensions.

One further thing remains to be done. We need to ensure that λ remains dimensionless for all d. Because the dimension of ϕ is

$$[\phi] = \mathrm{Mass}^{d/2-1}, \tag{4.15}$$

which follows from the dimensional character of the free action, we must introduce an arbitrary mass μ into the interaction term in order to make it dimensionless:

$$W_{\mathrm{int}} = \lambda\mu^x \int d^dx\, \phi^4, \quad \Rightarrow x = 4 - d. \tag{4.16}$$

That is, we replace

$$\lambda \to \lambda(\mu^2)^{2-d/2}. \tag{4.17}$$

We are now going to evaluate all the $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$ graphs using dimensional regularization. We start with the 2-point function. The mass operator (3.109) becomes

$$\begin{aligned}
\Sigma^{(1)} &= -12\lambda(\mu^2)^{2-d/2} \int \frac{d^dl}{(2\pi)^d} \frac{1}{l^2+m^2} \\
&= -12\lambda(\mu^2)^{2-d/2} \frac{m^{d-2}}{(4\pi)^{d/2}} \Gamma(1-d/2) \\
&= -\frac{12\lambda m^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{2-d/2} \Gamma(1-d/2),
\end{aligned} \tag{4.18}$$

which uses Eq. (4.13). Now expand $\Gamma(1 - d/2)$ about $d = 4$ by introducing $d = 4 - \epsilon$, $|\epsilon| \ll 1$:

$$\Gamma(1-d/2) = \Gamma(-1+\epsilon/2) = \frac{1}{-1+\epsilon/2}\frac{1}{\epsilon/2}\Gamma(1+\epsilon/2) \approx -\frac{2}{\epsilon} - 1 - \Gamma'(1), \tag{4.19}$$

since $\Gamma(1 + x) = x\Gamma(x)$.

Now we introduce the diagamma function,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{4.20}$$

with the property

$$\psi(1+x) = \frac{d}{dx}\ln\Gamma(1+x) = \frac{d}{dx}\left[\ln\Gamma(x) + \ln x\right] = \psi(x) + \frac{1}{x}. \tag{4.21}$$

Then we can rewrite Eq. (4.18) as an infinite part plus a remainder:

$$\begin{aligned}
\Sigma^{(1)} &= -\frac{12\lambda m^2}{(4\pi)^2}\left(1+\frac{\epsilon}{2}\ln\frac{4\pi\mu^2}{m^2}+\ldots\right)\left(-\frac{2}{\epsilon}-\psi(2)\right)\\
&= -\frac{12\lambda m^2}{(4\pi)^2}\left(-\frac{2}{\epsilon}-\ln\frac{4\pi\mu^2}{m^2}-\psi(2)\right). \qquad (4.22)
\end{aligned}$$

The finite part is totally arbitrary, since it depends on the arbitrary artificial parameter μ.

Next we turn to the evaluation of the last graph in Fig. 3.10, which is according to Eq. (3.108)

$$\begin{aligned}
\Sigma^{(2b)} &= 144\lambda^2(\mu^2)^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+m^2)^2}\int\frac{d^dl'}{(2\pi)^d}\frac{1}{l'^2+m^2}\\
&= 144\lambda^2(\mu^2)^{4-d}\frac{m^{d-4}}{(4\pi)^{d/2}}\Gamma(2-d/2)\frac{m^{d-2}}{(4\pi)^{d/2}}\Gamma(1-d/2)\\
&= \frac{144\lambda^2m^2}{(4\pi)^4}\left(\frac{4\pi\mu^2}{m^2}\right)^{4-d}\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-1+\frac{\epsilon}{2}\right)\\
&= \frac{9\lambda^2m^2}{16\pi^4}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\left\{\frac{2}{\epsilon}+\psi(1)+\frac{\epsilon}{4}\left[\psi^2(1)+\psi'(1)\right]\right\}\\
&\quad\times\left\{-\frac{2}{\epsilon}-[\psi(1)+1]-\frac{\epsilon}{4}\left[2\psi(1)+\psi^2(1)+\psi'(1)+2\right]\right\}\\
&= \frac{9\lambda^2m^2}{16\pi^4}\Bigg\{-\frac{4}{\epsilon^2}-\frac{2}{\epsilon}\left[2\psi(1)+1+2\ln\frac{4\pi\mu^2}{m^2}\right]-2\ln^2\frac{4\pi\mu^2}{m^2}\\
&\quad-2\left[2\psi(1)+1\right]\ln\frac{4\pi\mu^2}{m^2}-\frac{1}{2}\left[2\psi(1)+1\right]^2-\frac{1}{2}\left[2\psi'(1)+1\right]\Bigg\},
\end{aligned}$$
(4.23)

where, in terms of Euler's constant,

$$\psi(1)=-\gamma=-0.5772156649,\quad\text{and}\quad\psi'(1)=\frac{\pi^2}{6}. \qquad (4.24)$$

Here both the coefficient of $1/\epsilon$ and the constant are ambiguous.

