Chapter 5

Non-Abelian Gauge Fields

The simplest example starts with two Fermions (Dirac particles) ψ_1, ψ_2, degenerate in mass, and hence satisfying in the absence of interactions

$$(\gamma\frac{1}{i}\partial + m)\psi_1 = 0, \quad (\gamma\frac{1}{i}\partial + m)\psi_2 = 0. \tag{5.1}$$

We can define a two-component object $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with the associated action

$$W = \int (dx)\mathcal{L}, \quad \mathcal{L} = -\overline{\psi}\left(\gamma\frac{1}{i}\partial + m\right)\psi. \tag{5.2}$$

If U is a *constant* 2×2 matrix, $\mathcal{L}$ is invariant under the replacement $\psi \to U\psi$, provided U is unitary,

$$U^\dagger U = UU^\dagger = 1. \tag{5.3}$$

We know that the most general unitary 2×2 matrix, apart from a pure phase factor [$U(1)$ transformation], can be written in terms of the Pauli matrices $\boldsymbol{\tau}$ as

$$\begin{aligned} U \;&=\; e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\tau}} = \cos|\boldsymbol{\lambda}| + i\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\tau}}{|\boldsymbol{\lambda}|}\sin|\boldsymbol{\lambda}| \\ &= \begin{pmatrix} \cos\lambda + i\hat{\boldsymbol{\lambda}}_3\sin\lambda & (i\hat{\boldsymbol{\lambda}}_1 + \hat{\boldsymbol{\lambda}}_2)\sin\lambda \\ (i\hat{\boldsymbol{\lambda}}_1 - \hat{\boldsymbol{\lambda}}_2)\sin\lambda & \cos\lambda - i\hat{\boldsymbol{\lambda}}_3\sin\lambda \end{pmatrix}, \end{aligned} \tag{5.4}$$

where $\boldsymbol{\lambda}$ is a arbitrary vector. The generators of these transformations are the Pauli matrices, which obey the algebra

$$[\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c. \tag{5.5}$$

Because

$$\det U = \cos^2\lambda + (\hat{\boldsymbol{\lambda}}_3^2 + \hat{\boldsymbol{\lambda}}_1^2 + \hat{\boldsymbol{\lambda}}_2^2)\sin^2\lambda = 1, \tag{5.6}$$

this actually represents an SU(2) transformation.

Now suppose we gauge the symmetry, by letting $\boldsymbol{\lambda} \to \boldsymbol{\lambda}(x)$. Then $\mathcal{L}$ is not invariant,

$$\delta\mathcal{L} = -\overline{\psi}U^\dagger\gamma^\mu\frac{1}{i}(\partial_\mu U)\psi \approx -\partial_\mu\delta\boldsymbol{\lambda}\cdot\overline{\psi}\gamma^\mu\boldsymbol{\tau}\psi, \tag{5.7}$$

for $\delta\boldsymbol{\lambda}$ infinitesimal. We see here the conserved "isospin" current [compare (2.33)]

$$\delta W = -2\int(dx)\partial_\mu\delta\boldsymbol{\lambda}\cdot\mathbf{j}^\mu = 0, \tag{5.8}$$

where

$$\mathbf{j}^\mu = \frac{1}{2}\overline{\psi}\gamma^\mu\boldsymbol{\tau}\psi, \quad \text{or} \quad j^a_\mu = \frac{1}{2}\overline{\psi}\gamma_\mu\tau^a\psi, \tag{5.9}$$

which is conserved, by the stationary action principle,

$$\partial_\mu j^\mu = 0. \tag{5.10}$$

How can we cancel $\delta\mathcal{L}$ identically? Evidently, by couping to this current a *triplet* of vector fields,

$$\mathbf{A}_\mu = (A^1_\mu, A^2_\mu, A^3_\mu), \tag{5.11}$$

as follows:

$$\mathcal{L}^I_{\text{int}} = g\mathbf{A}_\mu\cdot\overline{\psi}\gamma^\mu\frac{\boldsymbol{\tau}}{2}\psi, \tag{5.12}$$

where under an infinitesimal gauge transformation

$$\mathbf{A}_\mu \to \mathbf{A}_\mu + \partial_\mu\delta\boldsymbol{\omega}, \tag{5.13}$$

where $\delta\boldsymbol{\omega}$ is related to $\delta\boldsymbol{\lambda}$:

$$\delta\boldsymbol{\lambda} = \frac{g}{2}\delta\boldsymbol{\omega}. \tag{5.14}$$

