Mesons, then are described by a wavefunction

$$\Phi = \overline{q}_a q_a, \tag{6.23}$$

and baryons by

$$\Psi = \epsilon_{abc} q_a q_b q_c. \tag{6.24}$$

This resolves the old paradox that ground state wavefunctions tend to be symmetric, in violation of the Pauli principle. For example, the famous spin 3/2 baryon resonance, composed of three up quarks,

$$\Delta^{++}(1232) = uuu, \tag{6.25}$$

has a symmetric wavefunction in space and spin ($l = 0$), because

$$S_z = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}. \tag{6.26}$$

However, the requirement of the Fermi-Dirac statistics that the state be totally antisymmetric under the interchange of its fermionic components is satisfied because it is antisymmetric in color.

6.1 Quadratic Casimir Invariants

For a representation R, let the generators be $T_a(R)$. Then we define the quadratic Casimir invariant by

$$\sum_a T_a(R)^2 = C_2(R)I. \tag{6.27}$$

For example, for the fundamental representation $\mathbf{3}$ of SU(3),

$$T_a(\mathbf{3}) = \frac{\lambda^a}{2}, \tag{6.28}$$

so

$$\mathrm{Tr} \sum_a \left(\frac{\lambda^a}{2}\right)^2 = \sum_a \frac{1}{2}\delta^{aa} = 4 = 3C_2(\mathbf{3}), \tag{6.29}$$

or

$$C_2(\mathbf{3}) = \frac{4}{3}. \tag{6.30}$$

For the fundamental representation $\mathbf{8}$

$$(T_a(\mathbf{8}))_{bc} = i f_{abc}, \tag{6.31}$$

so

$$-\sum_a f_{abc} f_{acd} = C_2(\mathbf{8})\delta_{bd}, \tag{6.32}$$

so setting $b = d$ and summing on all three indices

$$8C_2(\mathbf{8}) = -f_{abc} f_{acb} = f_{abc}^2 = 6\left(1 + \frac{6}{4} + 2\frac{3}{4}\right) = 24, \tag{6.33}$$

where the last step is supplied by your homework, or

$$C_2(\mathbf{8}) = 3. \tag{6.34}$$

quark	$\xrightarrow{p}$	$-i\frac{m-\gamma p}{p^2+m^2-i\epsilon}$
gluon	q	$-i\frac{1}{q^2-i\epsilon}\left(g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2-i\epsilon}(1-\xi)\right)$
ghost	$\overset{p}{- - - - \rightarrow}$	$-i\frac{1}{p^2-i\epsilon}$

Figure 6.2: Propagators in QCD. Here ξ is an arbitrary gauge parameter. Ghosts occur only inside loops, and like fermions have a -1 associated with the loop.

6.2 Asymptotic Freedom

From the QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \sum_f \overline{q}_f\left(\gamma\frac{1}{i}D+m\right)q_f + \mathcal{L}_{\mathrm{gf}}, \tag{6.35}$$

where the added term is the gauge fixing term (for example, see Peskin and Schroeder) we can deduce the Feynman rules shown in Figs. 6.2 and 6.3.

Renormalization proceeds similarly to the scalar case discussed in Chapter 3. The bare fields are related to the renormalized ones by

$$A_0^{a,\mu} = Z_3^{1/2}A^{a,\mu}, \quad \psi_0 = Z_2^{1/2}\psi. \tag{6.36}$$

The quark mass is renormalized according to

$$m_0 Z_2 = m + \delta m, \tag{6.37}$$

where we recall that the relation between the bare mass and the renormalized mass terms in the Lagrangian is

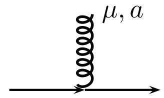
$$m_0\overline{\psi}_0\psi_0 = m\overline{\psi}\psi + \delta m\overline{\psi}\psi. \tag{6.38}$$

Consider the corresponding equivalence between the quark-gluon coupling terms:

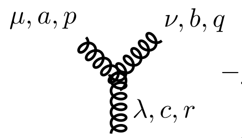
$$g_0\overline{\psi}_0\gamma^\mu\frac{\lambda^a}{2}\psi_0(A_\mu)_0 = (g+\delta g)\overline{\psi}\gamma^\mu\frac{\lambda^a}{2}\psi A_\mu. \tag{6.39}$$