Next, let us consider the contribution to the four-point function in Fig. 4.1 given by Eq. (4.7), or in d dimensions

$$\begin{aligned}
\Gamma^{(4)} &= \frac{288\lambda^2\mu^{4-d}}{(4\pi)^2}\int_0^1 du\left[\frac{u(1-u)(p_1+p_2)^2+m^2}{4\pi\mu^2}\right]^{d/2-2}\Gamma(2-d/2)\\
&= \frac{288\lambda^2\mu^{\epsilon}}{(4\pi)^2}\int_0^1 du\left\{\frac{2}{\epsilon}+\psi(1)-\ln\left[\frac{u(1-u)(p_1+p_2)^2+m^2}{4\pi\mu^2}\right]\right\}.
\end{aligned}$$
(4.25)

It is easy to carry out the integration, by integrating by parts, and then partial fractioning. We leave the details to the homework. The result is

$$\int_0^1 du \ln\left[u(1-u)\frac{q^2}{m^2}+1\right] = -2 - \sqrt{1+\frac{4m^2}{q^2}}\ln\frac{\sqrt{1+\frac{4m^2}{q^2}}-1}{\sqrt{1+\frac{4m^2}{q^2}}+1}. \qquad (4.26)$$

Thus the graph in question has the value ($q = p_1 + p_2$)

$$\begin{aligned}\Gamma^{(4)}(p_1,p_2) &= \frac{288\lambda^2}{(4\pi)^2}\mu^\epsilon\left[\frac{2}{\epsilon}+\psi(1)+2+\ln\frac{4\pi\mu^2}{m^2}\right.\\ &\left.\quad -\sqrt{1+\frac{4m^2}{q^2}}\ln\frac{\sqrt{1+\frac{4m^2}{q^2}}+1}{\sqrt{1+\frac{4m^2}{q^2}}-1}\right], \qquad (4.27)\end{aligned}$$

There are actually three similar diagrams contributing to $\Gamma^{(4)}(p_1,p_2,p_3,p_4)$, depending on the attachment of the external momenta. In terms of the quantity computed above, we have

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \Gamma^{(4)}(p_1,p_2)+\Gamma^{(4)}(p_1,p_3)+\Gamma^{(4)}(p_1,p_4). \qquad (4.28)$$

Finally, we turn to the middle graph in Fig. 3.10,

$$\Sigma^{(2a)}(p) = 96\lambda^2(\mu^2)^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{d^dl'}{(2\pi)^d}\frac{1}{l^2+m^2}\frac{1}{l'^2+m^2}\frac{1}{(p-l-l')^2+m^2}, \qquad (4.29)$$

according to second line of Eq. (3.108). This diagram has what are called overlapping divergences, and must be handled with care. The standard trick is to insert the factor

$$1 = \frac{1}{2d}\left(\frac{dl^\mu}{dl^\mu}+\frac{dl'^\mu}{dl'^\mu}\right), \qquad (4.30)$$

and integrate by parts, dropping the surface term (see below). This gives

$$\begin{aligned}\Sigma^{(2a)}(p) &= -\frac{48\lambda^2}{d}(\mu^2)^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{d^dl'}{(2\pi)^d}2\left[-\frac{2l^\mu l_\mu}{(l^2+m^2)^2}\right.\\ &\quad\times\frac{1}{(l'^2+m^2)[(p-l-l')^2+m^2]}\\ &\left.\quad+\frac{1}{(l^2+m^2)(l'^2+m^2)}\frac{2l^\mu(p-l-l')_\mu}{[(p-l-l')^2+m^2]^2}\right], \qquad (4.31)\end{aligned}$$

where a factor of two arises from two terms which are identical after the replacement of l by l'. In the first integral, make a shift of variable,

$$l \to p-l-l', \quad p-l-l' \to l, \qquad (4.32)$$

which amounts to

$$l^2 \to (p-l-l')^2 = p\cdot(p-l-l') - l\cdot(p-l-l') - l'\cdot(p-l-l'). \qquad (4.33)$$