But this is not the whole story! That is because the ψ variation of $\mathcal{L}^I_{\text{int}}$ is

$$\begin{aligned}\delta_\psi\mathcal{L}^I_{\text{int}} &= -\overline{\psi}\gamma^\mu\left[\frac{\boldsymbol{\tau}}{2}\cdot g\mathbf{A}_\mu, \frac{i}{2}g\boldsymbol{\tau}\cdot\delta\boldsymbol{\omega}\right]\psi \\ &= g^2\overline{\psi}\gamma^\mu\frac{\boldsymbol{\tau}}{2}\cdot(\mathbf{A}_\mu\times\delta\boldsymbol{\omega})\psi \neq 0. \end{aligned} \tag{5.15}$$

This will be cancelled if we modify our $\mathbf{A}_\mu$ variation to

$$\mathbf{A}_\mu \to \mathbf{A}_\mu + \partial_\mu\delta\boldsymbol{\omega} - g\delta\boldsymbol{\omega}\times\mathbf{A}_\mu. \tag{5.16}$$

This last is in fact the transformation law for a vector under an ordinary rotation. Mathematically, we say that these *gauge fields* transform as the spin-1 (adjoint) representation of SU(2).

Now assemble the Fermion parts of $\mathcal{L}$:

$$\mathcal{L}_f = -\overline{\psi}\left(\gamma\frac{1}{i}D + m\right)\psi, \tag{5.17}$$

where the gauge covariant derivative is

$$D_\mu = \partial_\mu - \frac{i}{2} g \boldsymbol{\tau} \cdot \mathbf{A}_\mu. \tag{5.18}$$

Under a gauge transformation,

$$D_\mu \to U^\dagger D_\mu^U U = D_\mu. \tag{5.19}$$

That is,

$$\begin{aligned} D_\mu^U &= \partial_\mu - ig\frac{\boldsymbol{\tau}}{2} \cdot \mathbf{A}_\mu^U \\ &= U D_\mu U^\dagger = U\left(\partial_\mu - \frac{i}{2} g \boldsymbol{\tau} \cdot \mathbf{A}_\mu\right) U^\dagger \\ &= \partial_\mu - \frac{i}{2} g U \boldsymbol{\tau} U^\dagger \cdot \mathbf{A}_\mu + U \partial_\mu U^\dagger, \end{aligned} \tag{5.20}$$

which says

$$\frac{\boldsymbol{\tau}}{2} \cdot \mathbf{A}_\mu^U = U \frac{\boldsymbol{\tau}}{2} U^\dagger \cdot \mathbf{A}_\mu + \frac{i}{g} U \partial_\mu U^\dagger, \tag{5.21}$$

which generalizes the infinitesimal transformation given in (5.16). Indeed if

$$U = 1 + ig\delta\boldsymbol{\omega} \cdot \frac{\boldsymbol{\tau}}{2}, \tag{5.22}$$

we get

$$\begin{aligned} \frac{\boldsymbol{\tau}}{2} \cdot \mathbf{A}_\mu^U &= \frac{\boldsymbol{\tau}}{2} \cdot \mathbf{A}_\mu - \left[\frac{\boldsymbol{\tau}}{2}, ig\delta\boldsymbol{\omega} \cdot \frac{\boldsymbol{\tau}}{2}\right] \cdot \mathbf{A}_\mu - \frac{i}{g} ig \frac{\boldsymbol{\tau}}{2} \cdot \partial_\mu \delta\boldsymbol{\omega} \\ &= \frac{\boldsymbol{\tau}}{2} \cdot \mathbf{A}_\mu - i\frac{\boldsymbol{\tau}}{2} \cdot ig\mathbf{A}_\mu \times \delta\boldsymbol{\omega} + \frac{\boldsymbol{\tau}}{2} \cdot \partial_\mu \delta\boldsymbol{\omega}, \end{aligned} \tag{5.23}$$

which agrees with (5.16).