From this we see that the relation between the bare and renormlized coupling constants is

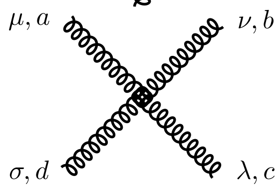
$$g_0 Z_2 Z_3^{1/2} = g + \delta g \equiv Z_1 g, \tag{6.40}$$



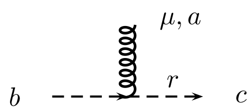
$$-ig\gamma^\mu \frac{\lambda^a}{2}$$



$$-gf_{abc}[g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-r)_\mu + g_{\lambda\mu}(r-p)_\nu]$$



$$\begin{aligned}-ig^2[&f_{abe}f_{cde}(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})\\ &+ f_{ace}f_{bde}(g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\sigma}g_{\nu\lambda})\\ &+ f_{ade}f_{cbe}(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\nu}g_{\lambda\sigma})]\end{aligned}$$



$$-gf_{abc}r_\mu$$

Figure 6.3: Feynman rules for vertices in QCD. Here the convention is that all momenta flow into the vertex.

or

$$g_0 = Z_1 Z_2^{-1} Z_3^{-1/2} g. \tag{6.41}$$

This is because the vertex involves both vertex and proper vertex corrections.

Let us illustrate the process by computing the quark contribution to Z_3. It is the generalization of the vacuum polarization graph we computed before in Section 4.5, and is in d dimensions

$$i\Pi^{ab}_{\alpha\beta} = -\left(-\frac{ig}{2}\right)^2 \int \frac{d^dp}{(2\pi)^d} \mathrm{Tr}\left[\gamma_\alpha \lambda^a \frac{-i}{m - \gamma p - i\epsilon}\gamma_\beta \lambda^b \frac{-i}{m - \gamma p + \gamma q - i\epsilon}\right]. \tag{6.42}$$

Recall $\mathrm{Tr}\,\lambda^a\lambda^b = 2\delta^{ab}$, and the Dirac trace is

$$\begin{aligned}&\mathrm{Tr}\,[\gamma_\alpha(m+\gamma p)\gamma_\beta(m+\gamma p - \gamma q)]\\ =\ & m^2 \mathrm{Tr}\,\gamma_\alpha\gamma_\beta + \mathrm{Tr}\,\gamma_\alpha\gamma p\gamma_\beta\gamma(p-q)\\ =\ & -4m^2 g_{\alpha\beta} - 4[p_\alpha(p-q)_\beta + p_\beta(p-q)_\alpha - g_{\alpha\beta}p(p-q)].\end{aligned} \tag{6.43}$$

Anticipating the change of variable that will follow, we note that the last parenthetical term can be written as

$$[(p-qu)+qu]_\alpha[p-qu-q(1-u)]_\beta + (\alpha \leftrightarrow \beta) - g_{\alpha\beta}[(p-qu)+qu][(p-qu)-q(1-u)]. \tag{6.44}$$

Now the denominators may be combined by the usual exponential trick

$$\frac{1}{m^2+p^2-i\epsilon}\frac{1}{m^2+(p-q)^2-i\epsilon} = -\int_0^\infty ds\, s\int_0^1 du\, e^{-is\chi(u)}, \tag{6.45}$$

where

$$\chi(u) = (1-u)(m^2+p^2)+u[m^2+(p-q)^2] = m^2+(p-qu)^2+q^2u(1-u). \tag{6.46}$$

Now when we integrate this over momentum we get

$$\begin{aligned} &-\int_0^\infty ds\, s\frac{i}{(2\pi)^d}\left(\frac{\pi}{is}\right)^{d/2}\int_0^1 du\, e^{-is[m^2+q^2u(1-u)]} \\ &= \frac{i}{2^d\pi^{d/2}}\Gamma(2-d/2)\int_0^1 du[m^2+q^2u(1-u)]^{d/2-2}. \end{aligned} \tag{6.47}$$

