Physics 5970, Advanced Quantum Field Theory
Homework 4

Due Wednesday, March 9, 2005

1. It is oftentimes convenient to work in a so-called Majorana representation, where $\psi^\dagger = \psi$. (Transposition is understood from context.) Then consider the response of the spin-1/2 Lagrange function

$$\mathcal{L} = -\frac{1}{2}\psi\gamma^0\left(\gamma\frac{1}{i}\partial + m\right)\psi$$

(the 1/2 is inserted to eliminate double counting) to the infinitesimal variation

$$\delta\psi = -\delta x^\nu(x)\partial_\nu\psi(x) - \frac{i}{4}\sigma^{\mu\nu}\psi(x)\partial_\mu\delta x_\nu(x).$$

This is a local generalization of a space-time translation, and incorporates Lorentz transformations. Show that by virtue of the properties of the γ matrices, and the fact, to be established later, that the ψ's are anticommuting Grassmann elements, $\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha$,

$$\delta\mathcal{L} = -\partial_\nu(\delta x^\nu\mathcal{L}) + t^{\mu\nu}\partial_\mu\delta x_\nu,$$

and display $t^{\mu\nu}$, the energy-momentum tensor. Show by the stationary action principle that

$$\partial_\mu t^{\mu\nu} = 0.$$

2. Scale invariance is violated by the energy-momentum tensor derived for a scalar in class, because, as $m \to 0$, $t = t^\mu{}_\mu \not\to 0$. To what limit

does t converge? We can, however, introduce a new energy-momentum tensor, $\theta^{\mu\nu}$, by adding to $t^{\mu\nu}$ an identically conserved term,

$$\theta^{\mu\nu} = t^{\mu\nu} + \xi \left(\partial^\mu \partial^\nu - g^{\mu\nu}\partial^2\right)\phi^2.$$

Determine the number ξ so that $\theta = \theta^\mu{}_\mu \to 0$ as $m \to 0$. This is the so-called conformal stress tensor. Argue that $\theta^{\mu\nu}$ and $t^{\mu\nu}$ are equally as good at describing the energy and momentum of the system. [There is an inherent arbitrariness in defining the energy-momentum tensor.]

3. Show that under a Euclidean-space transformation

$$i|x^0 - x'^0| \to |x_4 - x_4'|$$

the propagation function becomes

$$\Delta_+(x - x') \to i\Delta_E(x - x'),$$

where the Euclidean Green's function is

$$\Delta_E(x - x') = \int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}') - p^0|x_4 - x_4'|}, \quad p^0 = \sqrt{\mathbf{p}^2 + m^2}.$$

Now establish the identity

$$\frac{1}{2p^0} e^{-p^0|x_4 - x_4'|} = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{e^{ip_4(x_4 - x_4')}}{p_4^2 + (p^0)^2}.$$

Use this identity to prove

$$\Delta_E(x - x') = \int \frac{(dp)_E}{(2\pi)^4} \frac{e^{ip\cdot(x-x')_E}}{p_E^2 + m^2},$$

where the subscript E indicates Euclidean scalar products and measures,

$$p \cdot (x - x')_E = \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}') + p_4(x_4 - x_4'),$$
$$(dp)_E = (d\mathbf{p})dp_4.$$

4. What happens, if instead of rotating back,

$$|x_4 - x_4'| \to i|x^0 - x'^0|,$$

to recover Δ_+, we continue rotating in the same sense,

$$|x_4 - x_4'| \to -i|x^0 - x'^0|?$$

Show that then

$$\Delta_E(x - x') \to i\Delta_-(x - x')$$

where

$$\Delta_-(x - x') = \Delta_-(x' - x) = -i \int d\tilde{p}\, e^{-ip(x-x')}, \quad (x - x')^0 > 0,$$

where

$$d\tilde{p} = \frac{(d\mathbf{p})}{(2\pi)^3} \frac{1}{2p^0}$$

Show that

$$\Delta_-(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2 + m^2 + i\epsilon},$$

where now Minkowski inner products and measures are used.

5. It is particularly easy to get explicit limiting cases for the Euclidean propagation function. Let $|x-x'| = R$ and choose the four-dimensional coordinate system so that

$$\mathbf{x} - \mathbf{x}' = 0, \quad x_4 - x_4' = R.$$

Then show that

(a) if $mR \ll 1$,

$$\Delta_E(R) \sim \frac{1}{4\pi^2 R^2},$$

and

(b) if $mR \gg 1$,

$$\Delta_E(R) \sim \frac{\sqrt{2m}}{(4\pi R)^{3/2}} e^{-mR}.$$