This allows us to write the result with a single denominator,

$$\begin{aligned}\Sigma^{(2a)}(p) &= \frac{48\lambda^2}{d}(\mu^2)^{4-d}4\int \frac{d^dl}{(2\pi)^d}\frac{d^dl'}{(2\pi)^d} \\ &\quad\times\frac{[(p-l-l')^2+\frac{1}{2}(p-l-l')^2+\frac{3}{2}m^2]-\frac{3}{2}m^2-\frac{1}{2}p\cdot(p-l-l')}{(l^2+m^2)(l'^2+m^2)[(p-l-l')^2+m^2]^2} \\ &= \frac{3}{d}\Sigma^{(2a)}(p)+\frac{96\lambda^2}{d}(\mu^2)^{4-d}\left[-3m^2K(p)-p^\mu K_\mu(p)\right], \qquad (4.34)\end{aligned}$$

where

$$K(p) = \int \frac{d^dl}{(2\pi)^d}\frac{d^dl'}{(2\pi)^d}\frac{1}{(l^2+m^2)^2}\frac{1}{l'^2+m^2}\frac{1}{(p-l-l')^2+m^2}, \qquad (4.35a)$$

$$K_\mu(p) = \int \frac{d^dl}{(2\pi)^d}\frac{d^dl'}{(2\pi)^d}\frac{1}{l^2+m^2}\frac{1}{l'^2+m^2}\frac{(p-l-l')_\mu}{[(p-l-l')^2+m^2]^2}, \qquad (4.35b)$$

so that Eq. (4.34) can be rewritten as

$$\Sigma^{(2a)}(p) = \frac{96\lambda^2}{3-d}(\mu^2)^{4-d}\left[3m^2K(p)+p^\mu K_\mu(p)\right]. \qquad (4.36)$$

Integration by parts has resulted in superficially more convergent integrals.

Let us make some remarks about this formula.

- We obtained it by working with d in a region ($d < 3/2$) where the integrals exist. Thus the surface term in the integration by parts is zero there.
- When $p = 0$ the formula may be obtained more simply by differentiation with respect to m^2: On the one hand, since the mass operator has dimensions of mass-squared, we must have

$$\Sigma^{(2a)}(0) = (m^2)^{d-3}(\mu^2)^{4-d} \times \text{ number }, \qquad (4.37)$$

where

$$\Sigma^{(2a)}(0) = 96\lambda^2(\mu^2)^{4-d}\int \frac{d^dl_1\,d^dl_2\,d^dl_3}{(2\pi)^{3d}}\frac{(2\pi)^d\delta(l_1+l_2+l_3)}{(l_1+m^2)(l_2+m^2)(l_3+m^2)}, \qquad (4.38)$$

and so from Eq. (4.35a)

$$\frac{\partial^2}{\partial m^2}\Sigma^{(2a)}(0) = (d-3)\frac{\Sigma^{(2a)}(0)}{m^2} = -96\lambda^2(\mu^2)^{4-d}3K(0), \qquad (4.39)$$

or

$$\Sigma^{(2a)}(0) = 96\lambda^2(\mu^2)^{4-d}\frac{3m^2}{3-d}K(0), \qquad (4.40)$$

which is the value given by Eq. (4.36).

Let us now proceed to work out $K(p)$. We first recall that

$$\begin{aligned}\frac{1}{l'^2+m^2}\frac{1}{(p-l-l')^2+m^2} &= \int_0^\infty ds\, s\int_0^1 du\, e^{-su[(p-l-l')^2+m^2]}e^{-s(1-u)(l'^2+m^2)}\\ &= \int_0^\infty ds\, s\int_0^1 du\, e^{-s[l'^2+m^2-2u\,l'\cdot(p-l)+u(p-l)^2]}\\ &= \int_0^\infty ds\, s\int_0^1 du\, e^{-s\{[l'-u(p-l)]^2+u(1-u)(p-l)^2+m^2\}},\end{aligned} \tag{4.41}$$

so that according to Eq. (4.12),

$$\int\frac{d^dl'}{(2\pi)^d}\frac{1}{l'^2+m^2}\frac{1}{(p-l-l')^2+m^2} = \int_0^1 du\frac{[m^2+u(1-u)(p-l)^2]^{d/2-2}}{(4\pi)^{d/2}} \times\Gamma(2-d/2). \tag{4.42}$$

