Chapter 2

Schwinger's Quantum Action Principle

We now turn to the dynamics of quantum mechanics. We begin by considering the transformation function $\langle a', t+dt|b', t\rangle$. Here $|b', t\rangle$ is a state specified by the values $b' = \{b'\}$ of a complete set of dynamical variables $B(t)$, while $|a', t+dt\rangle$ is a state specified by values $a' = \{a'\}$ of a (different) complete set of dynamical variables $A(t+dt)$. We suppose that A and B do not possess any explicit time dependence—that is, their definition does not depend upon t. Here

$$\langle a', t+dt| = \langle a', t|U, \tag{2.1}$$

where the infinitesimal time translation operator is related to the generator of time translations as follows,

$$U = 1 + iG = 1 - i\,dt\,H. \tag{2.2}$$

The Hamiltonian H is a function of dynamical variables, which we write generically as $\chi(t)$, and of t explicitly. Thus

$$\langle a', t+dt|b', t\rangle = \langle a', t|1 - i\,dt\,H(\chi(t), t)|b', t\rangle. \tag{2.3}$$

We next translate states and operators to time zero:

$$\langle a', t| = \langle a'|U(t), \quad |b', t\rangle = U^{-1}(t)|b'\rangle, \tag{2.4a}$$
$$\chi(t) = U^{-1}(t)\chi U(t), \tag{2.4b}$$

where $\chi = \chi(0)$, etc. Then,

$$\langle a', t+dt|b', t\rangle = \langle a'|1 - i\,dt\,H(\chi, t)|b'\rangle, \tag{2.5}$$

or, as a differential equation

$$\begin{aligned}\delta_{\rm dyn}\langle a', t+dt|b', t\rangle &= i\langle a'|\delta_{\rm dyn}[-dtH]|b'\rangle \\ &= i\langle a', t+dt|\delta_{\rm dyn}[-dt\,H(\chi(t), t)]|b', t\rangle,\end{aligned} \tag{2.6}$$

where $\delta_{\rm dyn}$ corresponds to changes in initial and final times, δt_2 and δt_1, and in the structure of H, δH. [By reintroducing dt in the state on the left, we make a negligible error of $\mathcal{O}(dt^2)$.]

However, we can also consider *kinematical* changes. To understand these, consider a system defined by coordinates and momenta, $\{q_a(t)\}$, $\{p_a(t)\}$, $a = 1,\dots,n$, which satisfy the canonical commutation relations,

$$\begin{aligned}
[q_a(t), p_b(t)] &= i\delta_{ab}, \quad (\hbar = 1) &\text{(2.7a)}\\
[q_a(t), q_b(t)] &= [p_a(t), p_b(t)] = 0. &\text{(2.7b)}
\end{aligned}$$

A spatial displacement δq_a is induced by

$$U = 1 + iG_q, \quad G_q = \sum_{a=1}^{n} p_a \delta q_a. \tag{2.8}$$

In fact (δq_a is a number, not an operator),

$$\begin{aligned}
U^{-1} q_a U &= q_a - \frac{1}{i}[q_a, G_q]\\
&= q_a - \delta q_a,
\end{aligned}\tag{2.9}$$

while

$$U^{-1} p_a U = p_a - \frac{1}{i}[p_a, G_q] = p_a. \tag{2.10}$$

The (dual) symmetry between position and momentum,

$$q \to p, \quad p \to -q, \tag{2.11}$$

gives us the form for the generator of a displacement in p:

$$G_p = -\sum_a q_a \delta p_a. \tag{2.12}$$

A *kinematic* variation in the states is given by the generators

$$\begin{aligned}
\delta_{\rm kin}\langle\ | &= \overline{\langle\ |} - \langle\ | = \langle\ |iG, &\text{(2.13a)}\\
\delta_{\rm kin}|\ \rangle &= \overline{|\ \rangle} - |\ \rangle = -iG|\ \rangle, &\text{(2.13b)}
\end{aligned}$$

so, for example, under a δq variation, the transformation function changes by

$$\delta_q \langle a', t+dt | b', t\rangle = i\langle a', t+dt| \sum_a \left[p_a(t+dt)\delta q_a(t+dt) - p_a(t)\delta q_a(t)\right] |b', t\rangle. \tag{2.14}$$

Now the dynamical variables at different times are related by Hamilton's equations,

$$\begin{aligned}
\frac{dp_a(t)}{dt} &= \frac{1}{i}[p_a(t), H(q(t), p(t), t)]\\
&= -\frac{\partial H}{\partial q_a}(t),
\end{aligned}\tag{2.15}$$

so

$$p_a(t+dt) - p_a(t) = dt\frac{dp_a(t)}{dt} = -dt\frac{\partial H}{\partial q_a}(t). \tag{2.16}$$

Similarly, the other Hamilton's equation

$$\frac{dq_a}{dt} = \frac{\partial H}{\partial p_a} \tag{2.17}$$

implies that

$$q_a(t+dt) - q_a(t) = dt\frac{\partial H}{\partial p_a}(t). \tag{2.18}$$

From this we deduce first the q variation of the transformation function,

$$\begin{aligned}
&\delta_q\langle a', t+dt|b', t\rangle \\
&= i\langle a', t+dt|\sum_a p_a(t)[\delta q_a(t+dt) - \delta q_a(t)] - dt\frac{\partial H}{\partial q_a}\delta q_a(t) + \mathcal{O}(dt^2)|b', t\rangle \\
&= i\langle a', t+dt|\delta_q\left[\sum_a p_a(t).[q_a(t+dt) - q_a(t)] - dt\, H(q(t), p(t), t)\right]|b', t\rangle,
\end{aligned} \tag{2.19}$$

where the dot denotes symmetric multiplication of the p and q operators.

