Thus the required rotational invariance statement is verified:

$$[\mathbf{J}, H] = [\mathbf{L} + \frac{1}{2}\mathbf{\Sigma}, H] = i\boldsymbol{\alpha} \times \mathbf{p} - i\boldsymbol{\alpha} \times \mathbf{p} = 0. \tag{1.49}$$

1.3 Translational Invariance

One of the invariances of any isolated physical system is the freedom to change the origin of time. Let us imagine a change in the time coordinate,

$$t \to \overline{t} = t - \delta t, \quad \text{where } \delta t \text{ is constant.} \tag{1.50}$$

In going from t to $\overline{t}$, the origin of time is shifted *forward* by an amount δt. Under such a change, states and operators do not change. However, we want to introduce *new* states and operators which have the same properties relative to the new time coordinate $\overline{t}$ as the old states and operators had relative to the old time coordinate t:

$$t\text{ coordinate}: \quad X, \quad |\,\rangle, \quad \langle\,|, \tag{1.51a}$$
$$\overline{t}\text{ coordinate}: \quad \overline{X}, \quad \overline{|\,\rangle}, \quad \overline{\langle\,|}. \tag{1.51b}$$

The new states and operators have the same inter-relations as the old states and operators; therefore, the two sets are related by a *unitary transformation*:

$$\overline{X} = U^{-1} X U, \quad U^{-1} = U^\dagger, \tag{1.52a}$$
$$\overline{\langle\,|} = \langle\,|U, \quad \overline{|\,\rangle} = U^\dagger|\,\rangle. \tag{1.52b}$$

What can we say about the unitary operator U? If $\delta t = 0$, the change in the states and operators is zero, so $U = 1$. If $\delta t \neq 0$, but very small, U must differ infinitesimally from 1. We therefore write

$$U = 1 - \frac{i}{\hbar}\delta t\, H. \tag{1.53}$$

We'll see in a moment why it's convenient to have the $-i/\hbar$ factor. What are the properties of H? U must be unitary so

$$1 = U^\dagger U = \left(1 + \frac{i}{\hbar}\delta t\, H^\dagger\right)\left(1 - \frac{i}{\hbar}\delta t\, H\right) = 1 - \frac{i}{\hbar}(H - H^\dagger)\delta t + O(\delta t^2). \tag{1.54}$$

Therefore H must be Hermitian, $H = H^\dagger$, which is why the i was put in front. We conclude that H is a physical quantity. It corresponds to the energy of the system; we call it the energy operator or the *Hamiltonian.* It has the right dimensions to be an energy, since $[\hbar] = \text{energy} \times \text{time}$. The Hamiltonian is the generator of time translations.

A *dynamical variable* is an operator characterizing in part a dynamical system, which changes as time evolves. An example we've discussed so far is the angular momentum $\mathbf{J}$. Let $v(t)$ be some dynamical variable. What happens under a displacement of the time origin?

$$t \to \overline{t} = t - \delta t, \tag{1.55a}$$

$$v(t) \to v(\overline{t} + \delta t) = \overline{v}(\overline{t}), \tag{1.55b}$$

where $\overline{v}$ is the new (transformed) variable. By definition, the new variable at the new time is the old variable at the old time. This is what is meant by saying that the new operators have the same properties relative to the new time coordinate as the old operators have relative to the old coordinate. Now, simply changing the name of the variable,

$$\overline{v}(t) = v(t + \delta t) = v(t) + \delta t \frac{d}{dt} v(t) = v(t) - \delta v(t), \tag{1.56}$$

where $\delta v(t)$ is just the change in the operator at the same value of the time coordinate,

$$\delta v(t) = v(t) - \overline{v}(t). \tag{1.57}$$

On the other hand, we can compute δv from the unitary operator U:

$$\begin{aligned} \overline{v}(t) &= U^\dagger v(t) U \\ &= \left(1 + \frac{i}{\hbar}\delta t\, H\right) v(t) \left(1 - \frac{i}{\hbar}\delta t\, H\right) \\ &= v(t) + \frac{1}{i\hbar}\delta t [v(t), H]. \end{aligned} \tag{1.58}$$

Hence,

$$\delta v(t) = -\delta t \frac{1}{i\hbar}[v(t), H], \tag{1.59}$$

or from (1.56)

$$\frac{d}{dt} v(t) = \frac{1}{i\hbar}[v(t), H], \tag{1.60}$$

which is the Heisenberg equation of motion.