We have one remaining loop integration to perform. Eq. (4.35a) becomes

$$\begin{aligned}K(p) &= \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+m^2)^2}\int_0^1 du\,[u(1-u)]^{d/2-2}\\ &\quad\times\left[(p-l)^2+\frac{m^2}{u(1-u)}\right]^{d/2-2}\\ &= \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 du\,[u(1-u)]^{d/2-2}\int\frac{d^dl}{(2\pi)^d}\int_0^\infty ds'\,s'e^{-s'(l^2+m^2)}\\ &\quad\times\frac{1}{\Gamma(2-d/2)}\int_0^\infty ds''\,s''^{1-d/2}e^{-s''\{(p-l)^2+m^2/[u(1-u)]\}}.\end{aligned} \tag{4.43}$$

Now we let

$$s'=s(1-v),\quad s''=sv,\quad ds'\,ds''=s\,ds\,dv, \tag{4.44}$$

and so

$$\begin{aligned}K(p) &= \frac{1}{(4\pi)^{d/2}}\int_0^\infty ds\,s\,s^{2-d/2}\int_0^1 du\,[u(1-u)]^{d/2-2}\int_0^1 dv\,v^{1-d/2}(1-v)\\ &\quad\times\int\frac{d^dl}{(2\pi)^d}e^{-s\{(1-v)(l^2+m^2)+v[(p-l)^2+m^2/u(1-u)]\}}.\end{aligned} \tag{4.45}$$

The exponent here can be rewritten as

$$(l-pv)^2+v(1-v)p^2+m^2\left[1-v+\frac{v}{u(1-u)}\right], \tag{4.46}$$

so using Eq. (4.12) again, we have

$$\begin{aligned}K(p) &= \frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 du\,[u(1-u)]^{d/2-2}\int_0^1 dv\,(1-v)v^{1-d/2}\\ &\quad\times\left[v(1-v)p^2+m^2\left(1-v+\frac{v}{u(1-u)}\right)\right]^{d-4}.\end{aligned} \tag{4.47}$$

We now evaluate this for d near 4. Again we write $d = 4 - \epsilon$, and (the result converges for $\epsilon > 0$)

$$
\begin{aligned}
K(p) &= \frac{\Gamma(\epsilon)}{(4\pi)^{4-\epsilon}} \int_0^1 du\,[u(1-u)]^{-\epsilon/2} \int_0^1 dv\, v^{-1+\epsilon/2}(1-v) \\
&\quad \times \left[v(1-v)p^2 + m^2 \left(1 - v + \frac{v}{u(1-u)}\right)\right]^{-\epsilon} \\
&\approx \frac{\Gamma(\epsilon)}{(4\pi)^{4-\epsilon}} m^{-2\epsilon} \frac{2}{\epsilon}\left(1 + \frac{\epsilon}{2}\right) + \mathcal{O}(1).
\end{aligned} \tag{4.48}
$$

Here the $2/\epsilon$ occurs because the v integration is singular as at $v = 0$. The details of the derivation of this result are relegated to the homework.

Next, we calculate $K_\mu(p)$, Eq. (4.35b). First we combine the two denominators containing l':

$$
\begin{aligned}
\frac{1}{l'^2 + m^2} \frac{1}{[(p-l-l')^2 + m^2]^2} &= \int_0^\infty ds'\, ds''\, s'' e^{-s'(l'^2+m^2)} e^{-s''[(p-l-l')^2+m^2]} \\
&= \int_0^\infty ds\, s^2 \int_0^1 du\, u\, e^{-s\{(1-u)(l'^2+m^2)+u[(p-l-l')^2+m^2]\}} \\
&= \int_0^\infty ds\, s^2 \int_0^1 du\, u\, e^{-s\{[l'-u(p-l)]^2+u(1-u)(p-l)^2+m^2\}}
\end{aligned} \tag{4.49}
$$

and the corresponding loop integral

$$
\int \frac{d^d l'}{(2\pi)^d} \frac{1}{l'^2 + m^2} \frac{(p-l-l')_\mu}{[(p-l-l')^2 + m^2]^2} \tag{4.50}
$$

is done by first changing variables,

$$
l' - u(p-l) \to l', \quad p - l - l' \to (p-l)(1-u) - l', \tag{4.51}
$$

and then integrating symmetrically on the new l' variable, so that the term linear in l' integrates to zero. According to Eq. (4.12) we are left then with

$$
\begin{aligned}
&(p-l)_\mu \int_0^\infty ds\, s^2 \int_0^1 du\, u(1-u) \frac{1}{(4\pi)^{d/2} s^{d/2}} e^{-s[m^2+u(1-u)(p-l)^2]} \\
&= (p-l)_\mu \frac{\Gamma(3-d/2)}{(4\pi)^{d/2}} \int_0^1 du\, u(1-u)[m^2 + u(1-u)(p-l)^2]^{d/2-3}.
\end{aligned} \tag{4.52}
$$