It is obviously convenient to define a matrix representation for the gauge fields:

$$A_\mu = \frac{\boldsymbol{\tau}}{2} \cdot \mathbf{A}_\mu. \tag{5.24}$$

Then the above gauge transformation (5.21) reads

$$A_\mu^U = U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger, \tag{5.25}$$

and the covariant derivative is

$$D_\mu = \partial_\mu - igA_\mu. \tag{5.26}$$

This last includes the Abelian case, where $U = e^{ie\lambda}$, $\boldsymbol{\tau}/2 \to 1$, which is a U(1) gauge group. To pick out the components of the gauge field, we recall that

$$\tau_a \tau_b = \delta_{ab} + i\epsilon_{abd}\tau_c, \tag{5.27a}$$

so because $\mathrm{Tr}\,1 = 2$ we have

$$\mathrm{Tr}\,\tau_a\tau_b = 2\delta_{ab}. \tag{5.27b}$$

Therefore

$$A^a_\mu = \mathrm{Tr}\,\left(\tau_a \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{A}\right) = \mathrm{Tr}\,(\tau_a A_\mu). \tag{5.28}$$

Is the above interaction, minimal substitution, the end of the story? No, because we must consider the gauge field part of the action. Now

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu].\end{aligned} \tag{5.29}$$

Because our fields are non-Abelian, this last commutator is nonzero. Explicitly,

$$[A_\mu, A_\nu] = \left[\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{A}_\mu, \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{A}_\nu\right] = \frac{i}{2}\boldsymbol{\tau}\cdot(\mathbf{A}_\mu\times\mathbf{A}_\nu). \tag{5.30}$$

We will define the commutator as the field strength,

$$[D_\mu, D_\nu] = -igF_{\mu\nu}, \tag{5.31}$$

where

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]. \tag{5.32}$$

In terms of components, related to $F_{\mu\nu}$ by

$$F_{\mu\nu} = \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{F}_{\mu\nu}, \tag{5.33}$$

we have

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu, \tag{5.34a}$$

or

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\mathbf{A}_\mu\times\mathbf{A}_\nu. \tag{5.34b}$$

Why is this a useful quantity? Because it transforms *covariatly*,

$$F_{\mu\nu} \to F^U_{\mu\nu} = \frac{i}{g}[D^U_\mu, D^U_\nu] = UF_{\mu\nu}U^\dagger, \tag{5.35}$$

unlike the potential, as seen in (5.25). In infinitesimal form this means

$$\mathbf{F}_{\mu\nu} \to \mathbf{F}_{\mu\nu} - g\delta\boldsymbol{\omega}\times\mathbf{F}_{\mu\nu}. \tag{5.36}$$

From the field strength, the gauge field part of the Lagrangian can be constructed. (Why? Because it's gauge invariant!)

$$\mathcal{L}_g = -\frac{1}{2}\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}. \tag{5.37}$$

Under a gauge transformation, $\mathcal{L}_g \to \mathcal{L}_g$. Explicitly,

$$\begin{aligned}\mathcal{L}_g &= -\frac{1}{4}(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\cdot(\partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu)\\ &\quad - \frac{g}{2}(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\nu)\cdot(\mathbf{A}^\mu \times \mathbf{A}^\nu)\\ &\quad - \frac{g^2}{4}(\mathbf{A}_\mu\cdot\mathbf{A}^\mu\mathbf{A}_\nu\cdot\mathbf{A}^\nu - \mathbf{A}_\mu\cdot\mathbf{A}_\nu\mathbf{A}^\mu\cdot\mathbf{A}^\nu),\end{aligned} \tag{5.38}$$

because

$$\epsilon^{abc}\epsilon^{ade} = \delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}. \tag{5.39}$$

Note that the requirement of gauge invariance necessarily leads to cubic and quartic self-interactions of the gauge field, with the same coupling constnat as appears in the gauge-field–fermion interaction.

