Physics 5970
Advanced Quantum Field Theory
Homework 6

Due Monday, April 4, 2005

1. To establish the unitarity of the free photon description, given by the vacuum persistence amplitude expression

$$\langle 0_+|0_-\rangle_0^J = \exp\left[\frac{i}{2}\int (dx)(dx')\, J^\mu(x)D_+(x-x')J_\mu(x')\right]$$

supplemented by the constraint

$$\partial^\mu J_\mu = 0,$$

proceed as follows. For a massless particle, p^μ is a null vector,

$$p^2 = 0.$$

Define a second null vector from the components of $p^\mu = (p^0, \mathbf{p})$:

$$\overline{p}^\mu = (p^0, -\mathbf{p}), \quad \overline{p}^2 = 0.$$

Show that $p + \overline{p}$ is a time-like vector, while $p - \overline{p}$ is space-like. Let the two remaining space-like directions be spanned by the orthonormal vectors e^μ_{p1}, e^μ_{p2}:

$$\begin{aligned} g^{\mu\nu} &= \frac{(p+\overline{p})^\mu (p+\overline{p})^\nu}{(p+\overline{p})^2} + \frac{(p-\overline{p})^\mu (p-\overline{p})^\nu}{(p-\overline{p})^2} + \sum_{\lambda=1}^{2} e^\mu_{p\lambda} e^{\nu *}_{p\lambda} \\ &= \frac{p^\mu \overline{p}^\nu + \overline{p}^\mu p^\nu}{(p\overline{p})} + \sum_{\lambda=1}^{2} e^\mu_{p\lambda} e^{\nu *}_{p\lambda}. \end{aligned} \tag{1}$$

Then show that

$$\int d\tilde{p}\, J^\mu(-p) J_\mu(p) = \sum_{\mathbf{p}\lambda} |J_{\mathbf{p}\lambda}|^2,$$

where

$$J_{\mathbf{p}\lambda} = \sqrt{d\tilde{p}}\, e^{\mu *}_{p\lambda} J_\mu(p),$$

and from this show that

$$|\langle 0_+|0_-\rangle_0^J|^2 \leq 1.$$

These next two problems have to do with the Euclidean postulate and fermionic charge.

2. Consider first spin 1. Show that under a Lorentz transformation a vector source transforms as

$$\delta J^\lambda(x) = -\delta\omega^{\mu\nu} x_\mu \partial_\nu J^\lambda(x) - \delta\omega^{\lambda\nu} J_\nu(x).$$

Show that this can be rewritten as

$$\delta J^\lambda = -i\frac{1}{2}\delta\omega^{\mu\nu}[L_{\mu\nu} + S_{\mu\nu}]^\lambda{}_\kappa J^\kappa,$$

where L is the orbital angular momentum

$$L_{\mu\nu} = x_\mu \frac{1}{i}\partial_\nu - x_\nu \frac{1}{i}\partial_\mu$$

and the spin $S_{\mu\nu}$ has matrix elements

$$(S_{\mu\nu})^\lambda{}_\kappa = \frac{1}{i}(\delta^\lambda_\mu g_{\nu\kappa} - \delta^\lambda_\nu g_{\mu\kappa}).$$

Show that S_{kl} are antisymmetrical and Hermitian, while S_{0k} are symmetrical and anti-Hermitian. The spin matrices cannot be all Hermitian, because the Lorentz group does not possess finite dimensional realizations. However, now pass to Euclidean space by letting $x_4 = ix^0$. Display then the Euclidean spin matrices $(S^E_{\mu\nu})_{\lambda\kappa}$ and show that they are all imaginary, antisymmetrical, and Hermitian. Therefore all four dimensions are on the same footing.

3. How goes it for spin 1/2? There the spin matrices are given in terms of

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu].$$

Show that σ_{kl} is antisymmetrical and Hermitian, while $\sigma^0{}_k$ is symmetrical and anti-Hermitian. In passing to Euclidean space we encounter $\sigma_{k4} = \gamma^0\gamma_k$ which is Hermitian, symmetrical, and hence real. Is it possible to find a unitary transformation U which preserves σ_{kl} but which changes the symmetry and reality of σ_{k4}? Answer this question in the negative by showing that γ^0, $i\gamma_5$ and $\gamma^0\gamma_5$ are the only Hermitian matrices which commute with $\gamma_k\gamma_l$, and that these are all imaginary, so U is necessarily real. Thus in Euclidean space the time axis is remembered, a violation of the "Euclidean postulate."

There is a way out if the fermions possess an additional multiplicity ("charge") in which space an independent antisymmetric charge matrix

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

can act. Then show that $U = e^{i\alpha q\gamma^0}$ implies

$$U\sigma_{k4}U^{-1} = e^{2i\alpha q\gamma^0}\gamma^0\gamma_k.$$

Show that this transformed matrix is imaginary and antisymmetrical for a particular value of α. That value of α defines the Euclidean spin matrices σ^E_{ij}, $i, j = 1, \ldots 4$, which are all imaginary and antisymmetrical. Display σ^E_{ij} in terms of q and γ matrices. All memory of the time axis has thus been lost.

Thus, the Euclidean postulate requires that every spin-1/2 particle possesses a charge-like attribute, and this is certainly in accord with experiment. Electrons have electric charge, neutrinos have lepton number, neutrons have baryon number, etc. That is, fermions have distinct particles and antiparticles. This need not be the case with bosons (π^0, γ,...).