Since we are here interested only in ultraviolet behavior, let us drop the quark mass, and to keep this quantity properly dimensionless insert a factor of μ^{4-d}, where μ is some arbitrary mass scale. Then the above is simply

$$\frac{i}{2^d\pi^{d/2}}\left(\frac{q^2}{\mu^2}\right)^{d/2-2}\Gamma(2-d/2)\frac{\Gamma(d/2-1)^2}{\Gamma(d-2)} = \frac{i}{16\pi^2}\left(\frac{q^2}{\mu^2}\right)^{-\epsilon/2}\Gamma(\epsilon/2), \tag{6.48}$$

where we have let d approach 4: $d=4-\epsilon$. We still have to integrate over $p-qu$ in the numerator. A linear term of this will vanish by symmetry, and for the quadratic term we consider

$$\int\frac{d^dp}{(2\pi)^d}p^\alpha p^\beta e^{-isp^2} = Ag^{\alpha\beta}. \tag{6.49}$$

Take the trace of this:

$$Ad = \int\frac{d^dp}{(2\pi)^d}p^2e^{-isp^2} = -\frac{1}{i}\frac{d}{ds}\int\frac{d^dp}{(2\pi)^d}e^{-isp^2}. \tag{6.50}$$

This gives the following relative evaluation

$$\langle p^\alpha p^\beta\rangle = g^{\alpha\beta}\frac{1}{2}\frac{\Gamma(1-d/2)}{\Gamma(2-d/2)}q^2u(1-u). \tag{6.51}$$

Therefore in the massless limit,

$$\begin{aligned} i\Pi^{ab}_{\alpha\beta}(q) &= \frac{g^2}{4}2\delta^{ab}\frac{i}{16\pi^2}\frac{2}{\epsilon}\left(\frac{q^2}{\mu^2}\right)^{-\epsilon/2}(-4)\int_0^1 du[u(1-u)]^{-\epsilon/2} \\ &\quad\times\left[-2q_\alpha q_\beta u(1-u)-g_{\alpha\beta}q^2u(1-u)-g_{\alpha\beta}[-q^2u(1-u)-2q^2u(1-u)]\right] \\ &= ig^2\delta^{ab}\frac{1}{12\pi^2}\frac{1}{\epsilon}\left(\frac{q^2}{\mu^2}\right)^{-\epsilon/2}(q_\alpha q_\beta-g_{\alpha\beta}q^2)+\text{finite terms}. \end{aligned} \tag{6.52}$$

This is all there is in QED, where there are no non-Abelian couplings, and $Z_1 = Z_2$. The result for QED is obtained from this by the replacement $g^2\delta^{ab} \to 2e^2$. If we sum the geometric series

$$\frac{-i}{q^2} + \frac{-i}{q^2} i\Pi(q) \frac{-i}{q^2} + \ldots = \frac{-i}{q^2 - \Pi(q)}, \tag{6.53}$$

and so we see

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon}. \tag{6.54}$$

Actually, we have already obtained this result in (4.98), where we recognize the coefficient of $-\frac{1}{4}F^2$ in the first line as Z_3^{-1} where

$$Z_3 = 1 - \frac{e^2}{12\pi^2}\int_{s_0}^{\infty} \frac{ds}{s} e^{-m^2 s} = 1 + \frac{e^2}{12\pi^2}\ln m^2 s_0. \tag{6.55}$$

This is the same as the above when we identify $2/\epsilon$ with $\ln m^2 s_0$. The corresponding β function is

$$\beta_{\text{QED}} = s_0 \frac{\partial}{\partial s_0} Z_3 e = \frac{e^3}{12\pi^2} > 0. \tag{6.56}$$

For QCD, we must multiply the above by n_f, the number of quark flavors, and add the other contributions to the vacuum polarization, for gluon and ghost loops, which you will calculated in homework:

$$i\Pi^{ab}_{\alpha\beta}(q) = i\delta^{ab}(q_\alpha q_\beta - g_{\alpha\beta}q^2)\frac{g^2}{8\pi^2}\left(\frac{q^2}{\mu^2}\right)^{-\epsilon/2}\frac{1}{\epsilon}\left[\frac{n_f}{2}C_2(\mathbf{3}) - \frac{5}{3}C_2(\mathbf{8})\right], \tag{6.57}$$

or

$$Z_3 = 1 - \frac{g^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{\epsilon}\frac{1}{\epsilon}\left[\frac{n_f}{2}C_2(\mathbf{3}) - \frac{5}{3}C_2(\mathbf{8})\right] + O(g^2), \tag{6.58}$$

where μ_R is the renormalization scale. From the quark and gluon vertices you will find

$$Z_1 = 1 - [C_2(\mathbf{3}) + C_2(\mathbf{8})]\frac{g^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{\epsilon}\frac{1}{\epsilon}, \tag{6.59}$$

while from the quark propagator we get

$$Z_2 = 1 - C_2(\mathbf{3})\frac{g^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{\epsilon}\frac{1}{\epsilon}. \tag{6.60}$$