For p variations we have a similar result:

$$\begin{aligned}
&\delta_p\langle a', t+dt|b', t\rangle \\
&= -i\langle a', t+dt|\sum_a [q_a(t+dt)\delta p_a(t+dt) - q_a(t)\delta p_a(t)]|b', t\rangle \\
&= -i\langle a', t+dt|\sum_a q_a(t)[\delta p_a(t+dt) - \delta p_a(t)] + dt\,\frac{\partial H}{\partial p_a}(t)\delta p_a(t)|b', t\rangle \\
&= i\langle a', t+dt|\delta_p\left[-\sum_a q_a(t).(p_a(t+dt) - p_a(t)) - dt\, H(q(t), p(t), t)\right]|b', t\rangle.
\end{aligned} \tag{2.20}$$

That is, for q variations

$$\delta_q\langle a', t+dt|b', t\rangle = i\langle a', t+dt|\delta_q\,[dtL_q]\,|b', t\rangle, \tag{2.21a}$$

with the quantum Lagrangian

$$L_q = \sum_a p_a.\dot{q}_a - H(q, p, t), \tag{2.21b}$$

while for p variations

$$\delta_p\langle a', t+dt|b', t\rangle = i\langle a', t+dt|\delta_p\,[dtL_p]\,|b', t\rangle, \tag{2.22a}$$

with the quantum Lagrangian

$$L_p = -\sum_a q_a \cdot \dot{p}_a - H(q, p, t). \tag{2.22b}$$

We see here two alternative forms of the quantum Lagrangian. Note that the two forms differ by a total time derivative,

$$L_q - L_p = \frac{d}{dt}\sum_a p_a \cdot q_a. \tag{2.23}$$

We now can unite the kinematic transformations considered here with the dynamic ones considered earlier, in Eq. (2.6):

$$\delta = \delta_{\text{dyn}} + \delta_{\text{kin}} : \quad \delta\langle a', t+dt|b', dt\rangle = i\langle a', t+dt|\delta[dt\, L]|b', t\rangle. \tag{2.24}$$

Suppose, for concreteness, that our states are defined by values of q, so that

$$\delta_p\langle a', t+dt|b't\rangle = 0. \tag{2.25}$$

This is consistent, as a result of Hamilton's equations,

$$\delta_p L_q = \sum_a \delta p_a \left(\dot{q}_a - \frac{\partial H}{\partial p_a}\right) = 0. \tag{2.26}$$

In the following we will use L_q.

It is immediately clear that we can iterate the infinitesimal version (2.24) of the quantum action principle by inserting at each time step a complete set of intermediate states (to simplify the notation, we ignore their quantum numbers):

$$\langle t_1|t_2\rangle = \langle t_1|t_1 - dt\rangle\langle t_1 - dt|t_1 - 2dt\rangle \cdots \langle t_2 + 2dt|t_2 + dt\rangle\langle t_2 + dt|t_2\rangle, \tag{2.27}$$

So in this way we deduce the general form of *Schwinger's quantum action principle:*

$$\delta\langle t_1|t_2\rangle = i\langle t_1|\delta\int_{t_2}^{t_1} dt\, L|t_2\rangle. \tag{2.28}$$

This summarizes all the properties of the system.

Suppose the dynamical system is given, that is, the structure of H does not change. Then

$$\delta\langle t_1|t_2\rangle = i\langle t_1|G_1 - G_2|t_2\rangle, \tag{2.29}$$

where the generator G_i depends on p and q at time t_i. Comparing with the action principle (2.28) we see

$$\delta\int_{t_2}^{t_1} dt\, L = G_1 - G_2, \tag{2.30}$$

which has exactly the form of the classical action principle (1.3), except that the Lagrangian L and the generators G are now operators. If no changes occur at the endpoints, we have the *principle of stationary action*,

$$\delta \int_{t_2}^{t_1} \left(\sum_a p_a . dq_a - H\, dt \right) = 0. \tag{2.31}$$

As in the classical case, let us introduce a time parameter τ, $t = t(\tau)$, such that τ_2 and τ_1 are fixed. The the above variation reads

$$\begin{aligned} &\sum_a [\delta p_a . dq_a + p_a . d\delta q_a - \delta H\, dt - H\, d\delta t] \\ &= d\left[\sum_a p_a . \delta q_a - H\, \delta t\right] + \sum_a [\delta p_a . dq_a - dp_a . \delta q_a] - \delta H\, dt + dH\, \delta t, \end{aligned} \tag{2.32}$$

so the action principle says

$$G = \sum_a p_a . \delta q_a - H\, \delta t, \tag{2.33a}$$

$$\delta H = \frac{dH}{dt} \delta t + \sum_a \left(\delta p_a . \frac{dq_a}{dt} - \delta q_a . \frac{dp_a}{dt} \right). \tag{2.33b}$$

