Physics 5970. Homework 1
Due Friday, February 4, 2005

January 23, 2005

1. Verify that the representation presented in lecture,

$$\beta = \tau_3 \otimes \sigma_2, \quad \alpha_1 = 1 \otimes \sigma_1, \quad \alpha_2 = \tau_2 \otimes \sigma_2, \quad \alpha_3 = 1 \otimes \sigma_3$$

   satisfies the Dirac algebra as well as the Hermiticity and reality requirements.

2. Prove that Σ_k defined by

$$\frac{1}{2}[\alpha_i, \alpha_j] = i\epsilon_{ijk}\Sigma_k$$

   satisfies the spin-1/2 algebra

$$\Sigma_k \Sigma_l = \delta_{kl} + i\epsilon_{klm}\Sigma_m.$$

   What is the explicit form of $\Sigma_{1,2,3}$ in terms of the representation in problem 1?

3. Verify explicitly the rotational invariance of the Dirac equation

$$i\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \left(\frac{1}{i}\vec{\alpha} \cdot \vec{\nabla} + \beta m\right)\psi(\mathbf{x}, t).$$

4. Using the commutation relations for $\mathbf{J}$ and $\mathbf{N}$ show that

$$[J_{\mu\nu}, J_{\kappa\lambda}] = i(g_{\mu\kappa}J_{\nu\lambda} - g_{\nu\kappa}J_{\mu\lambda} + g_{\nu\lambda}J_{\mu\kappa} - g_{\mu\lambda}J_{\nu\kappa},$$

and that this is satisfied by

$$J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

Show that

$$\sigma^{ij} = \epsilon^{ijk}\Sigma^k$$

and

$$\frac{1}{2}\sigma^{0i} = -N^i.$$

5. (a) Prove that $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}J_{\lambda\sigma}P_\nu$ indeed transforms as a four-vector, and hence that $W^2 = W^\mu W_\mu$ is a scalar (invariant).

   (b) By letting $m \to 0$ carefully, show that $W^\mu = \lambda P^\mu$ where $\lambda = \mathbf{P} \cdot \mathbf{S}/P^0$ is a Lorentz invariant, the helicity.