- $(0,0)$ is a scalar.
- $(1/2,0)$ is a left-handed spinor.
- $(0,1/2)$ is a right-handed spinor.
- $(1/2,1/2)$ is a vector.

Before discussing spinors in detail, let us mention the *discrete* transformations that preserve the form $\mathbf{x} - t^2$. These are

$$\text{Parity} \quad P: \qquad \mathbf{x} \to -\mathbf{x}, \quad t \to t, \tag{1.116a}$$
$$\text{Time reversal} \quad T: \qquad \mathbf{x} \to \mathbf{x}, \quad t \to -t. \tag{1.116b}$$

(We know experimentally that these are not exact symmetries of nature.) The Lorentz group generators transform as follows:

$$P: \qquad \mathbf{J} \to \mathbf{J} \quad \text{(pseudoscalar or axial vector)}, \tag{1.117a}$$
$$\mathbf{N} \to -\mathbf{N} \quad \text{(vector or polar vector)}, \tag{1.117b}$$
$$T: \qquad \mathbf{J} \to -\mathbf{J}, \tag{1.117c}$$
$$\mathbf{N} \to -\mathbf{N}, \tag{1.117d}$$

so

$$P: \quad \mathbf{J}^{(+)} \leftrightarrow \mathbf{J}^{(-)}, \quad T: \quad \mathbf{J}^{(\pm)} \to -\mathbf{J}^{(\pm)}. \tag{1.118}$$

In particular, under P, the representations change accoring to

$$(s_+, s_-) \to (s_-, s_+), \tag{1.119}$$

that is, the left-handed spinor changes into the right-handed spinor, $(1/2,0) \to (0,1/2)$. So if we want a description of spin-1/2 particles which respects parity, we must use

$$\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right), \tag{1.120}$$

which is what is called a Dirac spinor,

Incidentally, note that we can construct everything from spinors by using

$$(s_+, s_-) \otimes (s'_+, s'_-) = \oplus \sum_{r_\pm = |s_\pm - s'_\pm|}^{|s_\pm + s'_\pm|} (r_+, r_-). \tag{1.121}$$

The transformation properties of a left-handed spinor $(1/2,0)$ are given by the explicit realization of the generators,

$$\mathbf{J}^{(+)} = \frac{1}{2}\boldsymbol{\sigma}_+, \quad \mathbf{J}^{(-)} = \mathbf{0}, \tag{1.122}$$

which means that under finite rotations and boosts

$$\left(\frac{1}{2},0\right): \quad U_R^{(+)} = \exp\left[i\frac{1}{2}\theta\,\mathbf{e}\cdot\boldsymbol{\sigma}_+\right], \qquad U_B^{(+)} = \exp\left[-\frac{1}{2}\phi\,\mathbf{e}\cdot\boldsymbol{\sigma}_+\right]. \tag{1.123}$$

Similarly, the transformations of a right-handed spinor $(0,1/2)$ are given by

$$\left(0,\frac{1}{2}\right): \quad U_R^{(-)} = \exp\left[i\frac{1}{2}\theta\,\mathbf{e}\cdot\boldsymbol{\sigma}_-\right], \qquad U_B^{(-)} = \exp\left[\frac{1}{2}\phi\,\mathbf{e}\cdot\boldsymbol{\sigma}_-\right]. \tag{1.124}$$

Therefore, a Dirac spinor $(1/2,0)\otimes(0,1/2)$ transforms by

$$U_R = U_R^{(+)}U_R^{(-)} \approx 1 + i\delta\theta\,\mathbf{e}\cdot\frac{1}{2}(\boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-), \tag{1.125a}$$

$$U_B = U_B^{(+)}U_B^{(-)} \approx 1 - \delta\phi\,\mathbf{e}\cdot\frac{1}{2}(\boldsymbol{\sigma}_+ - \boldsymbol{\sigma}_-), \tag{1.125b}$$

that is, the rotation and boost generators are

$$\mathbf{J} = \frac{1}{2}(\boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-), \quad \mathbf{N} = -i\frac{1}{2}(\boldsymbol{\sigma}_+ - \boldsymbol{\sigma}_-). \tag{1.126}$$