We will again assume δp_a, δq_a are not operators (that is, they are proportional to the unit operator); then we recover Hamilton's equations,

$$\frac{\partial H}{\partial t} = \frac{dH}{dt}, \tag{2.34a}$$

$$\frac{\partial H}{\partial p_a} = \frac{dq_a}{dt}, \tag{2.34b}$$

$$\frac{\partial H}{\partial q_a} = -\frac{dp_a}{dt}. \tag{2.34c}$$

(In the homework, you will explore the possibility of operator variations.) We learn from the generators,

$$G_t = -H\, \delta t, \quad G_q = \sum_a p_a \delta q_a, \tag{2.35}$$

that the change in some function F of the dynamical variable is

$$\delta F = \frac{dF}{dt} \delta t + \frac{1}{i} [F, G], \tag{2.36}$$

so we deduce

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i} [F, H], \tag{2.37a}$$

$$\frac{\partial F}{\partial q_a} = \frac{1}{i} [F, p_a]. \tag{2.37b}$$

Note that from this the canonical commutation relations follow,

$$[q_a, p_b] = i\delta_{ab}, \quad [p_a, p_b] = 0, \tag{2.38}$$

as well as Newton's law,

$$\dot{p}_a = -\frac{1}{i}[H, p_a] = -\frac{\partial H}{\partial q_a}. \tag{2.39}$$

If we had used L_p instead of L_q, we would have obtained the same equations of motion, but in place of G_q, we would have obtained

$$G_p = -\sum_a q_a \delta p_a, \tag{2.40}$$

which implies

$$\frac{\partial F}{\partial p_a} = -\frac{1}{i}[F, q_a]. \tag{2.41}$$

From this can be deduced the remaining canonical commutator,

$$[q_a, q_b] = 0, \tag{2.42}$$

as well as the remaining Hamilton equation,

$$\dot{q}_a = \frac{1}{i}[q_a, H] = \frac{\partial H}{\partial p_a}. \tag{2.43}$$

In homework, you will show that the effect of changing the Lagrangian by a total time derivative (which is what is done in passing from L_q to L_p) is to change the generators.

We now turn to examples.

2.1 Harmonic Oscillator

The harmonic oscillator is defined in terms of creation and annihilation operators, $a^\dagger$ and a, and the corresponding Hamiltonian H,

$$[a, a^\dagger] = 1, \tag{2.44a}$$

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right). \tag{2.44b}$$

The equations of motion are

$$\frac{da}{dt} = \frac{1}{i}[a, H] = \frac{1}{i}\omega a, \tag{2.45a}$$

$$\frac{da^\dagger}{dt} = \frac{1}{i}[a^\dagger, H] = -\frac{1}{i}\omega a^\dagger. \tag{2.45b}$$

Eigenstates of a and $a^\dagger$ exist, as right and left vectors, respectively,

$$\begin{aligned} a|a'\rangle &= a'|a'\rangle, && (2.46a)\\ \langle a^{\dagger\prime}|a^\dagger &= a^{\dagger\prime}\langle a^{\dagger\prime}|, && (2.46b) \end{aligned}$$

while $\langle a'|$ and $|a^{\dagger\prime}\rangle$ do not exist.[1] These are the famous "coherent states."

The transformation function we seek is therefore

$$\langle a^{\dagger\prime}, t_1|a'', t_2\rangle. \tag{2.48}$$

If we regard a as a "coordinate," the corresponding "momentum" is $ia^\dagger$:

$$\dot a = \frac{1}{i}\omega a = \frac{\partial H}{\partial ia^\dagger}, \quad i\dot a^\dagger = -\omega a^\dagger = -\frac{\partial H}{\partial a}. \tag{2.49}$$

The corresponding Lagrangian is therefore[2]

$$L = ia^\dagger.\dot a - H. \tag{2.51}$$

Because we use a as our state variable at the initial time, and $a^\dagger$ at the final time, we must exploit our freedom to redefine our generators to write (see homework)

$$W_{12} = \int_2^1 dt\, L - ia^\dagger(t_1).a(t_1). \tag{2.52}$$

Then the variation of the action is

$$\begin{aligned} \delta W_{12} &= -i\delta(a_1^\dagger.a_1) + G_1 - G_2\\ &= -i\delta a_1^\dagger.a_1 - ia_1^\dagger.\delta a_1 + ia_1^\dagger.\delta a_1 - ia_2^\dagger.\delta a_2 - H\,\delta t_1 + H\,\delta t_2\\ &= -i\delta a_1^\dagger.a_1 - ia_2^\dagger.\delta a_2 - H(\delta t_1 - \delta t_2). \end{aligned} \tag{2.53}$$

Then the quantum action principle says

$$\delta\langle a^{\dagger\prime}, t_1|a'', t_2\rangle = i\langle a^{\dagger\prime}, t_1| - i\delta a_1^{\dagger\prime}a_1 - ia_2^\dagger\delta a_2'' - \omega a_1^{\dagger\prime}a_1(\delta t_1 - \delta t_2)|a'', t_2\rangle, \tag{2.54}$$

since by assumption the variations in the dynamical variables are numerical:

$$[\delta a_1^\dagger, a_1] = [a_2^\dagger, \delta a_2], \tag{2.55}$$

and we have dropped the zero-point energy (see homework). Now use the equations of motion (2.45a) and (2.45b) to deduce that

$$a_1 = e^{-i\omega(t_1-t_2)}a_2, \quad a_2^\dagger = e^{-i\omega(t_1-t_2)}a_1^\dagger \tag{2.56}$$