Finally we must integrate over l:

$$
\begin{aligned}
K_\mu(p) &= \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + m^2} (p-l)_\mu \frac{\Gamma(3-d/2)}{(4\pi)^{d/2}} \int_0^1 du\, u(1-u) \\
&\qquad \times [m^2 + u(1-u)(p-l)^2]^{d/2-3} \\
&= \frac{1}{(4\pi)^{d/2}} \int_0^1 du\, [u(1-u)]^{1+d/2-3} \int \frac{d^d l}{(2\pi)^d} (p-l)_\mu \int_0^\infty ds' e^{-s'(l^2+m^2)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^\infty ds''(s'')^{2-d/2} e^{-s''[(p-l)^2+m^2/u(1-u)]} \\
&= \frac{1}{(4\pi)^{d/2}} \int_0^\infty ds\, s^{3-d/2} \int_0^1 du\, [u(1-u)]^{d/2-2} \int_0^1 dv\, v^{2-d/2} \\
&\times \int \frac{d^d l}{(2\pi)^d} (p-l)_\mu e^{-s\{(l-vp)^2+v(1-v)p^2+m^2[1-v+v/u(1-u)]\}}. \qquad (4.53)
\end{aligned}
$$

Again, we change variables,

$$
l - vp \to l, \quad p - l \to (1-v)p - l, \qquad (4.54)
$$

so symmetric integration in the new l variable gives

$$
\begin{aligned}
K_\mu(p) &= \frac{p_\mu}{(4\pi)^d} \int_0^\infty ds\, s^{3-d} \int_0^1 du\, [u(1-u)]^{d/2-2} \\
&\quad \times \int_0^1 dv\, v^{2-d/2}(1-v) e^{-s\{p^2 v(1-v)+m^2[1-v+v/u(1-u)]\}} \\
&= \frac{p_\mu}{(4\pi)^d} \int_0^1 du\, [u(1-u)]^{d/2-2} \int_0^1 dv\, (1-v) v^{2-d/2} \Gamma(4-d) \\
&\quad \times \left[p^2 v(1-v) + m^2 \left(1 - v + \frac{v}{u(1-u)} \right) \right]^{d-4} \\
&= \frac{p_\mu \Gamma(\epsilon)}{(4\pi)^{4-\epsilon}} \int_0^1 du\, [u(1-u)]^{-\epsilon/2} \int_0^1 dv\, (1-v) v^{\epsilon/2} \\
&\quad \times \left[p^2 v(1-v) + m^2 \left(1 - v + \frac{v}{u(1-u)} \right) \right]^{-\epsilon} \\
&\approx \frac{p_\mu \Gamma(\epsilon)}{(4\pi)^{4-\epsilon}} m^{-2\epsilon} \int_0^1 du \int_0^1 dv\, (1-v) = \frac{p_\mu \Gamma(\epsilon)}{(4\pi)^{4-\epsilon}} \frac{m^{-2\epsilon}}{2} + \mathcal{O}(1), \qquad (4.55)
\end{aligned}
$$

the details in the last line again being supplied in the homework.

Putting together the results for $K(p)$, Eq. (4.48), and $K_\mu(p)$, Eq. (4.55), we find that Eq. (4.36) becomes ($\epsilon \to 0$)

$$
\begin{aligned}
\Sigma^{(2a)}(p) &\approx \frac{1}{\epsilon - 1} 96 \lambda^2 (\mu^2)^\epsilon \frac{\Gamma(\epsilon)}{(4\pi)^4} \left(\frac{4\pi}{m^2} \right)^\epsilon \left[3m^2 \frac{2}{\epsilon} \left(1 + \frac{\epsilon}{2} \right) + \frac{p^2}{2} \right] \\
&\approx -\frac{3\lambda^2}{8\pi^4} \left\{ \frac{6m^2}{\epsilon^2} \left[1 + \epsilon \left(\frac{3}{2} + \psi(1) + \ln \frac{4\pi\mu^2}{m^2} \right) \right] + \frac{p^2}{2\epsilon} + \mathcal{O}(1) \right\}. \qquad (4.56)
\end{aligned}
$$

We now have to deal with the meaning of these explicitly divergent expressions.