For a finite time displacement, say by an amount t, the time evolution operator is

$$U(t) = e^{-iHt/\hbar}. \tag{1.61}$$

In a very similar way, we can talk about the translation of the origin of spatial coordinates. Consider an infinitesimal shift of the origin by an amount $\delta\mathbf{r}$. Consequently, a point located at position $\mathbf{r}$ in the original coordinate system is located at $\overline{\mathbf{r}}$ in the new coordinate system, where

$$\overline{\mathbf{r}} = \mathbf{r} - \delta\mathbf{r}. \tag{1.62}$$

As before, let the barred states and operators have the same properties relative to the new coordinate system as the unbarred states do relative to the old:

$$\overline{\mathbf{r}} \text{ coordinates}: \quad \overline{X}, \quad \overline{|\,\rangle}, \quad \overline{\langle\,|}, \tag{1.63a}$$

$$\mathbf{r} \text{ coordinates}: \quad X, \quad |\,\rangle, \quad \langle\,|, \tag{1.63b}$$

and these two sets are related by a unitary transformation:

$$\overline{X} = U^\dagger X U, \quad \overline{\langle\ |} = \langle\ |U, \quad \overline{|\ \rangle} = U^\dagger |\ \rangle. \tag{1.64}$$

For an infinitesimal transformation of coordinates we write the unitary operator as

$$U = 1 + \frac{i}{\hbar}\delta\mathbf{r}\cdot\mathbf{P}, \quad \mathbf{P} = \mathbf{P}^\dagger, \tag{1.65}$$

where $\mathbf{P}$, the generator of spatial translations, is the linear momentum operator. Since if we first make a displacement along the x axis, and then a displacement along the y axis, we end up with the same coordinate system as if we had first displaced along the y axis and then along the x axis,

$$U_{\delta x}U_{\delta y} = U_{\delta y}U_{\delta x}, \tag{1.66}$$

or

$$\left(1 + \frac{i}{\hbar}\delta x P_x\right)\left(1 + \frac{i}{\hbar}\delta y P_y\right) = \left(1 + \frac{i}{\hbar}\delta y P_y\right)\left(1 + \frac{i}{\hbar}\delta x P_x\right), \tag{1.67}$$

or

$$P_x P_y = P_y P_x, \tag{1.68}$$

$$[P_x, P_y] = 0. \tag{1.69}$$

The different components of $\mathbf{P}$ commute with each other.

Operators transform according to

$$\begin{aligned}\overline{X} &= U^\dagger X U = \left(1 - \frac{i}{\hbar}\delta\mathbf{r}\cdot\mathbf{P}\right) X \left(1 + \frac{i}{\hbar}\delta\mathbf{r}\cdot\mathbf{P}\right)\\ &= X - \frac{1}{i\hbar}[X,\mathbf{P}]\cdot\delta\mathbf{r} = X - \delta X,\end{aligned} \tag{1.70}$$

where

$$\delta X = \frac{1}{i\hbar}[X,\mathbf{P}]\cdot\delta\mathbf{r}. \tag{1.71}$$

A finite displacement of the coordinate system by $\mathbf{r}$ is implemented by

$$U(\mathbf{r}) = e^{i\mathbf{P}\cdot\mathbf{r}/\hbar}. \tag{1.72}$$

All of this is precisely the same as in nonrelativistic quantum mechanics.