[1] If $a\langle a'|a = a'\langle a'|$ then we would have an evident contradiction:

$$1 = \langle a'|[a, a^\dagger]|a'\rangle = a'\langle a'|a^\dagger|a'\rangle - \langle a'|a^\dagger|a'\rangle a' = 0. \tag{2.47}$$

[2] We might note that in terms of (dimensionless) position and momentum operators

$$ia^\dagger.\dot a = \frac{i}{2}(q - ip).(\dot q + i\dot p) = \frac{1}{2}(p.\dot q - q.\dot p) + \frac{i}{4}\frac{d}{dt}(q^2 + p^2), \tag{2.50}$$

where the first term in the final form is the average of the Legendre transforms in L_q and L_p.

and hence

$$\begin{aligned}\delta\langle a^{\dagger\prime},t_1|a'',t_2\rangle &= \langle a^{\dagger\prime},t_1|\delta a^{\dagger\prime}e^{-i\omega(t_1-t_2)}a''+a^{\dagger\prime}e^{-i\omega(t_1-t_2)}\delta a'' \\ &\quad -i\omega a^{\dagger\prime}e^{-i\omega(t_1-t_2)}(\delta t_1-\delta t_2)a''|a'',t_2\rangle \\ &= \langle a^{\dagger\prime},t_1|a'',t_2\rangle\delta\left[a^{\dagger\prime}e^{-i\omega(t_1-t_2)}a''\right]. \end{aligned} \tag{2.57}$$

From this we can deduce that the transformation function has the exponential form

$$\langle a^{\dagger\prime},t_1|a'',t_2\rangle = \exp\left[a^{\dagger\prime}e^{-i\omega(t_1-t_2)}a''\right], \tag{2.58}$$

which has the correct boundary condition at $t_1=t_2$ (see homework); and in particular, $\langle 0|0\rangle = 1$.

On the other hand,

$$\langle a^{\dagger\prime},t_1|a'',t_2\rangle = \langle a^{\dagger\prime}|e^{-iH(t_1-t_2)}|a''\rangle, \tag{2.59}$$

where both states are expressed at the common time t_2, so, upon inserting a complete set of energy eigenstates, we obtain ($t=t_1-t_2$)

$$\sum_E \langle a^{\dagger\prime}|E\rangle e^{-iEt}\langle E|a''\rangle, \tag{2.60}$$

which we compare to the Taylor expansion of the previous formula,

$$\sum_{n=0}^{\infty}\frac{(a^{\dagger\prime})^n}{\sqrt{n!}}e^{-in\omega t}\frac{(a'')^n}{\sqrt{n!}}. \tag{2.61}$$

This gives all the eigenvectors and eigenvalues:

$$E_n = n\omega, \quad n=0,1,2,\dots, \tag{2.62a}$$

$$\langle a^{\dagger\prime}|E_n\rangle = \frac{(a^{\dagger\prime})^n}{\sqrt{n!}}, \tag{2.62b}$$

$$\langle E_n|a''\rangle = \frac{(a'')^n}{\sqrt{n!}}. \tag{2.62c}$$

These correspond to the usual construction of the eigenstates from the ground state:

$$|E_n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{2.63}$$

2.2 Forced Harmonic Oscillator

Now we add a driving term to the Hamiltonian,

$$H = \omega a^\dagger a + aK^*(t) + a^\dagger K(t), \tag{2.64}$$

where $K(t)$ is an external force (*Kraft* is force in German). The equation of motion is

$$i\frac{da}{dt} = \frac{\partial H}{\partial a^\dagger} = [a, H] = \omega a + K(t), \tag{2.65}$$

while $a^\dagger$ satisfies the adjoint equation. In the presence of $K(t)$, we wish to compute $\langle a^{\dagger\prime}, t_1|a'', t_2\rangle^K$.

Consider a variation of K. According to the action principle

$$\begin{aligned}\delta_K\langle a^{\dagger\prime}, t_1|a'', t_2\rangle^K &= \langle a^{\dagger\prime}, t_1|i\delta_K W_{12}|a'', t_2\rangle^K \\ &= -i\langle a^{\dagger\prime}, t_1|\int_{t_2}^{t_1} dt[\delta K a^\dagger + \delta K^* a]|a'', t_2\rangle^K. \end{aligned} \tag{2.66}$$

We can solve this differential equation by noting that the equation of motion (2.65) can be rewritten as

$$i\frac{d}{dt}\left[e^{i\omega t}a(t)\right] = e^{i\omega t}K(t), \tag{2.67}$$

which is integrated to read

$$e^{i\omega t}a(t) - e^{i\omega t_2}a(t_2) = -i\int_{t_2}^t dt'\, e^{i\omega t'}K(t'), \tag{2.68}$$

or

$$a(t) = e^{-i\omega(t-t_2)}a_2 - i\int_{t_2}^t dt'\, e^{-i\omega(t-t')}K(t'), \tag{2.69}$$

and the adjoint[3]

$$a^\dagger(t) = e^{-i\omega(t_1-t)}a_1^\dagger - i\int_t^{t_1} dt'\, e^{-i\omega(t'-t)}K^*(t'). \tag{2.72}$$

Thus our differential equation (2.66) reads

$$\frac{\delta_K\langle a^{\dagger\prime}, t_1|a'', t_2\rangle^K}{\langle a^{\dagger\prime}, t_1|a'', t_2\rangle^K} = \delta_K \ln\langle a^{\dagger\prime}, t_1|a'', t_2\rangle^K$$