1.4.1 Direct Product versus Direct Sum

We are familiar with adding two angular momenta in quantum mechanics, for example

$$|\frac{1}{2},m_1\rangle|\frac{1}{2}.m_2\rangle = |1,m_1+m_2\rangle\langle 1,m_1+m_2|\frac{1}{2}m_1,\frac{1}{2},m_2\rangle + |0,0\rangle\langle 0,0|\frac{1}{2}m_1,\frac{1}{2},m_2\rangle, \tag{1.127}$$

where the coefficients of the two vectors of the right-hand-side of this equation are the Clebsch-Gordan coefficients. This is a *direct product*, the product of two two-component objects. Thus, the generator of rotations is

$$\mathbf{J} = \frac{1}{2}(\boldsymbol{\sigma}+\boldsymbol{\tau}) = \frac{1}{2}(\boldsymbol{\sigma}\times 1 + 1\times\boldsymbol{\tau}), \tag{1.128}$$

the third component of which is

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.129}$$

The possible eigenvalues of J_3 are $(+1,0,-1)$ and 0, corresponding to the eigenvalue of $\mathbf{J}^2$ being $j(j+1)$ with $j=1$ or 0. Symbolically, we write

$$\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0. \tag{1.130}$$

This is the situation we have for the vector representation of the Lorentz group $(1/2,1/2)$.

A Dirac spinor is a direct sum, a four-component object, so the angular momentum operator is

$$\mathbf{J} = \frac{1}{2}(\boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-) = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma}_+ & 0 \\ 0 & \boldsymbol{\sigma}_- \end{pmatrix}, \tag{1.131}$$

so the third component if

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.132}$$

which has two eigenvalues of $+\frac{1}{2}$ and two eigenvalues of $-\frac{1}{2}$.

1.5 Lorentz transformations of the Dirac equation

Now let us return to the Dirac equation (1.14),

$$i\frac{\partial}{\partial t}\psi = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\psi, \quad \mathbf{p} = \frac{1}{i}\boldsymbol{\nabla}. \tag{1.133}$$

The αs are related to the angular momentum (spin) by (1.32), or

$$\frac{1}{2}[\alpha_i, \alpha_j] = i\epsilon_{ijk}\Sigma_k, \quad S_k = \frac{1}{2}\Sigma_k. \tag{1.134}$$

Comparing this to the boost equation (1.110) we must have

$$\frac{1}{2}\alpha_i = \pm i N_i, \tag{1.135}$$

which may be easily checked to satisfy (1.94) with $J_k \to S_k$. Thus, choosing the lower sign, we find that an infinitesimal Lorentz transformation on a Dirac spinor is given by

$$U = 1 + i\boldsymbol{\delta\omega}\cdot\frac{1}{2}\boldsymbol{\Sigma} - \delta\mathbf{v}\cdot\frac{1}{2}\boldsymbol{\alpha}. \tag{1.136}$$

Two relatively moving observers will write the Dirac equation as

$$O: \quad i\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\mathbf{x}} + \beta m\right)\psi(\mathbf{x}, t), \tag{1.137a}$$

$$\overline{O}: \quad i\frac{\partial}{\partial\overline{t}}\overline{\psi}(\overline{\mathbf{x}}, \overline{t}) = \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\overline{\mathbf{x}}} + \beta m\right)\overline{\psi}(\overline{\mathbf{x}}, \overline{t}), \tag{1.137b}$$

where

$$\overline{\psi}(\overline{\mathbf{x}}, \overline{t}) = U\psi(x, t). \tag{1.138}$$

