The first term here describes a modification of the electron propagator, which is involved in renormalization of the mass of the electron. The second term is what is of interest here:

$$\frac{\alpha}{2\pi}\frac{m}{s}u^2(1-u)e^{-ism^2u^2}eq\sigma F. \tag{4.73}$$

The integrals over the parameters s and u are as follows:

$$\int_0^\infty \frac{ds\, s}{s}\int_0^1 du\, u^2(1-u)e^{-isu^2(m^2-i\epsilon)} = -\frac{1}{im^2}\int_0^1 du\,(1-u) = \frac{i}{2m^2}, \tag{4.74}$$

and then the vacuum amplitude (4.55) is

$$\frac{i}{2}\int(dx)\psi(x)\gamma^0\frac{eq}{2m}\sigma F\frac{\alpha}{2\pi}\psi(x). \tag{4.75}$$

This is interpreted as a correction to the g-factor of the electron, where $g=2$ for a particle described by the Dirac equation:

$$\frac{g-2}{2}=\frac{\alpha}{2\pi}=\frac{1}{2\pi}\frac{1}{137.036}=0.0011614, \tag{4.76}$$

which is to be compared to the experimental value

$$\left(\frac{g-2}{2}\right)_{\text{exp}}=0.01159652187(4); \tag{4.77}$$

the discrepancy is entirely due to higher order QED effects, which have been computed out to 10th order! [For details, see *Quantum Electrodynamics*, ed. T. Kinoshita (World Scientific, Singapore, 1990).]

4.4.1 Mass renormalization

Let us return to the constant term in (4.72) It may be interpreted as a shift in the mass of the electron,

$$\delta m = -m\frac{\alpha}{2\pi}\int_{s_0}^\infty \frac{ds}{s}\int du(1+u)e^{-ism^2u^2}, \tag{4.78}$$

where we have recognized that the integral is divergent, and so have regulated it by inserted a lower limit on the proper time integral, $s_0 \to)+$. If I make the replacement to Euclidean proper time, $s\to -is$, we see that the integral over s has the divergent part $-lnm^2 s_0$. Thus the mass shift is

$$\delta m = \frac{3\alpha}{4\pi}m\ln\frac{1}{m^2 s_0}, \tag{4.79}$$

which may be interpreted as a mass counterterm.

4.5 Vacuum Polarization

We now return to the amputated graph shown in Fig. 4.2. According to (4.54) we write it as

$$\begin{aligned} i\Pi^{\mu\nu} &= -\mathrm{tr}\int\frac{(dp)}{(2\pi)^4}(-ie\gamma^\mu)\frac{-i}{m+\gamma p}(-ie\gamma^\nu)\frac{-i}{m+\gamma(p-k)} \\ &= -e^2\,\mathrm{tr}\int\frac{(dp)}{(2\pi)^4}\gamma^\mu\frac{m-\gamma p}{m^2+p^2}\gamma^\nu\frac{m+\gamma(k-p)}{m^2+(p-k)^2}. \end{aligned} \tag{4.80}$$

Here, the trace refers only to the Dirac space. Consider what happens when $\Pi^{\mu\nu}$ is contracted with k_μ. Because $\gamma k = m+\gamma p - m - \gamma(p-k)$, we cancel off one or the other denominator:

$$k_\mu\Pi^{\mu\nu} \propto \mathrm{tr}\left(\gamma^\nu\frac{1}{m+\gamma(p-k)} - \frac{1}{m+\gamma p}\gamma^\nu\right). \tag{4.81}$$

If we are permitted to shift the integration variable in the first term, $p \to p+k$, it is apparent that the two terms here cancel. Although that shift is actually illegitimate here because the individual integrals are quadratically divergent, we will actually impose this condition as a constraint in order to define the vacuum polarization operator:

$$k_\mu\Pi^{\mu\nu} = 0. \tag{4.82}$$

This states that the current to which the photon couples is conserved, and is a signal of gauge invariance. As a consequence, we can write

$$\Pi^{\mu\nu} = (k^\mu k^\nu - g^{\mu\nu}k^2)\Pi(k), \tag{4.83}$$

and our task is to compute the scalar function Π. That can be done by taking the trace:

$$\Pi^\mu{}_\mu = -3k^2\Pi = ie^2\,\mathrm{tr}\int\frac{(dp)}{(2\pi)^4}\gamma^\lambda\frac{m-\gamma p}{m^2+p^2}\gamma_\lambda\frac{m+\gamma(k-p)}{m^2+(p-k)^2}. \tag{4.84}$$

We can combine the denominators as before:

$$\frac{1}{m^2+p^2}\frac{1}{m^2+(p-k)^2} = -\int_0^\infty ds\,s\int_0^1 du\,e^{-is\chi}, \tag{4.85}$$

where

$$\begin{aligned} \chi &= u[m^2+(p-k)^2] + (1-u)(m^2+p^2) \\ &= (p-uk)^2 + m^2 + k^2u(1-u). \end{aligned} \tag{4.86}$$

The trace over Dirac matrices may be easily worked out using

$$\mathrm{tr}\,1 = 4, \tag{4.87a}$$

$$\mathrm{tr}\,\gamma^\mu = 0, \tag{4.87b}$$

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu = \mathrm{tr}\,\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = -4g^{\mu\nu}. \tag{4.87c}$$