1.4 Lorentz Covariance

The Lorentz group is the group of transformations that preserves the length

$$x^\mu x_\mu = \mathbf{x}^2 - t^2, \tag{1.73}$$

which defines the light cone. Evidently, continuous transformations that do this are:

1. Ordinary rotations, for example, a rotation about the z axis,

$$x \to \overline{x} = x\cos\theta + y\sin\theta, \tag{1.74a}$$
$$y \to \overline{y} = y\cos\theta - x\sin\theta, \tag{1.74b}$$
$$z \to \overline{z} = z, \tag{1.74c}$$
$$t \to \overline{t} = t, \tag{1.74d}$$

   or, infinitesimally,

$$\mathbf{x} \to \mathbf{x} - \delta\mathbf{x}, \quad \delta\mathbf{x} = \boldsymbol{\delta\theta} \times \mathbf{x}, \tag{1.75}$$

   where, for an infinitesimal rotation about the z axis,

$$\boldsymbol{\delta\theta} = \delta\theta\hat{\mathbf{z}}, \tag{1.76}$$

   so in agreement with (1.74d),

$$\delta x = -y\delta\theta, \quad \delta y = x\delta\theta. \tag{1.77}$$

2. "Boosts," or imaginary rotations, which correspond to passing from one inertial frame to another one moving relative to the first with velocity $\mathbf{v}$. If the latter lies along the x axis, for example,

$$t \to \overline{t} = t\cosh\phi + x\sinh\phi, \tag{1.78a}$$
$$x \to \overline{x} = x\cosh\phi + t\sinh\phi, \tag{1.78b}$$

   which for an infinitesimal boost reads

$$\delta t = -x\delta\phi, \quad \delta x = -t\delta\phi, \tag{1.79}$$

   while y and z are unchanged. Consider a particle at rest in the original frame, $dx/dt = 0$. In the new frame it will have velocity

$$\frac{d\overline{x}}{d\overline{t}} = \frac{d\overline{x}}{dt}\frac{dt}{d\overline{t}} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi = -v, \tag{1.80}$$

   which is just the relative velocity of the two frames. Thus we recover the elementary form of the Lorentz transformation,

$$\sinh\phi = -\frac{v}{\sqrt{1-v^2}}, \quad \cosh\phi = \frac{1}{\sqrt{1-v^2}}. \tag{1.81}$$

Now under a transformation of the space-time coordinates,

$$\mathbf{x} \to \overline{\mathbf{x}} = \mathbf{x} - \delta\mathbf{x}, \quad \delta\mathbf{x} = \delta\mathbf{v}\, t \tag{1.82a}$$
$$t \to \overline{t} = t - \delta t, \quad \delta t = \delta\mathbf{v}\cdot\mathbf{x}, \tag{1.82b}$$

the quantum states undergo a unitary transformation

$$\langle\,| \to \overline{\langle\,|} = \langle\,|U, \tag{1.83a}$$
$$|\,\rangle \to \overline{|\,\rangle} = U^\dagger|\,\rangle, \tag{1.83b}$$

where U is unitary, that is

$$U^\dagger = U^{-1}. \tag{1.84}$$

Now the barred state vectors bear the same relation to the barred coordinate system as the unbarred state vectors do to the unbarred coordinates. Operators change as follows under a unitary transformation

$$X \to \overline{X} = U^\dagger X U. \tag{1.85}$$

For an infinitesimal transformation,

$$U = 1 + iG, \tag{1.86}$$

where the "generator" G of the transformation must be Hermitian,

$$G = G^\dagger, \tag{1.87}$$

so that

$$U^{-1} = U^\dagger = 1 - iG. \tag{1.88}$$

Thus, the infinitesimal transformation of an operator is

$$X \to X - \delta X, \quad \delta X = \frac{1}{i}[X, G]. \tag{1.89}$$

For an ordinary rotation given by

$$\delta\mathbf{x} = \delta\boldsymbol{\omega} \times \mathbf{x}, \tag{1.90}$$

The generator is taken to be $G = \mathbf{J} \cdot \delta\boldsymbol{\omega}$. Since $\mathbf{J}$ is a vector as well as the generator of rotations,

$$\delta\mathbf{J} = \boldsymbol{\delta\omega} \times \mathbf{J} = \frac{1}{i}[\mathbf{J}, \mathbf{J} \cdot \boldsymbol{\delta\omega}], \tag{1.91}$$

or in component form

$$\epsilon_{ijk}\delta\omega_j J_k = \frac{1}{i}[J_i, J_l\delta\omega_l], \tag{1.92}$$

from which follows the familiar commutation relation for the angular momentum operator,

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{1.93}$$

The generator for the boost $\delta\mathbf{v}$ is $G = \mathbf{N} \cdot \delta\mathbf{v}$. Since $\mathbf{N}$ is also a vector, we have immediately

$$[N_i, J_j] = i\epsilon_{ijk}N_k. \tag{1.94}$$

To figure out the commutation relations for N, we need to consider more closely the effects of successive transformations. Suppose we consider two successive transformations, labeled 1 and 2, but undo them in the opposite order.