For a boost, because

$$t = \overline{t} + \delta\mathbf{v}\cdot\overline{\mathbf{x}}, \quad \mathbf{x} = \overline{\mathbf{x}} + \delta\mathbf{v}\,\overline{t}, \tag{1.139}$$

we have

$$\frac{\partial}{\partial\overline{t}} = \frac{\partial x^\mu}{\partial\overline{t}}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} + \delta\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}, \tag{1.140a}$$

$$\frac{\partial}{\partial\overline{\mathbf{x}}} = \frac{\partial x^\mu}{\partial\overline{\mathbf{x}}}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial\mathbf{x}} + \delta\mathbf{v}\cdot\frac{\partial}{\partial t}. \tag{1.140b}$$

So the Dirac equation as seen by the barred observer is

$$\begin{aligned} &i\left(\frac{\partial}{\partial t}+\delta\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}\right)\left(1-\delta\mathbf{v}\cdot\frac{1}{2}\boldsymbol{\alpha}\right)\psi(\mathbf{x},t)\\ &\quad=\left(\frac{1}{i}\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\mathbf{x}}-i\boldsymbol{\alpha}\cdot\delta\mathbf{v}\frac{\partial}{\partial t}+\beta m\right)\left(1-\delta\mathbf{v}\cdot\frac{1}{2}\boldsymbol{\alpha}\right)\psi(\mathbf{x},t), \end{aligned} \tag{1.141}$$

Premultiply this equation by $(1-\delta\mathbf{v}\cdot\frac{1}{2}\boldsymbol{\alpha})$:

$$\begin{aligned} &i\left(\frac{\partial}{\partial t}+\delta\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}-\delta\mathbf{v}\cdot\boldsymbol{\alpha}\frac{\partial}{\partial t}\right)\psi(\mathbf{x},t)\\ &\quad=\left(\frac{1}{i}\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\mathbf{x}}-i\boldsymbol{\alpha}\cdot\delta\mathbf{v}\frac{\partial}{\partial t}+\beta m-\left\{\frac{1}{i}\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\mathbf{x}}+\beta m,\delta\mathbf{v}\cdot\frac{1}{2}\boldsymbol{\alpha}\right\}\right)\psi(\mathbf{x},t). \end{aligned} \tag{1.142}$$

In view of the commutation relations (1.16a) and (1.16b) we see that all the terms proportional to $\delta\mathbf{v}$ cancel, so the barred equation agrees with the unbarred one to order $\delta\mathbf{v}^2$. In homework, you will verify rotational invariance.

We can do all of this more compactly if we adopt covariant notation. Let us take new Dirac matrices defined by

$$\gamma^0=\beta,\quad \gamma^i=\beta\alpha^i. \tag{1.143}$$

γ^0 is Hermitian while γ^i is anti-Hermitian,

$$\gamma^{0\dagger}=\gamma^0,\quad \gamma^{i\dagger}=-\gamma^i. \tag{1.144}$$

Then the anticommutation relations become

$$\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}, \tag{1.145}$$

and the Dirac equation reads

$$\left(\gamma^\mu\frac{1}{i}\partial_\mu+m\right)\psi=0. \tag{1.146}$$

Now write a general infinitesimal Lorentz transformation as

$$\overline{x}^\mu=x^\mu-\delta x^\mu,\quad \delta x^\mu=\delta\omega^{\nu\mu}x_\nu,\quad \delta\omega^{\mu\nu}=-\delta\omega^{\nu\mu}. \tag{1.147}$$

In terms of the parameters corresponding to rotations and boosts,

$$\delta\omega^{ij}=\epsilon_{ijk}\delta\omega_k,\quad \delta\omega^{i0}=\delta v^i. \tag{1.148}$$

This induces a unitary transformation $U=1+iG$, where the generator is

$$G=\frac{1}{2}J^{\mu\nu}\delta\omega_{\mu\nu}, \tag{1.149}$$

where

$$[J_{\mu\nu}, J_{\kappa\lambda}] = i\left(g_{\mu\kappa}J_{\nu\lambda} - g_{\mu\lambda}J_{\nu\kappa} - g_{\nu\kappa}J_{\mu\lambda} + g_{\nu\lambda}J_{\mu\kappa}\right), \tag{1.150}$$

which you will prove in homework.