[3]The consistency of these two equations follows from

$$e^{i\omega t_1}a_1 = e^{i\omega t_2}a_2 - i\int_{t_2}^{t_1} dt'\, e^{i\omega t'}K(t'), \tag{2.70}$$

so that the adjoint of Eq. (2.69) is

$$\begin{aligned}[a(t)]^\dagger &= e^{i\omega t}\left[e^{-i\omega t_1}a_1^\dagger - i\int_{t_2}^{t_1} dt'\, e^{-i\omega t'}K^*(t')\right] + i\int_{t_2}^t dt'\, e^{-i\omega(t'-t)}K^*(t') \\ &= e^{i\omega(t-t_1)}a_1^\dagger + i\int_{t_1}^t dt'\, e^{-i\omega(t'-t)}K^*(t'), \end{aligned} \tag{2.71}$$

which is Eq. (2.72).

$$= -i\int_{t_2}^{t_1} dt\,\delta K(t)\left[a^{\dagger\prime}e^{-i\omega(t_1-t)} - i\int_t^{t_1} dt'\,e^{-i\omega(t'-t)}K^*(t')\right]$$
$$- i\int_{t_2}^{t_1} dt\,\delta K^*(t)\left[e^{-i\omega(t-t_2)}a'' - i\int_{t_2}^{t} dt'\,e^{-i\omega(t-t')}K(t')\right]. \quad (2.73)$$

Notice that in the terms bilinear in K and K^*, K always occurs earlier than K^*. Therefore, these terms can be combined to read

$$-\delta_K\int_{t_2}^{t_1} dt\,dt'\,K^*(t)\eta(t-t')e^{-i\omega(t-t')}K(t'), \quad (2.74)$$

where the step function is

$$\eta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \quad (2.75)$$

Since we already know the $K = 0$ value from Eq. (2.58), we may now immediately integrate our differential equation:

$$\begin{aligned}\langle a^{\dagger\prime}, t_1 | a'', t_2\rangle^K = \exp\Bigg[&a^{\dagger\prime}e^{-i\omega(t_1-t_2)}a'' \\ &- ia^{\dagger\prime}\int_{t_2}^{t_1} dt\,e^{-i\omega(t_1-t)}K(t) - i\int_{t_2}^{t_1} dt\,e^{-i\omega(t-t_2)}K^*(t)\,a'' \\ &- \int_{t_2}^{t_1} dt\,dt'\,K^*(t)\eta(t-t')e^{-i\omega(t-t')}K(t')\Bigg]. \end{aligned} \quad (2.76)$$

The ground state is defined by $a'' = a^{\dagger\prime} = 0$, so

$$\langle 0, t_1|0, t_2\rangle^K = \exp\left[-\int_{-\infty}^{\infty} dt\,dt'\,K^*(t)\eta(t-t')e^{-i\omega(t-t')}K(t')\right], \quad (2.77)$$

where we now suppose that the forces turn off at the initial and final times, t_2 and t_1, respectively.

A check of this result is obtained by computing the probability of the system remaining in the ground state:

$$\begin{aligned}|\langle 0, t_1|0, t_2\rangle^K|^2 &= \exp\left\{-\int_{-\infty}^{\infty} dt\,dt'\,K^*(t)e^{-i\omega(t-t')}[\eta(t-t')+\eta(t'-t)]K(t')\right\} \\ &= \exp\left[-\int_{-\infty}^{\infty} dt\,dt'\,K^*(t)e^{-i\omega(t-t')}K(t')\right] \\ &= \exp\left[-|K(\omega)|^2\right], \end{aligned} \quad (2.78)$$

where the Fourier transform of the force is

$$K(\omega) = \int_{-\infty}^{\infty} dt\,e^{i\omega t}K(t). \quad (2.79)$$

The probability requirement

$$|\langle 0, t_1|0, t_2\rangle^K|^2 \leq 1 \tag{2.80}$$

is thus satisfied. We see here a *resonance* effect: If the oscillator is driven close to its natural frequency, so $K(\omega)$ is large, there is a large probability of finding the system in an excited state, and therefore of not remaining in the ground state. Let us calculate this transition amplitude to an excited state. By setting $a'' = 0$ in Eq. (2.76) we obtain

$$\begin{aligned}\langle a^{\dagger\prime}, t_1|0, t_2\rangle^K &= \exp\left[-ia^{\dagger\prime}\int_{-\infty}^{\infty} dt\, e^{-i\omega(t_1-t)}K(t)\right]\langle 0, t_1|0, t_2\rangle^K \\ &= \sum_n \langle a^{\dagger\prime}, t_1|n, t_1\rangle\langle n, t_1|0, t_2\rangle^K,\end{aligned} \tag{2.81}$$

where we have inserted a sum over a complete set of energy eigenstates, which possess the amplitude [see Eq. (2.62b)]

$$\langle a^{\dagger\prime}|n\rangle = \frac{(a^{\dagger\prime})^n}{\sqrt{n!}}. \tag{2.82}$$

If we expand the first line of Eq. (2.81) in powers of $a^{\dagger\prime}$, we find

$$\langle n, t_1|0, t_2\rangle^K = \frac{(-i)^n}{\sqrt{n!}}e^{-in\omega t_1}[K(\omega)]^n\langle 0, t_1|0, t_2\rangle^K. \tag{2.83}$$

The corresponding probability is

$$p(n, 0) = |\langle n, t_1|0, t_2\rangle^K|^2 = \frac{|K(\omega)|^{2n}}{n!}e^{-|K(\omega)|^2}, \tag{2.84}$$

which is a Poisson distribution[4] with mean $\bar{n} = |K(\omega)|^2$.