The resulting unitary operator we denote as $U_{[12]}$, where, where for infinitesimal transformations

$$\begin{aligned} U_{[12]} &= U_1^{-1}U_2^{-1}U_1U_2 \\ &= (1-iG_1)(1-iG_2)(1+iG_1)(1+iG_2) \\ &= 1 - G_1G_2 + G_2G_1, \end{aligned} \tag{1.95}$$

where we have ignored quadratic terms but not bilinear ones. Thus the generator of the net transformation is

$$iG_{[12]} = -[G_1, G_2].‘ \tag{1.96}$$

Let us check this result for the case of rotations, where

$$i\mathbf{J}\cdot\boldsymbol{\delta\omega}_{[12]} = -[\mathbf{J}\cdot\boldsymbol{\delta\omega}_1, \mathbf{J}\cdot\boldsymbol{\delta\omega}_2]. \tag{1.97}$$

But geometrically,

$$\boldsymbol{\delta\omega}_{[12]} = \boldsymbol{\delta\omega}_2 \times \boldsymbol{\delta\omega}_1 \tag{1.98}$$

because we follow

$$\overline{\mathbf{x}} = \mathbf{x} - \boldsymbol{\delta\omega}_2 \times \mathbf{x} \tag{1.99}$$

by

$$\overline{\overline{\mathbf{x}}} = \overline{\mathbf{x}} - \boldsymbol{\delta\omega}_1 \times \overline{\mathbf{x}} = \mathbf{x} - \boldsymbol{\delta\omega}_2 \times \mathbf{x} - \boldsymbol{\delta\omega}_1 \times \mathbf{x} + \boldsymbol{\delta\omega}_1 \times (\boldsymbol{\delta\omega}_2 \times \mathbf{x}), \tag{1.100}$$

so

$$-[\delta_1, \delta_2]\mathbf{x} = \boldsymbol{\delta\omega}_1 \times (\boldsymbol{\delta\omega}_2 \times \mathbf{x}) - \boldsymbol{\delta\omega}_2 \times (\boldsymbol{\delta\omega}_1 \times \mathbf{x}). \tag{1.101}$$

But we recall the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{0}. \tag{1.102}$$

Thus

$$[\delta_1, \delta_2]\mathbf{x} = \boldsymbol{\delta\omega}_{[12]} \times \mathbf{x}, \tag{1.103}$$

according to (1.98). Thus from (1.97) follows the angular momentum commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{1.104}$$

Now, for two successive boosts, according to (1.82a) and (1.82b),

$$\delta_1\mathbf{x} = t\delta\mathbf{v}_1, \quad \delta_1 t = \delta\mathbf{v}_1 \cdot \mathbf{x}, \tag{1.105a}$$

$$\delta_2\mathbf{x} = t\delta\mathbf{v}_2, \quad \delta_2 t = \delta\mathbf{v}_2 \cdot \mathbf{x}, \tag{1.105b}$$

we have

$$\overline{\mathbf{x}} = \mathbf{x} - t\delta\mathbf{v}_2, \quad \overline{t} = t - \delta\mathbf{v}_2 \cdot \mathbf{x}, \tag{1.106a}$$

$$\overline{\overline{\mathbf{x}}} = \overline{\mathbf{x}} - \overline{t}\delta\mathbf{v}_1 = \mathbf{x} - t(\delta\mathbf{v}_1 + \delta\mathbf{v}_2) + \delta\mathbf{v}_1(\delta\mathbf{v}_2 \cdot \mathbf{x}), \tag{1.106b}$$