For spin 1/2, you will also prove in homework that

$$J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{1.151}$$

Now, under a general Lorentz transformation

$$\overline{\partial}_\mu = \frac{\partial x^\nu}{\partial \overline{x}^\mu}\frac{\partial}{\partial x^\nu} = \partial_\mu + \delta\omega_{\mu\nu}\partial^\nu. \tag{1.152}$$

Therefore, the transformed Dirac equation is ($\overline{\psi}(\overline{x}) = U\psi(x)$)

$$\begin{aligned}\left(\gamma^\mu\frac{1}{i}\overline{\partial}_\mu + m\right)\overline{\psi}(\overline{x}) &= \left[\gamma^\mu\frac{1}{i}(\partial_\mu + \delta\omega_{\mu\nu}\partial^\nu) + m\right]\left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\psi(x)\\ &= \left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\left(\gamma^\mu\frac{1}{i}\partial_\mu + m\right)\psi(x)\\ &\qquad + \left[\gamma^\mu, \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right]\frac{1}{i}\partial_\mu\psi(x) + \gamma^\mu\frac{1}{i}\delta\omega_{\mu\nu}\partial^\nu\psi(x).\end{aligned} \tag{1.153}$$

The commutator occuring here is

$$\begin{aligned}[\gamma^\mu, \sigma^{\alpha\beta}] &= \frac{i}{2}[\gamma^\mu, [\gamma^\alpha, \gamma^\beta]]\\ &= \frac{i}{2}\left(\{\gamma^\mu, \gamma^\alpha\}\gamma^\beta - \gamma^\alpha\{\gamma^\mu, \gamma^\beta\} - (\alpha \leftrightarrow \beta)\right)\\ &= 2i(g^{\mu\beta}\gamma^\alpha - g^{\mu\alpha}\gamma^\beta),\end{aligned} \tag{1.154}$$

so then we see that the second term in (1.153) cancels the third term exactly. Thus the Dirac equation transforms covariantly: If it holds in one inertial frame, it holds in all.

1.5.1 Discrete transformation

How does a Dirac field behave under a parity transformation, $\mathbf{x} \to -\mathbf{x}$? Since then $\partial_0 \to \partial_0$, $\partial_i \to -\partial_i$, we require $\psi \to U\psi$, where

$$U^{-1}\gamma_i U = -\gamma_i, \quad U^{-1}\gamma^0 U = \gamma^0. \tag{1.155}$$

A solution to this is

$$U = U^{-1} = \gamma^0, \tag{1.156}$$

so we take

$$\psi(\mathbf{x}, t) \to \gamma^0\psi(-\mathbf{x}, t). \tag{1.157}$$

Under time reversal, $\partial_0 \to -\partial_0$, $\partial_i \to \partial_i$, so now with $\psi \to U\psi$, we must have

$$U^{-1}\gamma^0 U = -\gamma^0, \quad U^{-1}\gamma^i U = \gamma^i \tag{1.158}$$

A solution is

$$U = U^{-1} = i\gamma_5 i\gamma^0, \tag{1.159}$$

where

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{1.160}$$

It may be easily shown that the matrices $i\gamma_5$ are Hermitian, and that

$$(i\gamma_5)^2 = 1, \quad \{i\gamma_5, \gamma^\mu\} = 0. \tag{1.161}$$

We also should recall that the implementation of time reversal on quantum states and operators is represented by an antiunitary operator,

$$\mathcal{T} = \mathcal{U}\mathcal{K}, \tag{1.162}$$

where $\mathcal{K}$ represents the operation of complex conjugation.