(Note that the coefficients of $C_2(\mathbf{3})$ are the same for Z_1 and Z_2 – hence the remark above.) Therefore, by simple addition we obtain from (6.41)

$$\frac{g}{g_0} = 1 + \frac{g^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{\epsilon}\frac{1}{\epsilon}\left[-\frac{n_f}{4}C_2(\mathbf{3}) + \frac{11}{6}C_2(\mathbf{8})\right]. \tag{6.61}$$

We define the beta function by the logarithmic derivative of g with respect to the renormalization point,

$$\beta = \mu_R \frac{\partial g}{\partial \mu_R} = -\frac{g^3}{16\pi^2}(11 - \frac{2}{3}n_f), \tag{6.62}$$

where we have now inserted the values for the Casimir invariants from (6.30) and (6.34). This will be negative provided $n_f \leq 16$. (Notice for QED, where there is only the fermionic contribution, β is always positive.) The running coupling constant $g(Q^2)$ is then given by

$$Q\frac{dg}{dQ} = -\beta_1 g^3, \tag{6.63}$$

where β_1 is the positive coefficient of $-g^3$ in β, which is integrated by

$$\frac{dg}{g^3} = -\beta_1 \frac{dQ}{Q}, \tag{6.64}$$

which implies

$$\frac{1}{2g^2} - \frac{1}{2g_0^2} = \beta_1 \ln\frac{Q}{Q_0} = \frac{\beta_1}{2}\ln\frac{Q^2}{Q_0^2}. \tag{6.65}$$

Let $g_0 = \infty$ at $Q = Q_0$; then we have

$$g^2 = \frac{1}{\beta_1 \ln Q^2/Q_0^2}. \tag{6.66}$$

Explicitly, we have the running strong coupling given by

$$\alpha_s = \frac{g^2}{4\pi} = \frac{12\pi}{(33 - 2n_f)\ln Q^2/\Lambda^2} \tag{6.67}$$

where we have now called $Q_0 = \Lambda$, which is the scale of QCD, experimentally about 200 MeV. This exhibits the property of asymptotic freedom, that the coupling decreases with increasing energy.

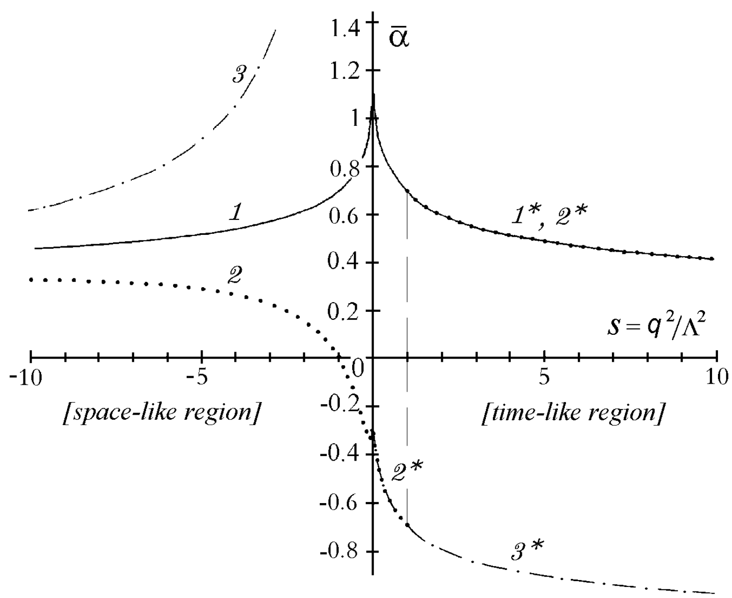


Figure 6.4: *The behavior of the running coupling constant calculated by different methods as a function of the dimensionless variable $s = q^2/\Lambda^2$. This figure is extracted from K. A. Milton and O. P. Solovtsova, Phys. Rev. D* **57***, 5402 (1998).*