$$\overline{\overline{t}} = \overline{t} - \delta\mathbf{v}_1 \cdot \overline{\mathbf{x}} = t - (\delta\mathbf{v}_1 + \delta\mathbf{v}_2) \cdot \mathbf{x} + \delta\mathbf{v}_1 \cdot \delta\mathbf{v}_2 t, \tag{1.106c}$$

from which we conclude that

$$-[\delta_1, \delta_2]\mathbf{x} = \delta\mathbf{v}_1(\delta\mathbf{v}_2 \cdot \mathbf{x}) - \delta\mathbf{v}_2(\delta\mathbf{v}_1 \cdot \mathbf{x}) = -(\delta\mathbf{v}_1 \times \delta\mathbf{v}_2) \times \mathbf{x}, \tag{1.107a}$$

$$-[\delta_1, \delta_2]t = (\delta\mathbf{v}_1 \cdot \delta\mathbf{v}_2)t - (\delta\mathbf{v}_2 \cdot \delta\mathbf{v}_1)t = 0. \tag{1.107b}$$

Thus the net effect of the succession of boosts, first in one order, and then in the reverse, is a rotation about the axis perpendicular to the axes of the individual boosts,

$$\boldsymbol{\delta\omega}_{[12]} = \delta\mathbf{v}_1 \times \delta\mathbf{v}_2. \tag{1.108}$$

From the composition property of the generators (1.96), we conclude

$$i\mathbf{J} \cdot \boldsymbol{\delta\omega}_{[12]} = -[\mathbf{N} \cdot \delta\mathbf{v}_1, \mathbf{N} \cdot \delta\mathbf{v}_2], \tag{1.109}$$

or, in view of (1.108),

$$[N_i, N_j] = -i\epsilon_{ijk} J_k. \tag{1.110}$$

Let us disentangle our algebra by defining

$$\mathbf{J}^{(\pm)} = \frac{1}{2}(\mathbf{J} \pm i\mathbf{N}), \tag{1.111}$$

so that the generators for rotations and boosts are written as

$$\mathbf{J} = \mathbf{J}^{(+)} + \mathbf{J}^{(-)}, \quad \mathbf{N} = -i(\mathbf{J}^{(+)} - \mathbf{J}^{(-)}), \tag{1.112}$$

where $\mathbf{J}^{(+)\dagger} = \mathbf{J}^{(-)}$. Then it is easy to check that

$$[J_i^{(\pm)}, J_j^{(\pm)}] = i\epsilon_{ijk} J_k^{(\pm)}, \tag{1.113a}$$

$$[J_i^{(\pm)}, J_j^{(\mp)}] = 0. \tag{1.113b}$$

That is, $\mathbf{J}^{(+)}$ and $\mathbf{J}^{(-)}$ constitute *two commuting angular momenta.*

The finite unitary transformation corresponding to rotations and boosts are

$$U_R(\theta, \mathbf{e}) = \exp\left\{i\theta\mathbf{e} \cdot [\mathbf{J}^{(+)} + \mathbf{J}^{(-)}]\right\}, \tag{1.114a}$$

$$U_B(\phi, \mathbf{e}) = \exp\left\{-\phi\mathbf{e} \cdot [\mathbf{J}^{(+)} - \mathbf{J}^{(-)}]\right\}, \tag{1.114b}$$

where $\mathbf{e}$ is the axis of the rotation, or the direction of the boost, respectively, and θ is the rotation angle, and ϕ the "rapidity" (1.81).

In group theory terms we have shown that that the Lorentz group equals

$$SO(3,1) = SU(2) \times SU(2), \tag{1.115}$$

The irreducible representations of the Lorentz group are therefore labelled by the eigenvalues of $(\mathbf{J}^{(+)})^2$, which are $s_+(s_+ +1)$ and of $(\mathbf{J}^{(-)})^2$, which are $s_-(s_- +1)$. We label the irreducible representations by the ordered pair (s_+, s_-) where $s_\pm$ are either integers or integers plus one-half.

Here are some examples:

- $(0,0)$ is a scalar.
- $(1/2,0)$ is a left-handed spinor.
- $(0,1/2)$ is a right-handed spinor.
- $(1/2,1/2)$ is a vector.