1.6 Poincaré Group

In addition to the Lorentz transformations, which are described by six generators $J^{\mu\nu} = -J^{\nu\mu}$ or $\mathbf{J}, \mathbf{N}$, we have, as discussed earlier in Section 1.3, translations in time and space. The four generators of translation are the energy and momentum,

$$G = P^\mu \delta\epsilon_\mu, \quad P^\mu = (H, \mathbf{P}). \tag{1.163}$$

The full set of commutators define the Poincaré group algebra:

$$[P_\mu, P_\nu] = 0, \quad \text{(translations commute)}, \tag{1.164a}$$

$$\frac{1}{i}[P_\mu, J_{\nu\lambda}] = g_{\mu\lambda}P_\nu - g_{\mu\nu}P_\lambda, \quad (P^\mu \text{ is a four-vector}), \tag{1.164b}$$

$$\frac{1}{i}[J_{\mu\nu}, J_{\kappa\lambda}] = g_{\mu\kappa}J_{\nu\lambda} - g_{\mu\lambda}J_{\nu\kappa} - g_{\nu\kappa}J_{\mu\lambda} + g_{\nu\lambda}J_{\mu\kappa}, \quad (J_{\mu\nu} \text{ is a second-rank tensor}). \tag{1.164c}$$

The irreducible representations of this group, which correspond to particles, are characterized by the values of

$$P^2 = \mathbf{P}^2 - (P^0)^2 = -m^2, \tag{1.165}$$

where m is the mass, and of

$$W^2 = W^\mu W_\mu, \quad W^\mu = P_\nu \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}J_{\lambda\sigma}, \quad P^\mu W_\mu = 0. \tag{1.166}$$

It is evident that W^μ is translationally invariant, $[P_\mu, W_\nu] = 0$. W^2 is a Lorentz scalar, $[J_{\mu\nu}, W^2]$, as you will explicitly show in homework. Here $\epsilon^{\mu\nu\lambda\sigma}$ is the totally antisymmetric invariant tensor,

$$\begin{aligned} \overline{\epsilon}^{\mu\nu\lambda\sigma} &= (g^{\mu\mu'} - \delta\omega^{\mu\mu'})(g^{\nu\nu'} - \delta\omega^{\nu\nu'})(g^{\lambda\lambda'} - \delta\omega^{\lambda\lambda'})(g^{\sigma\sigma'} - \delta\omega^{\sigma\sigma'})\epsilon_{\mu'\nu'\lambda'\sigma'} \\ &= \epsilon^{\mu\nu\lambda\sigma}, \end{aligned} \tag{1.167}$$

(because $\epsilon^{\mu\nu\lambda\sigma}$ vanishes if any two indices are the same), where $\epsilon^{0123} = +1$. In the rest frame of a particle,

$$\mathbf{P} = \mathbf{0}, \quad P^0 = E = m, \tag{1.168}$$

and so

$$W^0 = \frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = 0, \tag{1.169a}$$

$$W^i = -\frac{1}{2}\epsilon^{ijk0} J_{ij} m = m\frac{1}{2}\epsilon^{ijk} J_{jk} = mJ^i, \tag{1.169b}$$

where the latter is the spin. Thus, the eigenvalues of W^2 are

$$W^2 = m^2 s(s+1). \tag{1.170}$$

This means for a particle with nonzero rest mass, $m^2 > 0$, the irreducible representations belong to the values $s = 0, 1/2, 1, \ldots$. For a given s, the possible value of J_3 are $s_3 = -s, -s+1, -s+2, \ldots, s-1, s$. The massless limit has to be taken carefully (see homework):

$$m = 0: \quad W^\mu = \lambda P^\mu, \quad \lambda = \frac{\mathbf{P}\cdot\mathbf{S}}{P^0}. \tag{1.171}$$

λ is called the helicity, which is the spin projected along the direction of motion.

There are other representations of the Poincaré group, such as tachyons, where $m^2 < 0$, but they seem *not* to be realized in nature.