Finally, let us define the *Green's function* for this problem by

$$G(t-t') = -i\eta(t-t')e^{-i\omega(t-t')}. \tag{2.86}$$

It satisfies the differential equation

$$\left(i\frac{d}{dt} - \omega\right)G(t-t') = \delta(t-t'), \tag{2.87}$$

as it must because [see Eq. (2.65)]

$$\left(i\frac{d}{dt} - \omega\right)a(t) = K(t), \tag{2.88}$$

[4] A Poisson probability distribution has the form $p(n) = \gamma^n e^{-\gamma}/n!$. The mean value of n for this distribution is

$$\bar{n} = \sum_{n=0}^{\infty} n\, p(n) = \sum_{n=0}^{\infty}\frac{\gamma^n e^{-\gamma}}{(n-1)!} = \gamma\sum_{n=0}^{\infty}p(n) = \gamma. \tag{2.85}$$

where $a(t)$ is given by [see Eq. (2.69)]

$$a(t) = e^{-i\omega(t-t_2)}a_2 + \int_{-\infty}^{\infty} dt'\, G(t-t')K(t'). \tag{2.89}$$

Similarly, from Eq. (2.72)

$$a^\dagger(t) = e^{-i\omega(t_1-t)}a_1^\dagger + \int_{-\infty}^{\infty} dt'\, G(t'-t)K^*(t'). \tag{2.90}$$

We can now write the ground-state persistence amplitude (2.77) as

$$\langle 0,t_1|0,t_2\rangle^K = \exp\left[-i\int_{-\infty}^{\infty} dt\,dt'\, K^*(t)G(t-t')K(t')\right], \tag{2.91}$$

and the general amplitude (2.76) as

$$\begin{aligned}\langle a^{\dagger\prime},t_1|a'',t_2\rangle^K &= \exp\Bigg\{-i\int_{-\infty}^{\infty} dt\,dt'\,\left[K^*(t)+ia^{\dagger\prime}\delta(t-t_1)\right] \\ &\qquad \times G(t-t')\left[K(t')+ia''\delta(t'-t_2)\right]\Bigg\},\end{aligned} \tag{2.92}$$

which demonstrates that knowledge of $\langle 0,t_1|0,t_2\rangle^K$ for all K determines everything:

$$\langle a^{\dagger\prime},t_1|a'',t_2\rangle^K = \langle 0,t_1|0,t_2\rangle^{K(t)+ia''\delta(t-t_2)+ia^{\dagger\prime}\delta(t-t_1)}. \tag{2.93}$$

2.3 Feynman Path Integral Formulation

Although much more familiar, the path integral formulation of quantum mechanics is rather vaguely defined. We will here provide a formal "derivation" based on the Schwinger principle, in the harmonic oscillator context.

Consider a forced oscillator, defined by the Lagrangian (note in this section, H does not include the source terms)

$$L = ia^\dagger.\dot a - H(a,a^\dagger) - Ka^\dagger - K^*a. \tag{2.94}$$

As in the preceding section, the action principle says

$$\delta_K\langle 0,t_1|0,t_2\rangle^K = -i\langle 0,t_1|\int_{t_2}^{t_1} dt\,[\delta K a^\dagger + \delta K^* a]|0,t_2\rangle^K, \tag{2.95}$$

or for $t_2 < t < t_1$,

$$i\frac{\delta}{\delta K(t)}\langle 0,t_1|0,t_2\rangle^K = \langle 0,t_1|a^\dagger(t)|0,t_2\rangle^K, \tag{2.96a}$$

$$i\frac{\delta}{\delta K^*(t)}\langle 0,t_1|0,t_2\rangle^K = \langle 0,t_1|a(t)|0,t_2\rangle^K, \tag{2.96b}$$

where we have introduced the concept of the functional derivative. The equation of motion

$$i\dot{a} - \frac{\partial H}{\partial a^\dagger} - K = 0, \quad -i\dot{a}^\dagger - \frac{\partial H}{\partial a} - K^* = 0, \tag{2.97}$$

is thus equivalent to the functional differential equation,

$$0 = \left\{ i \left[K(t), W\left[i\frac{\delta}{\delta K^*}, i\frac{\delta}{\delta K} \right] \right] - K(t) \right\} \langle 0, t_1 | 0, t_2 \rangle^K, \tag{2.98}$$

where (the square brackets indicate functional dependence)

$$W[a, a^\dagger] = \int_{t_2}^{t_1} dt\, [ia^\dagger(t).\dot{a}(t) - H(a(t), a^\dagger(t))]. \tag{2.99}$$

The reason Eq. (2.98) holds is that by definition

$$\frac{\delta}{\delta K(t)} K(t') = \delta(t - t'), \tag{2.100}$$

so

$$\begin{aligned} i &\left[K(t), \int_{t_2}^{t_1} dt' \left(i\frac{i\delta}{\delta K(t')}.\frac{d}{dt'}\frac{i\delta}{\delta K^*(t')} - H\left(\frac{i\delta}{\delta K^*(t')}, \frac{i\delta}{\delta K(t')} \right) \right) \right] \\ &= i\frac{d}{dt}\frac{i\delta}{\delta K^*(t)} - \frac{\partial}{\partial(i\delta/\delta K(t))} H\left(\frac{i\delta}{\delta K^*(t)}, \frac{i\delta}{\delta K(t)} \right), \end{aligned} \tag{2.101}$$

which corresponds to the first two terms in the equation of motion (2.97), under the correspondence

$$a \leftrightarrow i\frac{\delta}{\delta K^*}, \quad a^\dagger \leftrightarrow i\frac{\delta}{\delta K}. \tag{2.102}$$