5.1 Summary

For an arbitrary gauge group, the Lagrangian is

$$\mathcal{L} = -\overline{\psi}\left(\gamma\frac{1}{i}D + m\right)\psi - \frac{1}{2}\mathrm{Tr}\,F^2, \tag{5.40}$$

where the gauge covariant derivative is

$$D_\mu = \partial_\mu - igA_\mu, \tag{5.41}$$

and the gauge-covariant field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{5.42}$$

This Lagrangian is invariant under the gauge transformations

$$\psi \to U\psi, \tag{5.43a}$$

$$A_\mu \to UA_\mu U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger, \tag{5.43b}$$

$$F_{\mu\nu} \to UF_{\mu\nu}U^\dagger. \tag{5.43c}$$

For SU(2), more explicitly, the Lagrangian is

$$\mathcal{L} = -\overline{\psi}\left[\gamma^\mu\frac{1}{i}\left(\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{A}_\mu\right) + m\right]\psi - \frac{1}{4}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}, \tag{5.44}$$

where the field strength is

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\mathbf{A}_\mu\times\mathbf{A}_\nu. \tag{5.45}$$

The gauge transformations are

$$U = e^{ig\boldsymbol{\omega}\cdot\boldsymbol{\tau}/2}, \tag{5.46}$$

so for an infinitesimal transformation, $\omega \to \delta\omega$,

$$\psi \to \left(1 + ig\delta\boldsymbol{\omega}\cdot\frac{\boldsymbol{\tau}}{2}\right)\psi, \tag{5.47a}$$

$$\mathbf{A}_\mu \to \mathbf{A}_\mu - g\delta\boldsymbol{\omega}\times\mathbf{A}_\mu + \partial_\mu\delta\boldsymbol{\omega}, \tag{5.47b}$$

$$\mathbf{F}_{\mu\nu} \to \mathbf{F}_{\mu\nu} - g\delta\boldsymbol{\omega}\times\mathbf{F}_{\mu\nu}. \tag{5.47c}$$

From the Lagrangian we can derive the equations of motion: Varying with respect to $\overline{\psi}$ gives the gauge-covariant Dirac equation,

$$\left(\gamma^\mu \frac{1}{i} D_\mu + m\right)\psi = 0. \tag{5.48}$$

Under a δA_μ transformation,

$$\delta\mathcal{L} = \overline{\psi}\gamma^\mu g\delta A_\mu\psi - \operatorname{Tr} F^{\mu\nu}\left(2\partial_\mu\delta A_\nu - 2igA_\mu\delta A_\nu - 2ig\delta A_\mu A_\nu\right), \tag{5.49}$$

so for SU(2) the change in the action is

$$\delta W = \int (dx)\delta\mathbf{A}_\mu\cdot\left(\overline{\psi}\gamma^\mu g\frac{\boldsymbol{\tau}}{2}\psi + \partial_\nu\mathbf{F}^{\nu\mu} + g\mathbf{F}^{\mu\nu}\times\mathbf{A}_\nu\right), \tag{5.50}$$

where we have used (5.45). Thus, the Yang-Mills equation (the generalization of Maxwell's equation) is

$$\partial_\nu\mathbf{F}^{\mu\nu} = \mathbf{j}^\mu, \tag{5.51}$$

with the current

$$\mathbf{j}^\mu = \overline{\psi}\gamma^\mu g\frac{\boldsymbol{\tau}}{2}\psi + g\mathbf{A}_\nu\times\mathbf{F}^{\nu\mu}. \tag{5.52}$$

The current has both fermion and gauge-boson pieces. Alternatively, we can define a gauge covariant derivative for the adjoint (isospin-1) representation of SU(2) by

$$\boldsymbol{\mathcal{D}}_\nu = \mathbf{1}\partial_\nu - g\mathbf{A}_\nu\times\,, \tag{5.53a}$$

$$\boldsymbol{\mathcal{D}}_\nu\cdot\mathbf{F}^{\mu\nu} = g\mathbf{j}^\mu_f = g\overline{\psi}\gamma^\mu\frac{\boldsymbol{\tau}}{2}\psi, \tag{5.53b}$$

where $\boldsymbol{\mathcal{D}}_\nu$ is a tensor, with components

$$(\boldsymbol{\mathcal{D}}_\nu)_{ab} = \delta_{ab}\partial_\nu + g\epsilon_{abc}A^c_\nu. \tag{5.53c}$$