Further noting that

$$\gamma^\lambda \gamma p \gamma_\lambda = -2p^\lambda \gamma_\lambda - \gamma p \gamma^\lambda \gamma_\lambda = -2\gamma p + 4\gamma p = 2\gamma p, \tag{4.88}$$

we find for the trace in (4.84)

$$\begin{aligned} \operatorname{tr} \gamma^\lambda (m - \gamma p)\gamma_\lambda [m + \gamma(k-p)] &= \operatorname{tr}(-4m - 2\gamma p)(m - \gamma(p-k)) \\ &= 4(-4m^2 - 2p(p-k)). \end{aligned} \tag{4.89}$$

Now we carry out the required integrals over p, following (4.64):

$$\int \frac{(dp)}{(2\pi)^4} e^{-is(p-ku)^2} = -\frac{i}{16\pi^2 s^2}, \tag{4.90a}$$

$$\int \frac{(dp)}{(2\pi)^4} (p-ku)^\mu e^{-is(p-ku)^2} = 0, \tag{4.90b}$$

$$\int \frac{(dp)}{(2\pi)^4} (p-ku)^2 e^{-is(p-ku)^2} = \frac{d}{ds}\frac{1}{16\pi^2 s^2} = -\frac{2i}{s}\frac{-i}{16\pi^2 s^2}. \tag{4.90c}$$

So using

$$2p(p-k) = 2[(p-uk)^2 - k(p-uk)(1-2u) - k^2 u(1-u)], \tag{4.91}$$

we find

$$-3k^2\Pi = i\frac{e^2}{\pi^2}\int_0^\infty \frac{ds}{s}\int_0^1 du \left[-im^2 - \frac{1}{s} + \frac{i}{2}k^2 u(1-u)\right] e^{-ism^2 - isu(1-u)k^2}. \tag{4.92}$$

Now we note that the first two terms in the square bracket appear as

$$\left(-im^2 - \frac{1}{s}\right)\frac{1}{s}e^{-ism^2} = \frac{d}{ds}\left(\frac{1}{s}e^{-ism^2}\right), \tag{4.93}$$

so integrating by parts on s in this term and omitting the surface term, with is either a constant or a term linear in k^2, we get

$$-3k^2\Pi = -\frac{3e^2}{2\pi^2}k^2\int_0^\infty \frac{ds}{s}\int_0^1 du\, u(1-u) e^{-is[m^2 + k^2 u(1-u)]}. \tag{4.94}$$

If we return to (4.83) and recall the external propagators, we have in the action

$$\begin{aligned} \frac{1}{2}J^\mu(-k)\frac{1}{k^2}(k_\mu k_\nu - g_{\mu\nu}k^2)\Pi\frac{1}{k^2}J^\nu(k) &= \frac{1}{2}A^\mu(-k)(k_\mu k_\nu - g_{\mu\nu}k^2)\Pi A^\nu(k) \\ &= -\frac{1}{4}F^{\mu\nu}(-k)F_{\mu\nu}(k)\Pi, \end{aligned} \tag{4.95}$$

so with the change of variable $u = (1+v)/2$, we have find the following effective correction to the action,

$$\begin{aligned} W^1 &= -\int \frac{(dk)}{(2\pi)^4}\frac{1}{4}F^{\mu\nu}(-k)F_{\mu\nu}(k) \\ &\quad \times \frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\int_0^1 dv\,(1-v^2)e^{-isk^2\frac{1}{4}(1-v^2)}. \end{aligned} \tag{4.96}$$

We may isolate the divergent term by integrating by parts on v, in which case the v integral here becomes

$$\frac{2}{3} - \frac{isk^2}{2}\int_0^1 dv\,\left(v^2 - \frac{v^4}{3}\right) e^{-isk^2\frac{1}{4}(1-v^2)}. \tag{4.97}$$

Then including the free Maxwell Lagrangian we find

$$\begin{aligned} W &= -\frac{1}{4}\int\frac{(dk)}{(2\pi)^4}F^{\mu\nu}(-k)F_{\mu\nu}(k)\left(1+\frac{e^2}{12\pi^2}\int_0^\infty\frac{ds}{s}e^{-ms^2}\right)\\ &\quad + \frac{e^2}{16\pi^2}\frac{1}{4}\int\frac{(dk)}{(2\pi)^4}F^{\mu\nu}(-k)F_{\mu\nu}(k)k^2\int_0^1 dv\frac{v^2\left(1-\frac{v^2}{3}\right)}{m^2+k^2\frac{1}{4}(1-v^2)}, \end{aligned} \tag{4.98}$$

where in the first line we have again made the Euclidean substitution, $s \to -is$, and in the second carried out the trivial s integration. The first line of W exhibits an infinite charge and wavefunction renormalization of the Maxwell action. We can recast the second, vacuum polarization term, in a more physical spectral form by the change of variables,

$$M^2 = \frac{4m^2}{1-v^2}, \tag{4.99}$$

with the result that the photon propagator is modified to

$$\overline{D}_+(k) = \frac{1}{k^2 - i\epsilon} + \frac{\alpha}{3\pi}\int_{4m^2}^\infty \frac{dM^2}{M^2}\frac{\left(1-\frac{4m^2}{M^2}\right)^{1/2}\left(1+\frac{2m^2}{M^2}\right)}{k^2+M^2-i\epsilon}. \tag{4.100}$$

For application of these results, see the homework.