Since $[K, W], W] = 0$, we can write the functional equation (2.98) as

$$0 = e^{iW[i\delta/\delta K^*, i\delta/\delta K]} K e^{-iW[i\delta/\delta K^*, i\delta/\delta K]} \langle 0, t_1 | 0, t_2 \rangle^K. \tag{2.103}$$

The above equation has a solution (up to a constant), because both equations (2.97) must hold,

$$\langle 0, t_1 | 0, t_2 \rangle^K = e^{iW[i\delta/\delta K^*, i\delta/\delta K]} \delta[K]\delta[K^*], \tag{2.104}$$

where $\delta[K]$, $\delta[K^*]$ are functional delta functions. The latter have functional Fourier decompositions (up to a multiplicative constant),

$$\delta[K] = \int [da^\dagger] e^{-i\int dt\, K(t)a^\dagger(t)}, \tag{2.105a}$$

$$\delta[K^*] = \int [da] e^{-i\int dt\, K^*(t)a(t)}, \tag{2.105b}$$

where $[da]$ represents an element of integration over all (numerical-valued) *functions* $a(t)$, and so we finally have

$$
\begin{aligned}
&\langle 0, t_1|0, t_2\rangle^{K,K^*} \\
&\quad = \int [da][da^\dagger] \exp\left(-i\int_{t_2}^{t_1} dt\,\left[K(t)a^\dagger(t) + K^*(t)a(t)\right] + iW[a, a^\dagger]\right) \\
&\quad = \int [da][da^\dagger] \exp\left(i\int_{t_2}^{t_1} dt\,\left[ia^\dagger\dot a - H(a, a^\dagger) - Ka^\dagger - K^* a\right]\right), \qquad (2.106)
\end{aligned}
$$

where a, $a^\dagger$ are now numerical, and the functional integration is over all possible functions, over all possible "paths." Of course, the classical paths, the ones for which $W - \int dt(Ka^\dagger + K^*a)$ is an extremum, receive the greatest weight, at least in the classical limit, where $\hbar \to 0$.

2.3.1 Example

Consider the harmonic oscillator Hamiltonian, $H = \omega a^\dagger a$. Suppose we wish to calculate, once again, the ground state persistence amplitude, $\langle 0, t_1|0, t_2\rangle^K$. It is perhaps easiest to perform a Fourier transform,

$$
a(\nu) = \int_{-\infty}^{\infty} dt\, e^{i\nu t} a(t), \quad a^*(-\nu) = \int_{-\infty}^{\infty} dt\, e^{-i\nu t} a^\dagger(t). \qquad (2.107)
$$

Then

$$
\begin{aligned}
\int_{-\infty}^{\infty} dt\, a^\dagger(t)a(t) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} a(\nu)a^*(-\nu), &(2.108a)\\
\int_{-\infty}^{\infty} dt\, ia^\dagger(t)\dot a(t) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \nu a(\nu)a^*(-\nu). &(2.108b)
\end{aligned}
$$

Thus Eq. (2.106) becomes

$$
\begin{aligned}
&\langle 0, t_1|0, t_2\rangle^{K,K^*} \\
&= \int [da][da^*] \exp\left\{i\int \frac{d\nu}{2\pi}\left[a(\nu)(\nu-\omega)a^*(-\nu) - a^*(-\nu)K(\nu) - a(\nu)K^*(-\nu)\right]\right\} \\
&= \int [da][da^*] \exp\left\{i\int \frac{d\nu}{2\pi}\left[a(\nu) - \frac{K(\nu)}{\nu-\omega}\right](\nu-\omega)\left[a^*(-\nu) - \frac{K^*(-\nu)}{\nu-\omega}\right]\right. \\
&\qquad \left. - i\int \frac{d\nu}{2\pi} K(\nu)\frac{1}{\nu-\omega}K^*(-\nu)\right\} \\
&= \int [da][da^*] \exp\left\{i\int \frac{d\nu}{2\pi} a(\nu)(\nu-\omega)a^*(-\nu)\right\} \\
&\qquad \times \exp\left\{-i\int \frac{d\nu}{2\pi} K(\nu)\frac{1}{\nu-\omega}K^*(-\nu)\right\} \\
&= \exp\left\{-i\int \frac{d\nu}{2\pi} K(\nu)\frac{1}{\nu-\omega}K^*(-\nu)\right\}, \qquad (2.109)
\end{aligned}
$$

since the first exponential in the penultimate line, obtained by shifting the integration variable,

$$a(\nu) - \frac{K(\nu)}{\nu - \omega} \to a(\nu), \tag{2.110a}$$

$$a^*(-\nu) - \frac{K^*(-\nu)}{\nu - \omega} \to a^*(-\nu), \tag{2.110b}$$

is $\langle 0, t_1 | 0, t_2 \rangle^{K=K^*=0} = 1$. How do we interpret the singularity at $\nu = \omega$ in the remaining integral? We should have inserted a convergence factor in the original functional integral:

