Chapter 6

Quantum Chromodynamics

The theory of strong interactions is the theory of quarks and gluons – QCD – based on the gauge group SU(3) which has $3^2 - 1 = 8$ generators:

$$U = e^{ig\omega^a \lambda^a/2}, \quad a = 1, \dots, 8, \tag{6.1}$$

where $T^a = \lambda^a/2$ satisfy the SU(3) Lie algebra

$$[T^a, T^b] = if^{abc}T^c, \tag{6.2}$$

f^{abc} being the totally antisymmetric structure constants of SU(3) (see homework). A set of such traceless matrices satisfying the standard normalization

$$\mathrm{Tr}\,\lambda^a\lambda^b = 2\delta^{ab} \tag{6.3}$$

was given by Gell-Mann:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{6.4}$$

QCD consists of *colored* quarks, occuring in a triplet,

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}, \tag{6.5}$$

transforming according to

$$q \to Uq = e^{ig\omega^a\lambda^a/2}q. \tag{6.6}$$

Color $\mathrm{SU}(3)_c$ has nothing to do with approximate flavor $\mathrm{SU}(3)_f$, which manifests itself in hadronic multiplets, and reflects the approximate zero-mass character of the three lightest quarks, u, d, and s. Each flavor of quark comes in three colors, as indicated above, which could be called red, green, and blue. The quarks interact with gluons, which are also colored,

$$A_\mu = \frac{\lambda^a}{2} A_\mu^a, \quad a = 1, \dots 8, \tag{6.7}$$

an octet of gluons, which transform according to

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger. \tag{6.8}$$

For an infinitesimal transformation, where $\omega \to \delta\omega$, $\delta\omega = \delta\omega^a \lambda^a/2$, the gauge transformation of the gluon field is

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \delta\omega + ig[\delta\omega, A_\mu] \\ &= A_\mu + \mathcal{D}_\mu \delta\omega, \end{aligned} \tag{6.9}$$

where the covariant derivative in the adjoint representation is

$$\mathcal{D}_\mu = \partial_\mu - ig[A_\mu, \ \ ]. \tag{6.10}$$

In component form, this reads

$$A_\mu^a \to A_\mu^a + \partial_\mu \delta\omega^a - g f^{abc} \delta\omega^b A_\mu^c. \tag{6.11}$$

The field strength

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\lambda^a}{2} = \frac{i}{g}[D_\mu, D_\nu] \tag{6.12}$$

transforms covariantly:

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger \approx F_{\mu\nu} + i[\delta\omega, F_{\mu\nu}], \tag{6.13}$$

where the latter form refers to the infinitesimal case. In component form this is

$$F_{\mu\nu}^a \to F_{\mu\nu}^a - g f^{abc} \delta\omega^b F_{\mu\nu}^c. \tag{6.14}$$

The SU(3) invariant QCD Lagrangian is

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2} \mathrm{Tr}\, F^2 - \sum_f \overline{q}_f \left(\gamma \frac{1}{i} D + m_f \right) q_f, \tag{6.15}$$

where the covariant derivative in the triplet representation is

$$D_\mu = \partial_\mu - ig A_\mu. \tag{6.16}$$

In component form, the QCD Lagrangian reads

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \sum_f \overline{q}_f \left[\gamma^\mu \frac{1}{i} \left(\partial_\mu - ig \frac{\lambda^a}{2} A_\mu^a \right) + m_f \right] q_f. \tag{6.17}$$

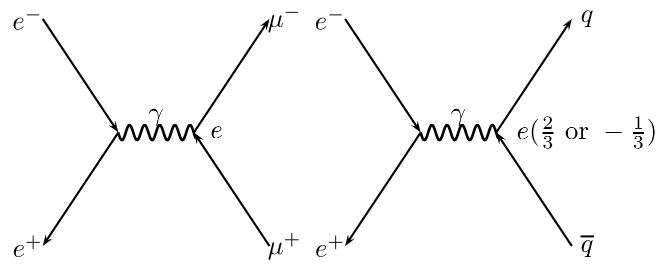

Figure 6.1: Feynman diagrams representing the production of $\mu^+\mu^-$ or of $q\overline{q}$ from electron-positron collisions.

A key consequence of the theory, we believe, is *confinement*. That is, the only physical states (hadrons) are color singlets:

$$q\overline{q} \quad \text{(meson)} \quad (\mathbf{3} \times \overline{\mathbf{3}} = \mathbf{8} + \mathbf{1}), \tag{6.18}$$

$$qqq \quad \text{(baryon)} \quad (\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{10} + \mathbf{8} + \mathbf{8} + \mathbf{1}), \tag{6.19}$$

where the boldface numbers refer to irreducible representations of SU(3), defined by their dimensionality. Phenomenologically this works well.[1] But it is somewhat of an embarassment that *glueball* states are not conspicous by their presence, whereas they should exist:

$$gg: \qquad \mathbf{8} \times \mathbf{8} = \mathbf{27} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{8} + \mathbf{8} + \mathbf{1}, \tag{6.20}$$

etc. In the hadrons there are, necessarily, virtual quarks and gluons (sea quarks and gluons) in addition to the valence quarks, and these play a significant role.

Before proceeding with a discussion of QCD, let us offer some experimental evidence of the existence of three colors. For example, at e^+e^- machines we can measure the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{6.21}$$

The corresponding amplitudes in lowest order are shown in Fig. 6.1. At high energy, masses are negligible, and

$$R = \sum_{f,c} Q^2 = N_c 3 \left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right] = \frac{5}{3} N_c, \tag{6.22}$$

if the energy is high enough to produce all six flavors of quark, including the top quark ($m_t \approx 175$ GeV). Below the top threshold, the ratio is reduced to $11N_c/9$. Experimentally, in that region, $R \approx 3.7$, which is consistent with $N_c = 3$. (There are small corrections to R due to radiative corrections, from which one can extract the value of $\alpha_s = g^2/4\pi \approx 0.2$ at this energy scale.)

[1] Recently, there has been much excitement because of various bits of evidence that pentaquark states, that is, states built of 5 quarks, exist in nature. But the evidence is weak, in that different experiments give disparate results. In my view, the whole effect will probably go away.

Mesons, then are described by a wavefunction

$$\Phi = \overline{q}_a q_a, \tag{6.23}$$

and baryons by

$$\Psi = \epsilon_{abc} q_a q_b q_c. \tag{6.24}$$

This resolves the old paradox that ground state wavefunctions tend to be symmetric, in violation of the Pauli principle. For example, the famous spin 3/2 baryon resonance, composed of three up quarks,

$$\Delta^{++}(1232) = uuu, \tag{6.25}$$

has a symmetric wavefunction in space and spin ($l = 0$), because

$$S_z = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}. \tag{6.26}$$

However, the requirement of the Fermi-Dirac statistics that the state be totally antisymmetric under the interchange of its fermionic components is satisfied because it is antisymmetric in color.