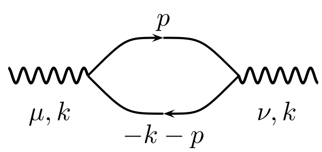

Figure 4.2: Lowest-order vacuum polarization graph

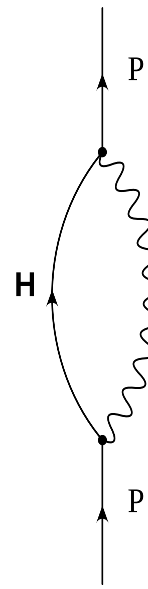

Figure 4.3: Radiative correction to the propagation of an electron in an external magnetic field H.

4.4 The Anomalous Magnetic Moment of the Electron

Here we offer a derivation of the electron's $g-2$ anomaly based on a correction the the electron propagator in an external magnetic field $\mathbf{H}$. Consider the process shown in Fig. 4.3. When $\mathbf{H}=\mathbf{0}$ the vacuum persistence amplitude for this process is given by

$$\frac{(ie)^2}{2}\int\frac{(dP)}{(2\pi)^4}\psi(-P)\gamma^0\gamma^\mu\int\frac{(dk)}{(2\pi)^4}\frac{-i}{k^2}\frac{-i}{m+\gamma\cdot(P-k)}\gamma_\mu\psi(P). \tag{4.55}$$

To incorporate the effects of the magnetic field, we make the minimal substitution, with q being the charge matrix,

$$P\to\Pi=P-eqA, \tag{4.56}$$

so the gauge-covariant momentum satisfies

$$[\Pi^\mu, \Pi^\nu] = ieqF^{\mu\nu}, \tag{4.57}$$

in terms of the field strength tensor, assumed here constant. Further, we compute

$$\begin{aligned}(\gamma \cdot \Pi)^2 &= \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\Pi_\mu\Pi_\nu + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\Pi_\mu\Pi_\nu \\ &= -\Pi^2 - i\sigma^{\mu\nu}\frac{i}{2}eqF_{\mu\nu} \\ &= -\Pi^2 + eq\sigma F, \quad \sigma F = \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} = \boldsymbol{\sigma} \cdot \mathbf{H},\end{aligned} \tag{4.58}$$

for the case of an external magnetic field. The electron propagator then is

$$\frac{1}{m + \gamma \cdot (\Pi - k)} = \frac{m - \gamma \cdot (\Pi - k)}{m^2 + (\Pi - k)^2 - eq\sigma F}. \tag{4.59}$$

It is useful to combine the denominators in an exponential representation. Write

$$\begin{aligned}\frac{1}{k^2}\frac{1}{(\Pi - k)^2 - eq\sigma F + m^2} &= -\int_0^\infty ds_1\, ds_2\, e^{-is_1k^2 - is_2[(\Pi-k)^2 - eq\sigma F + m^2]} \\ &= -\int_0^\infty ds\, s\int_0^1 du\, e^{-is\chi(u)},\end{aligned} \tag{4.60}$$

where we have introduced the proper time s, and the "Feynman" parameter u,

$$s_1 = s(1 - u), \quad s_2 = su, \tag{4.61}$$

where the Jacobian of the transformation is

$$\frac{\partial(s_1, s_2)}{\partial(s, u)} = \begin{vmatrix} 1 - u & u \\ -s & s \end{vmatrix} = s. \tag{4.62}$$

The exponential term in (4.60) is

$$\begin{aligned}\chi(u) &= (1 - u)k^2 + u[(\Pi - k)^2 - eq\sigma F + m^2] \\ &= (k - u\Pi)^2 + u(1 - u)\Pi^2 + u(m^2 - eq\sigma F).\end{aligned} \tag{4.63}$$

Now we carry out the k integration by a Euclidean rotation,

$$\int \frac{(dk)}{(2\pi)^4}e^{-isk^2} = i\int \frac{(dk)_E}{(2\pi)^4}e^{-isk_E^2} = -\frac{i}{16\pi^2 s^2}. \tag{4.64}$$

so then we have here for the basic integral

$$\int \frac{(dk)}{(2\pi)^4}e^{-is\chi(u)} = -\frac{i}{16\pi^2}\frac{1}{s^2}e^{-isu^2(m^2 - eq\sigma F)}e^{-is\mathcal{H}}, \tag{4.65}$$

where

$$\mathcal{H} = u(1-u)(\Pi^2 + m^2 - eq\sigma F) = u(1-u)[m^2 - (\gamma\cdot\Pi)^2]. \tag{4.66}$$

Here, in doing the k integration, we have ignored the noncommutativity of Π, because this would give rise to a term proportional to $[\Pi^\mu, \Pi^\nu]F_{\mu\nu} \propto F^2$, which is irrelevant for the magnetic moment term, which is linear in F.