$$\exp\left(i \int \frac{d\nu}{2\pi} [\ldots]\right) \to \exp\left(i \int \frac{d\nu}{2\pi} [\ldots + i\epsilon a(\nu) a^*(-\nu)]\right), \tag{2.111}$$

where ϵ goes to zero through positive values. Thus we have, in effect, $\nu - \omega \to \nu - \omega + i\epsilon$ and so we have for the ground-state persistence amplitude

$$\langle 0, t_1 | 0, t_2 \rangle^{K,K^*} = e^{-i \int dt\, dt'\, K^*(t) G(t-t') K(t')}, \tag{2.112}$$

which has the form of Eq. (2.91), with

$$G(t - t') = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{e^{-i\nu(t-t')}}{\nu - \omega + i\epsilon}, \tag{2.113}$$

which is evaluated by closing the ν contour in the upper half plane if $t - t' < 0$, and in the lower half plane when $t - t' > 0$. Since the pole is in the lower half plane we get

$$G(t - t') = -i\eta(t - t') e^{-i\omega(t-t')}, \tag{2.114}$$

which is exactly what we found in Eq. (2.86).

Now, let us rewrite the path integral (2.106) in terms of coördinates and momenta:

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger), \quad p = \sqrt{\frac{\omega}{2}} \frac{1}{i}(a - a^\dagger), \tag{2.115a}$$

$$a = \sqrt{\frac{\omega}{2}}\left(q + \frac{ip}{\omega}\right), \quad a^\dagger = \sqrt{\frac{\omega}{2}}\left(q - \frac{ip}{\omega}\right). \tag{2.115b}$$

Then the numerical Lagrangian appearing in (2.106) may be rewritten as

$$\begin{aligned}
L &= i a^\dagger \dot{a} - \omega a^\dagger a - K a^\dagger - K^* a \\
&= i\frac{\omega}{2}\left(q - i\frac{p}{\omega}\right)\left(\dot{q} + i\frac{\dot{p}}{\omega}\right) - \frac{\omega^2}{2}\left(q^2 + \frac{p^2}{\omega^2}\right) \\
&\qquad - \sqrt{\frac{\omega}{2}} K \left(q - \frac{ip}{\omega}\right) - \sqrt{\frac{\omega}{2}} K^* \left(q + \frac{ip}{\omega}\right) \\
&= i\frac{\omega}{4}\frac{d}{dt}\left(q^2 + \frac{p^2}{\omega^2}\right) + p\dot{q} - \frac{1}{2}\frac{d}{dt}(pq) - \frac{1}{2}(p^2 + \omega^2 q^2) - \sqrt{2\omega}\Re K q - \sqrt{\frac{2}{\omega}} \Im K p \\
&= \frac{d}{dt} w + L(q, \dot{q}, t),
\end{aligned} \tag{2.116}$$

where, if we set $\dot q = p$, the Lagrangian is

$$L(q,\dot q,t)=\frac{1}{2}\dot q^2-\frac{1}{2}\omega^2q^2+Fq, \tag{2.117}$$

if

$$\Im K=0,\quad F=-\sqrt{2\omega}\Re K. \tag{2.118}$$

In the path integral

$$[da][da^\dagger]=[dq][dp]\left|\frac{\partial(a,a^\dagger)}{\partial(q,p)}\right|, \tag{2.119}$$

where the Jacobian is

$$\left|\frac{\partial(a,a^\dagger)}{\partial(q,p)}\right|=\begin{vmatrix}\sqrt{\frac{\omega}{2}} & \sqrt{\frac{\omega}{2}}\\ \frac{i}{\sqrt{2\omega}} & -\frac{i}{\sqrt{2\omega}}\end{vmatrix}=1, \tag{2.120}$$

and so from the penultimate line of Eq. (2.116), the path integral (2.106) becomes

$$\begin{aligned}\langle 0,t_1|0,t_2\rangle^F &= \int [da][da^\dagger]\exp\left[i\int_{t_2}^{t_1}dt\,L(a,a^\dagger)\right]\\ &= \int[dq][dp]\exp\left[i\int_{t_2}^{t_1}dt\left(p\dot q-\frac{1}{2}p^2-\frac{1}{2}\omega^2q^2+Fq\right)\right].\end{aligned} \tag{2.121}$$

Now we can carry out the p integration, since it is Gaussian:

$$\begin{aligned}\int[dp]e^{i\int dt\left[-\frac{1}{2}p^2+p\dot q\right]} &= \int[dp]e^{i\int dt\left[-\frac{1}{2}(p-\dot q)^2+\frac{1}{2}\dot q^2\right]}\\ &= e^{i\int dt\frac{1}{2}\dot q^2}\prod_i\int_{-\infty}^{\infty}dp_i\,e^{-\frac{1}{2}ip_i^2\Delta t}.\end{aligned} \tag{2.122}$$

Here we have discretized time so that $p(t_i)=p_i$, so the final functional integral over p is just an infinite product of constants, each one of which equals $e^{-i\pi/4}\sqrt{2\pi/\Delta t}$. Thus we arrive at the form originally written down by Feynman,

$$\langle 0,t_1|0,t_2\rangle^F=\int[dq]\exp\left\{i\int_{t_2}^{t_1}dt\,L(q,\dot q,t)\right\}, \tag{2.123}$$

with the Lagrangian given by Eq. (2.117), where an infinite normalization constant has been absorbed into the measure.