What actually appears in the $P \to \Pi$ generalization of Eq. (4.55) is

$$\begin{aligned} e^2 \int \frac{(dk)}{(2\pi)^2}\gamma^\mu[m - \gamma\cdot(\Pi - k)]e^{-is\chi}\gamma_\mu \\ = e^2\int\frac{(dk)}{(2\pi)^4}\{[m+\gamma\cdot(\Pi-k)]\gamma^\mu + 2(\Pi-k)^\mu\}e^{-is\chi}\gamma_\mu. \end{aligned} \tag{4.67}$$

By virtue of the external Dirac field, we can set (on the outside) $\gamma\cdot\Pi + m \to 0$; then we can do the k integration by writing it in terms of

$$\int\frac{(dk)}{(2\pi)^4}(k - u\Pi)^\mu e^{-is\chi}. \tag{4.68}$$

This would be zero if the Πs were commuting variables, because $\chi(u)$ is even in $k - u\Pi$. Although they are not, we get here something proportional to $F^{\mu\nu}\Pi_\nu$, which is contracted with γ^μ:

$$\gamma_\mu F^{\mu\nu}\Pi_\nu = \frac{i}{2}[\sigma F, \gamma\cdot\Pi + m] \to 0, \tag{4.69}$$

using (1.154), where again we have ignored the F dependence in χ. So, in the numerator we may replace k^μ by $u\Pi^\mu$. The expression (4.67) is then, to $\mathcal{O}(F)$ is ($\alpha = e^2/4\pi$)

$$\begin{aligned} &-\frac{ie^2}{16\pi^2}\frac{1}{s^2}e^{-isu^2m^2}\left[-\gamma\cdot u\Pi\gamma^\mu + 2(1-u)\Pi^\mu\right]e^{-is\mathcal{H}}\left(1 + isu^2eq\sigma F\right)\gamma_\mu \\ &\to -\frac{i\alpha}{4\pi}\frac{1}{s^2}e^{-ism^2u^2}m\big[u\gamma^\mu e^{-is\mathcal{H}}(1+isu^2eq\sigma F)\gamma_\mu \\ &\qquad - 2(1-u)e^{-is\mathcal{H}}(1+isu^2eq\sigma F)\big], \end{aligned} \tag{4.70}$$

where we have again used Eq. (4.69). Now we evaluate this by putting the $isu^2eq\sigma F$ term in the exponent:

$$\begin{aligned} \gamma^\mu e^{-is(\mathcal{H}-u^2eq\sigma F)}\gamma_\mu &= \gamma^\mu\left[e^{-is[u(1-u)(\Pi^2+m^2)]}(1+isueq\sigma F)\right]\gamma_\mu \\ &= -4e^{-isu(1-u)(\Pi^2+m^2)} = -4e^{-isu(1-u)[m^2-(\gamma\cdot\Pi)^2+eq\sigma F]} \\ &= -4e^{-is\mathcal{H}}[1 - isu(1-u)eq\sigma F] \\ &\to -4\left[1 - isu(1-u)eq\sigma F\right], \end{aligned} \tag{4.71}$$

where in the second line we have used the fact that $\gamma^\lambda\sigma_{\alpha\beta}\gamma_\lambda = 0$. Thus we have from Eq. (4.70),

$$\begin{aligned} &-\frac{i\alpha}{4\pi}\frac{1}{s^2}e^{-ism^2u^2}m\left\{-4u[1-isu(1-u)eq\sigma F] - 2(1-u)(1+isu^2eq\sigma F)\right\} \\ &= -\frac{i\alpha}{4\pi}\frac{1}{s^2}me^{-ism^2u^2}\left[-2(1+u) + 2isu^2(1-u)eq\sigma F\right]. \end{aligned} \tag{4.72}$$

The first term here describes a modification of the electron propagator, which is involved in renormalization of the mass of the electron. The second term is what is of interest here:

$$\frac{\alpha}{2\pi}\frac{m}{s}u^2(1-u)e^{-ism^2u^2}eq\sigma F. \tag{4.73}$$

The integrals over the parameters s and u are as follows:

$$\int_0^\infty \frac{ds\,s}{s}\int_0^1 du\,u^2(1-u)e^{-isu^2(m^2-i\epsilon)} = -\frac{1}{im^2}\int_0^1 du\,(1-u) = \frac{i}{2m^2}, \tag{4.74}$$

and then the vacuum amplitude (4.55) is

$$\frac{i}{2}\int(dx)\psi(x)\gamma^0\frac{eq}{2m}\sigma F\frac{\alpha}{2\pi}\psi(x). \tag{4.75}$$

This is interpreted as a correction to the g-factor of the electron, where $g = 2$ for a particle described by the Dirac equation:

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} = \frac{1}{2\pi}\frac{1}{137.036} = 0.0011614, \tag{4.76}$$

which is to be compared to the experimental value

$$\left(\frac{g-2}{2}\right)_{\text{exp}} = 0.01159652187(4); \tag{4.77}$$

the discrepancy is entirely due to higher order QED effects, which have been computed out to 10th order! [For details, see *Quantum Electrodynamics*, ed. T. Kinoshita (World Scientific, Singapore, 1990).]